How to avoid a workout

(like a lazy mathematician)

A different kind of integral in 3D: "line integral"

- Work done by a force on a moving object:

$$W = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

Line integrals

$$\int_C \mathbf{F}(x,y) \cdot d\mathbf{r}$$

Line integrals

$$\int_C P(x,y)dx + Q(x,y)dy$$

where **F** = P **i** + Q **j**, and d**r** = dx **i** + dy **j**

Important! In a line integral we do ***not*** *hold x or y fixed while letting the other one vary.*

Line integrals

$$\vec{F}\cdot d\vec{r} = \begin{bmatrix} P(\vec{r}) \\ Q(\vec{r}) \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$\int_C P(x,y)dx + Q(x,y)dy + R(x,y,z)dz$$

where **F** = P **i** + Q **j**, and d**r** = dx **i** + dy **j** + dz $\hat{k}$

$$\vec{r}(t) \quad d\vec{r} = \frac{d\vec{r}}{dt}\,dt$$

Line integrals

$$\int_{t=a}^{t=b} P(x,y)\frac{dx}{dt}dt + Q(x,y)\frac{dy}{dt}dt$$

where **F** = P **i** + Q **j**, and d**r** = dx **i** + dy **j**

Examples from the previous episode.

- In all cases, let the curve be the unit circle, traversed counterclockwise.

1. **F**(x) = x **i** + y **j**

2. **F**(x) = y **i** + x **j**

3. **F**(x) = y **i** - x **j**. This time **F** · d**r** =(- $\sin^2 t$ - $\cos^2 t$) dt = - dt.
   The integral around the whole circle is -2π , even though the beginning and end points are the same.

   Under what conditions is it true that an integral from **a** to **b**=**a** gives us 0 - as in one dimension?

*Under what conditions is it true that an integral from **a** to **b=a** gives us 0 - as in one dimension?*

Related questions:

1. Is there such a thing as an antiderivative?
2. Does the value of the integral depend on the path you take?

The fundamental theorem for line integrals, at least some of them…

The fundamental theorem

- Assuming C is ___**?**____, f is ___**?**____, and **h** = ∇f on a set that is _____**?**______:

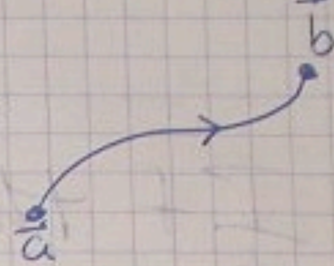
$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}).$$

Path-independence

- Can we ever reason that if the curve C goes from **a** to **b**, then the integral is just of the form f(**b**) - f(**a**), as in one dimension?

Reversal of path.

C:

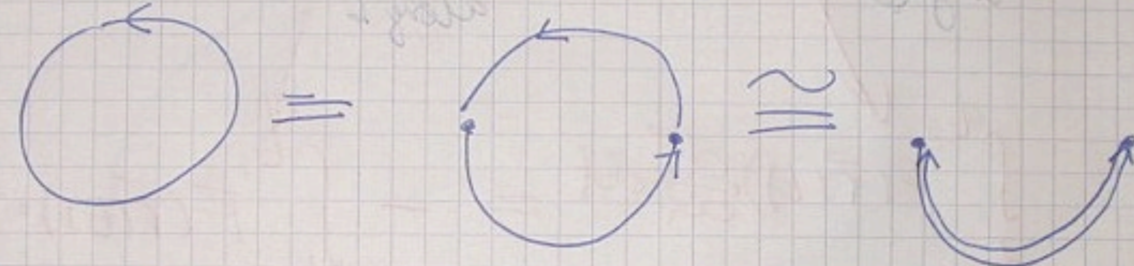


$$\int_{\vec a}^{\vec b} F(\vec r)\cdot d\vec r = -\int_{\vec b}^{\vec a} F(\vec r)\cdot dr$$

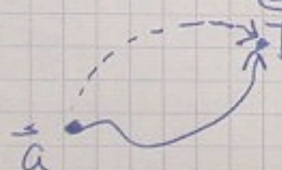
along C   along C

$$\int_{t_1}^{t_2} F(\vec r(t))\cdot \frac{d\vec r}{dt}\,dt = -\int_{t_2}^{t_1} F(\vec r(t))\cdot \frac{dr}{dt}\,dt$$

If the integral does not depend on the path, then the integral over any closed loop is 0.



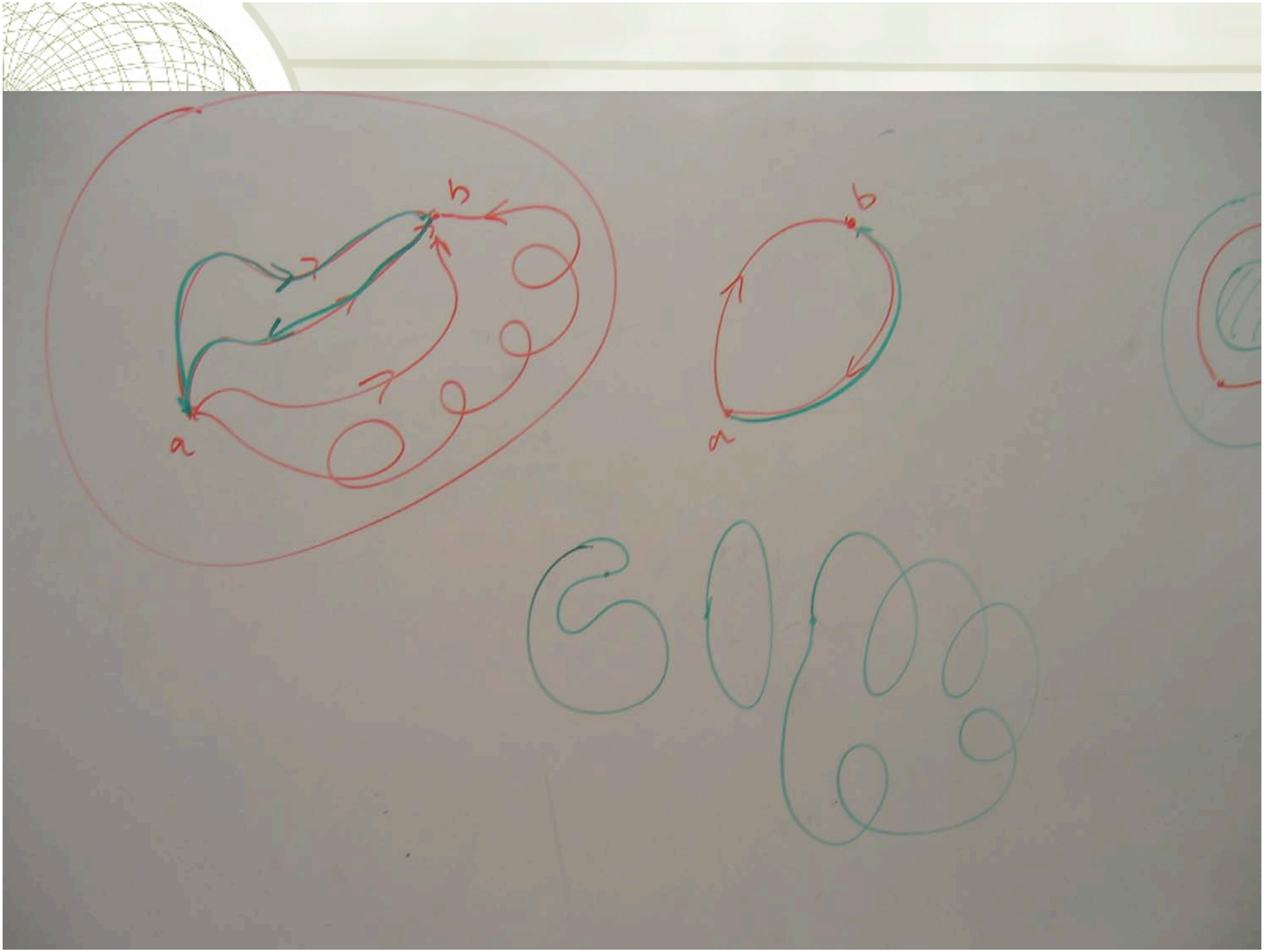
Vice versa If the integral over every closed loop is 0, then the integral from $\bar{a}$ to $\vec{b}$ is independent of the path.



⇐ If we reverse the second path and combine, we get a loop

Path-independence

- The technicalities of the theorem that path-independence is equivalent to the fact that integrals over all loops are zero.
  - Paths stay within an open, “simply connected” domain.
  - Curve and vector function **F** are sufficiently nice to change variables. Say, continuously differentiable.


b
a
b
a

The fundamental theorem

- Assuming C is ____________, f is _________, and **h** = ∇f on a set that is ______________:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}).$$

Examples

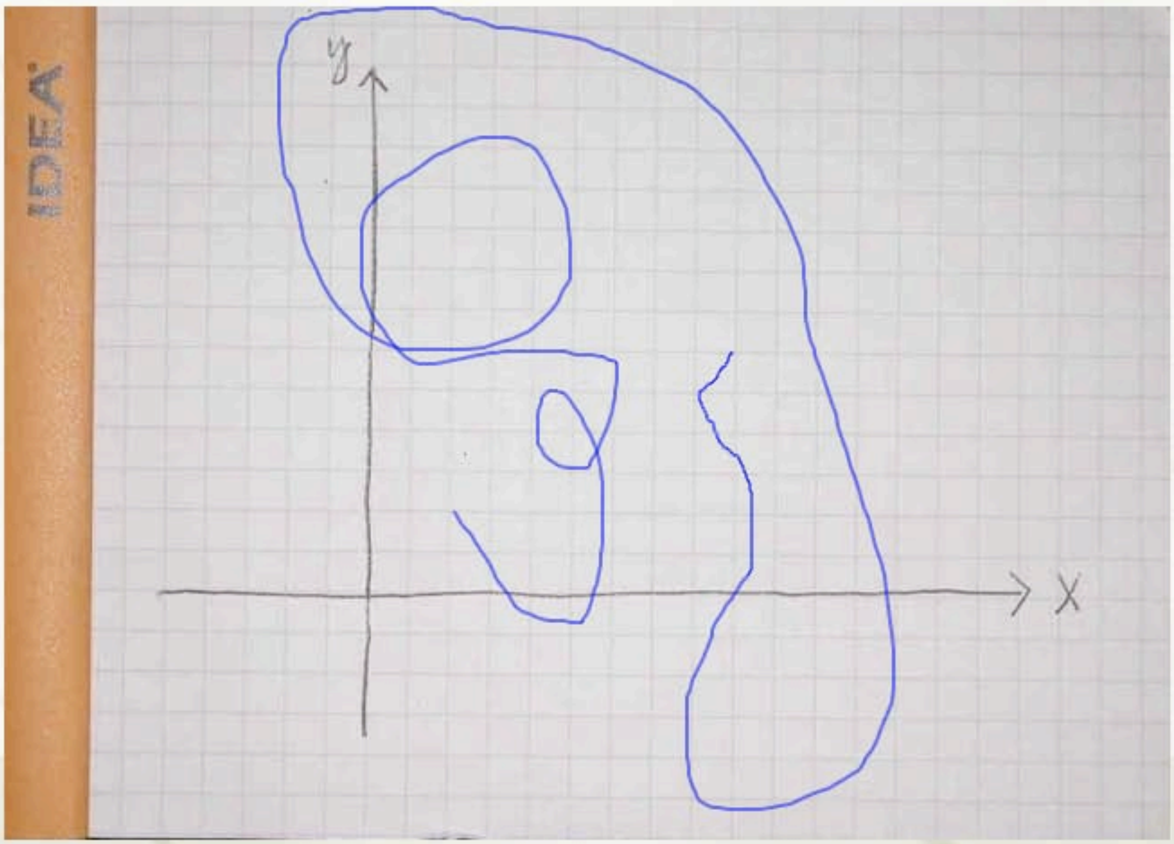
1. **F**(**r**) = x **i** + y **j**. **F**(**r**) = $\nabla(x^2+y^2)/2$ at each point

2. **F**(x) = y **i** + x **j**. **F**(**r**) = ∇ xy

3. **F**(x) = y **i** - x **j**. **F**(**r**) is not a gradient.

Examples

- Integrate $\mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ over the path shown, $\mathbf{F}(\mathbf{r}) = y\,\mathbf{i} + x\,\mathbf{j}$.



Examples

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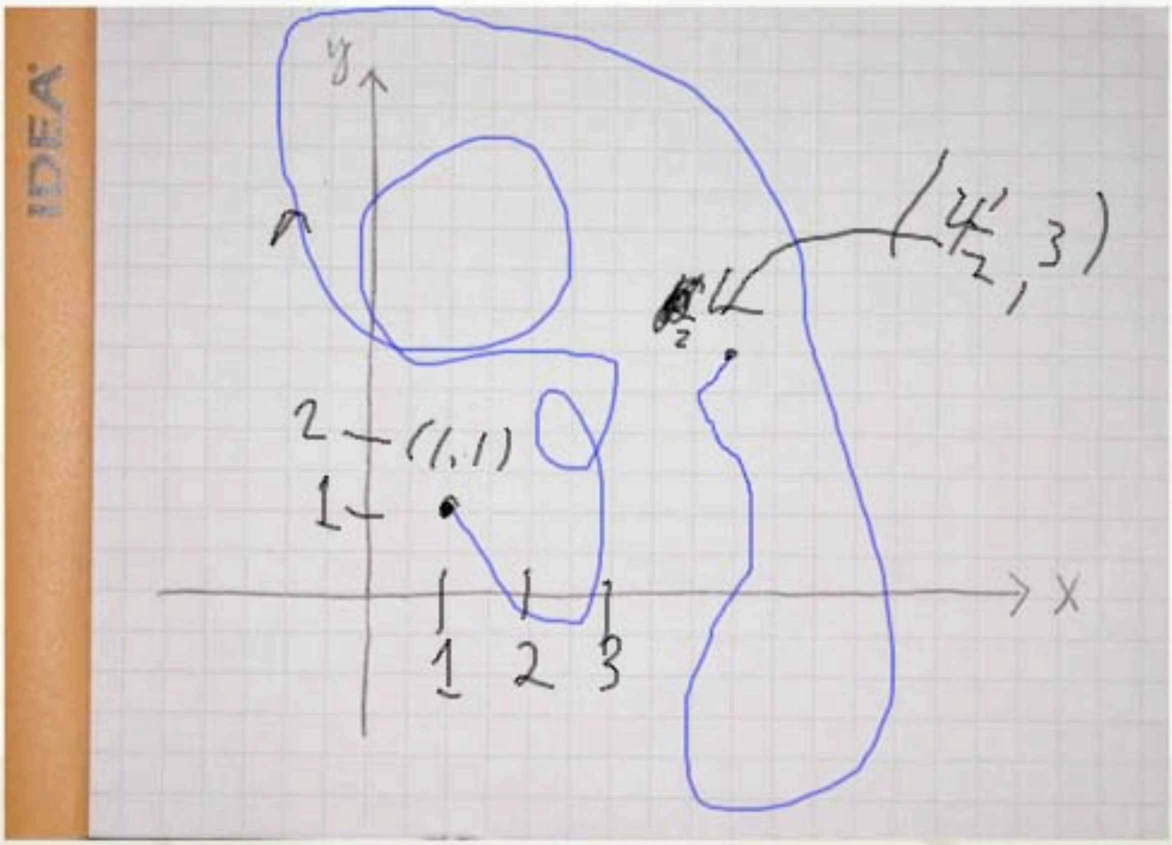


The fundamental theorem

- Assuming C is a piecewise smooth curve, f is continuously differentiable, and **h** = ∇f on a set that is open and simply connected:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}).$$

The fundamental theorem

- Why is this true? Strategy: reduce this question to a one-dimensional integral:
- f(**r**(t)) is a scalar-valued function of one variable. What’s its derivative?

The fundamental theorem

- Why is this true? Strategy: reduce this question to a one-dimensional integral:
- $f(\mathbf{r}(t))$ is a scalar-valued function of one variable. What's its derivative?
  - $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$. By the fundamental theorem of alculus, the integral of this function from t_1 to t_2 is

    $$f(\mathbf{r}(t_2)) - f(\mathbf{r}(t_1)) = f(\mathbf{b}) - f(\mathbf{a}).$$

    Quoth a rat demon, "strand 'em."

*The easy way to do line integrals, if **h** = ∇f*

1. **h**(**r**) = x **i** + y **j**. **h**(**r**) = $\nabla(x^2+y^2)/2$ at each point

2. **h**(x) = y **i** + x **j**. **h**(**r**) = ∇ xy

Remind me - how do you find f if **h** = ∇f?

A typical example

- $\mathbf{h}(\mathbf{r}) = (2\,x\,y^3 - 3\,x^2)\,\mathbf{i} + (3\,x^2\,y^2 + 2\,y)\,\mathbf{j}$
- Integral would be

$$\int (2\,x\,y^3 - 3\,x^2)dx + (3\,x^2\,y^2 + 2\,y)dy$$

1. Check that **h**(**r**) is a gradient.
2. Fix y, integrate P w.r.t. x.
3. Fix x, integrate Q w.r.t. y.
4. Compare and make consistent.

A typical example

- $\mathbf{h}(\mathbf{r}) = (2\ x\ y^3 - 3\ x^2)\ \mathbf{i} + (3\ x^2\ y^2 + 2\ y)\ \mathbf{j}$
  - $P_y = 6\ x\ y^2 = Q_x$, so we know $\mathbf{h} = \nabla f$ for some f.
  - To find f, integrate P in x, treating y as fixed. We get $x^2\ y^3 - x^3 + \phi$, but we don't really know ϕ is constant as regards y. It can be any function $\phi(y)$ and we still have $\partial\phi/\partial x = 0$.

A typical example

- $\mathbf{h}(\mathbf{r}) = (2\,x\,y^3 - 3\,x^2)\,\mathbf{i} + (3\,x^2\,y^2 + 2\,y)\,\mathbf{j}$
  - Now that we know $f(x,y) = x^2 y^3 - x^3 + \phi$, let's figure out ϕ by integrating Q in the variable y:
  - The integral of Q in y, treating x as fixed is $x^2 y^3 + y^2 + \psi$, but ψ won't necessarily be constant as regards x. It can be any function $\psi(x)$ and we still have $\partial\psi/\partial y = 0$.
  - Compare:
    - $f(x,y) = x^2 y^3 - x^3 + \phi(y) = x^2 y^3 + y^2 + \psi(x)$
    - So we can take $\phi(y) = y^2 + C_1$, $\psi(x) = x^3 + C_2$,
    - Conclusion: $\boxed{f(x,y) = x^2 y^3 - x^3 + y^2 + C}$ (combining the two arbitrary constants $C_{1,2}$ into one).

The fundamental theorem

- Assuming C is a piecewise smooth curve, f is continuously differentiable, and **h** = ∇f on a set that is open and simply connected:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}).$$


Dern! The pesky little auk up and grabbed the slides from that really cool example done in class and flew off to Baffin Island with' em!

Conservation of energy

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r})$$

U is the “potential energy.” **F** is a “conservative force.”

Conservation of energy

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r})$$

If **r**(t) is a curve, then E (t) is a function of t.

$$E = \frac{1}{2}m|\mathbf{v}|^2 + U(\mathbf{r})$$

In principle.

Total energy = kinetic + potential

Conservation of energy

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}) \qquad E = \frac{1}{2}m|\mathbf{v}|^2 + U(\mathbf{r})$$

$$v = \frac{d\vec{r}}{dt} \qquad \vec{v}\cdot\vec{v} \qquad \vec{a} = \frac{d}{dt}\vec{v}$$

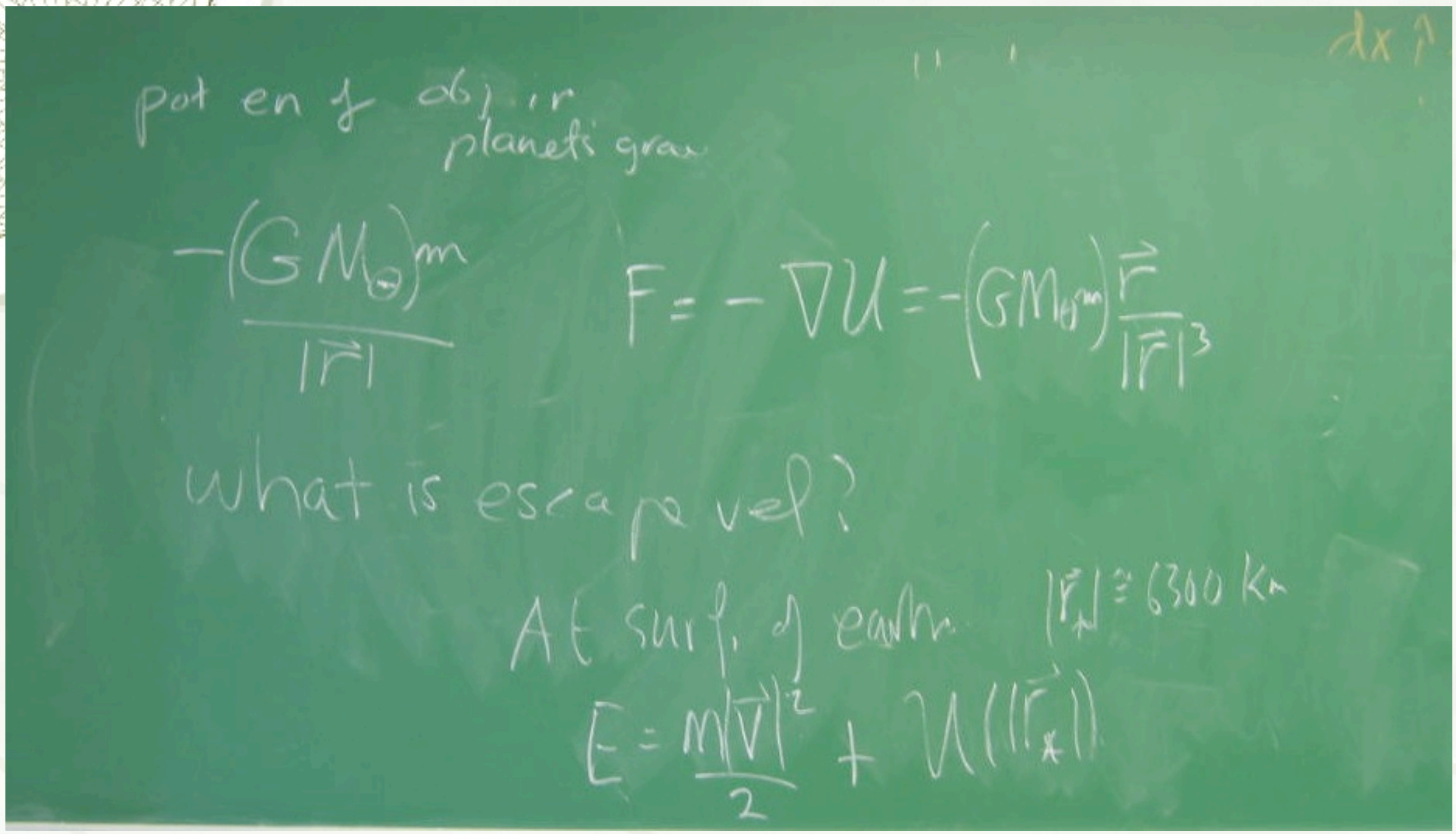
$$\frac{dE}{dt} = m(\vec{v}\cdot\vec{a}) + \nabla U\cdot\frac{d\vec{r}}{dt}$$

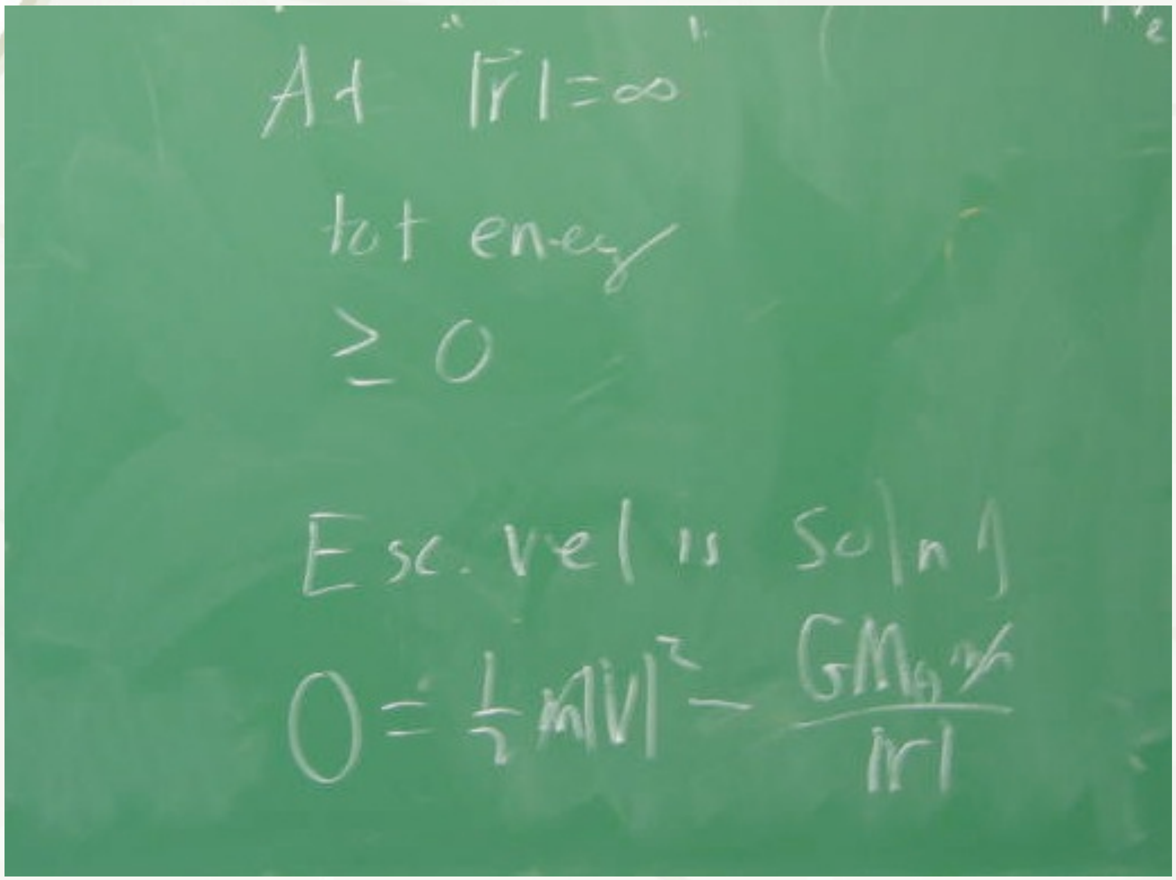
$$= \vec{v}\cdot(m\vec{a} + \nabla U) = \vec{v}\cdot(m\vec{a} - \vec{F})$$

$$= \vec{v}\cdot\vec{0} = 0$$

Application: Escape veolcity

How fast do you need to blast off to be lost in space? " Ground Control to Major Tom...."


pot en of obj in
planets grav
$-\frac{(GM_{\oplus})m}{|\vec{r}|}$
$F = -\nabla U = -(GM_{\oplus}m)\frac{\vec{r}}{|\vec{r}|^3}$
what is escape vel?
At surf. of earth $|\vec{r}_*| \cong 6300$ km
$E = \frac{m|\vec{v}|^2}{2} + U(|\vec{r}_*|)$


At |r| = ∞
tot energy
≥ 0
Esc. vel is soln
0 = ½ m|v|² − GM m / |r|

I. Grav. force is conservati.

$$\vec{F} = -\frac{G_m M_\odot m}{\rho^2}\,\hat{e}_\rho$$

↙ outward normal u-vec.

$$|\vec{F}| = \frac{G_m M_\odot}{(6.4\times10^6)^2} = 9.8 \text{ (mks)}$$

Initial

$$\begin{pmatrix} KE & \tfrac{1}{2}m\vec{v}^2 \\ PE & -\dfrac{G M_\odot m}{\rho_0} \end{pmatrix} =$$

$$\vec{v} = \sqrt{2\cdot 9.8\cdot 6.4\times10^6}$$

$$\vec{v} = \sqrt{\frac{2GM_\odot}{\rho_0}}$$