Going Green

Math stories

- Two mathematicians meet in the Skiles Building. The first asks the second how his family is, and the second answers: "They're great. My three daughters all had birthdays last week. The product of their ages is 72. The sum of their ages is the same as your office number.

- “Hmm...” says the first mathematician. I need to know something more before I can tell you how old your children are." The second answers: "Oh, my youngest daughter has a pet gerbil." The first mathematician says: “Aha, now I know.”

Math stories

- Two mathematicians meet in the Skiles Building. The first asks the second how her family is, and the second answers: "They're great. My three sons all had birthdays last week. The sum of their ages is 13. The product of their ages is the same as your street number.

- “Hmm...” says the first mathematician. I need to know something more before I can tell you how old your children are." The second answers: "Oh, my eldest son plays the violin." The first mathematician says: “Aha, now I know.”

Reminder…

There's a test on Thursday!

Recall - The fun. theorem

- Assuming C is a piecewise smooth curve, f is continuously differentiable, and **h** = ∇f on a set that is open and simply connected:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}).$$

Line integrals and arc length

Example: mass of a wire. Suppose that the mass density of a wire is the distance from the origin and the wire is in the form of a spiral. What would the total mass be?

θ $r = \theta$

Wire: $x = t\cos t$

$y = t\sin t$

$\lambda(x,y) = \sqrt{x^2+y^2}$

$$ds = |d\vec{x}| = \sqrt{(\cos(t) - t\sin t)^2 + (\sin t + t\cos t)^2}\,dt$$

$$= \sqrt{1+t^2}\,dt$$

λ

$$\text{Mass} = \int_a^b \sqrt{t^2(\cos^2 t + \sin^2 t)}\,\sqrt{1+t^2}\,dt = \int_a^b \sqrt{1+t^2}\,t\,dt$$

For example, $a = 0$, $b = 4\pi$:

$$\int_0^{4\pi} \sqrt{1+t^2}\,t\,dt = \int_1^{1+16\pi^2} u^{\frac{1}{2}}\,\tfrac{1}{2}\,du = \tfrac{1}{3}\left((1+16\pi^2)^{3/2} - 1\right)$$

Meet George Green

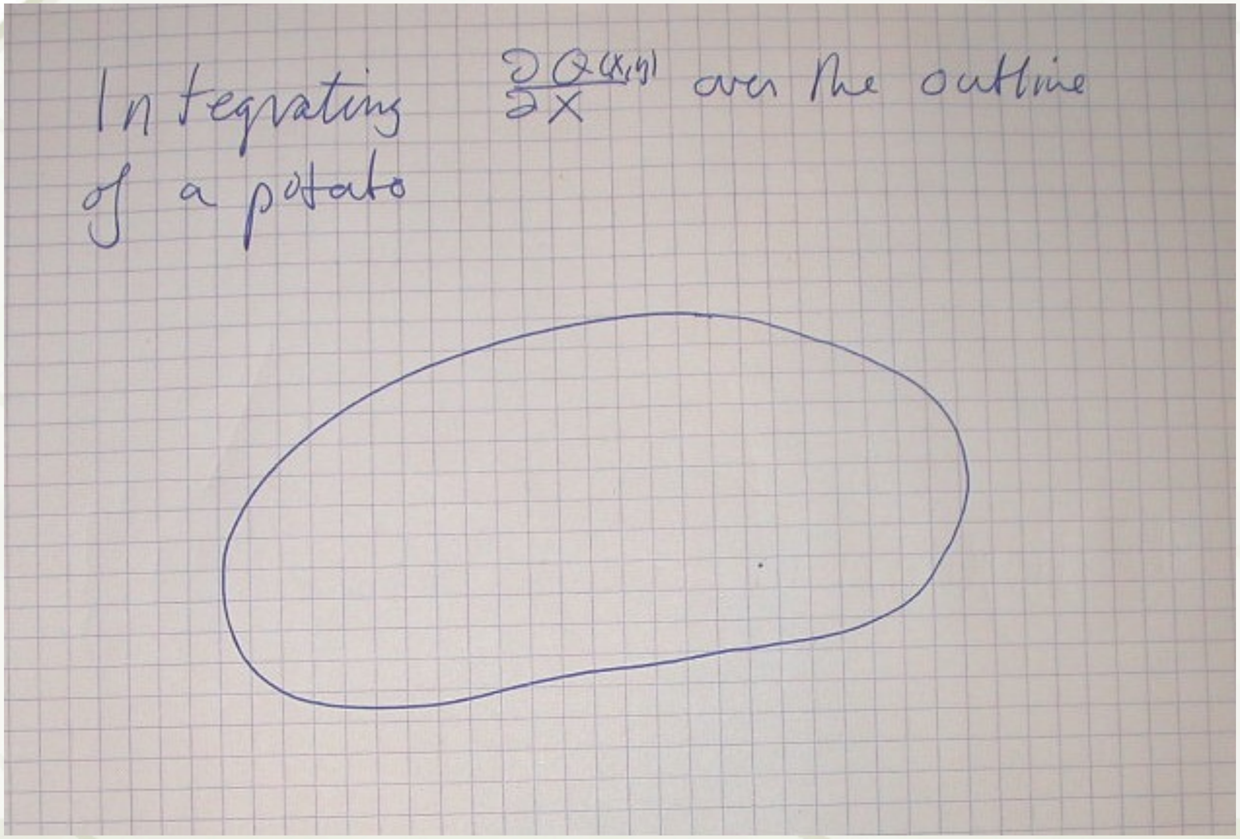
http://www-history.mcs.st-andrews.ac.uk/Biographies/Green.html

The next best thing to the fundamental theorem of calculus for double integrals

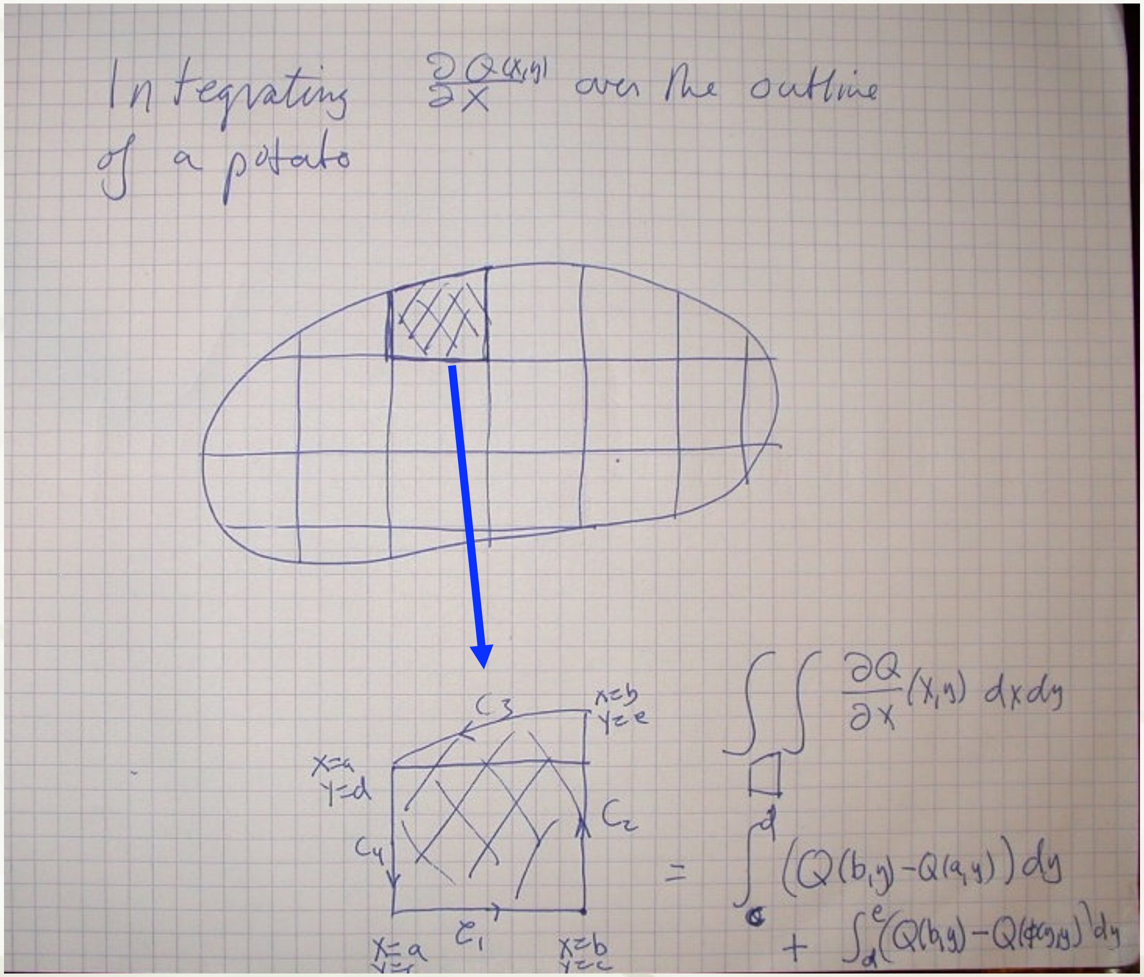
Usual fundamental theorem - an integral can "cancel off" a derivative.

New: If you have a double integral of a derivative, you can cancel one integral off against the derivative. Carefully.

Consider the humble potato, Ω



As usual, we chop the potato up:



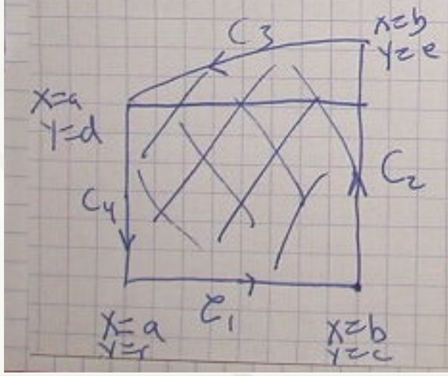
$$\iint_{\square} \frac{\partial Q}{\partial x}(x,y)\,dx\,dy$$

$$\int_c^d \big(Q(b,y)-Q(a,y)\big)\,dy + \int_d^e \big(Q(b,y)-Q(\phi(y),y)\big)\,dy$$

$$= \int_{C_2} Q(x,y)\,dy \;\text{(up)} + \int_{C_4} Q(x,y)\,dy \;\text{(down)}$$

$$+ \int_{C_3} Q(x,y)\,dy \;\text{(left-ish)} + \int_{C_1} Q(x,y)\,dy$$

because 0!



=

In sum,

$$\int\int \frac{\partial Q}{\partial x}(x,y)\,dx\,dy = \oint_{C_1 \cup C_2 \cup C_3 \cup C_4} Q\hat{j}\cdot d\vec{r}$$

C.C.W

Likewise…

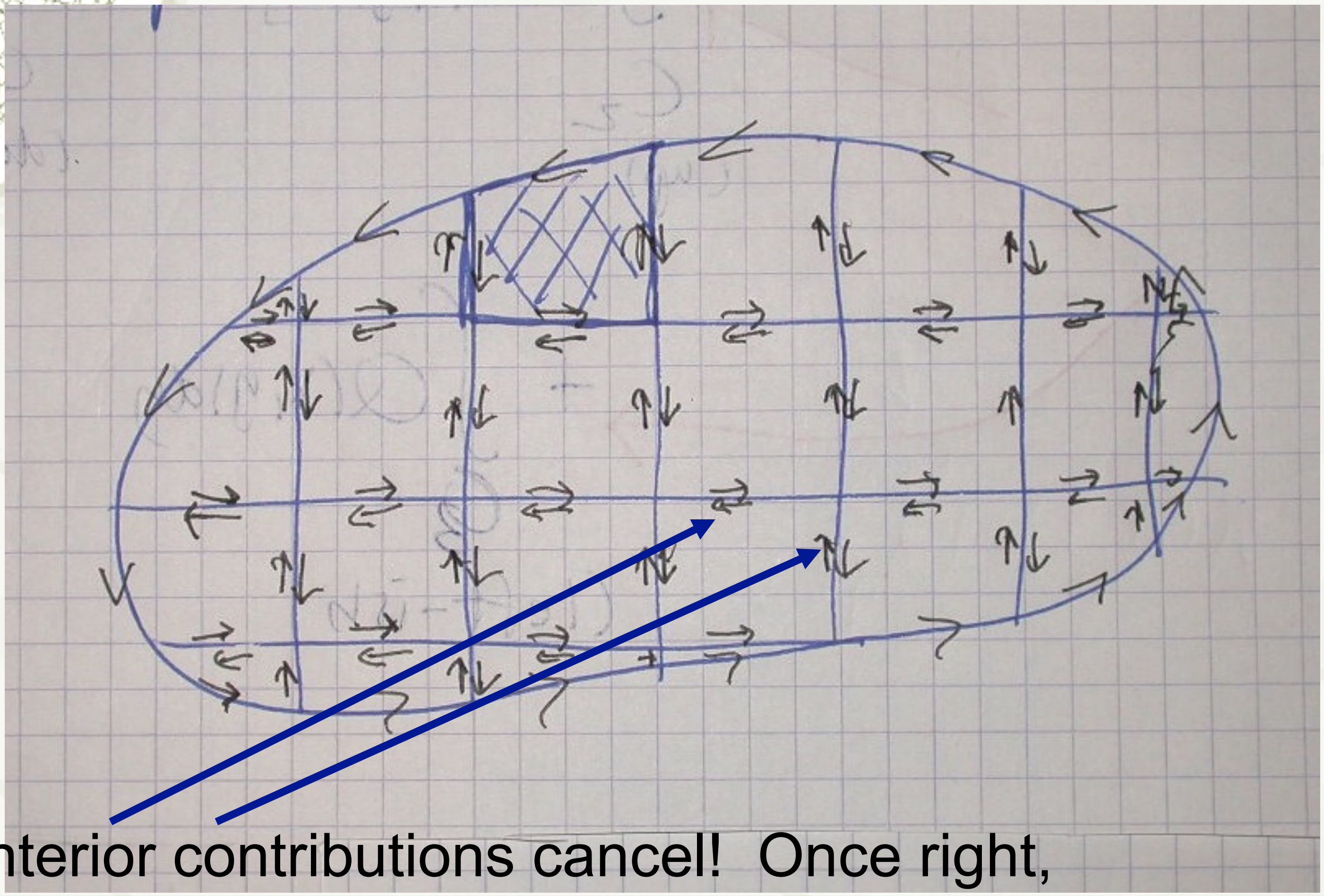
$$\int_{\Omega_N}\int \frac{\partial P}{\partial y}(x,y)\,dx\,dy = -\int_{\partial\Omega_N} P(x,y)\,dx$$

In sum,

- The counterclockwise integral of P **i** around the edge of a little sort-of rectangle is a double integral of – P_y. over the sort-of rectangle.
- The counterclockwise integral of Q **j** around the edge of a little sort-of rectangle is a double integral of Q_x. over the sort-of rectangle.

- So… there is a formula for the line integral of **F**•d**r** = (P **i** +Q **j**)•d**r**.

Now integrate over the whole potato



All interior contributions cancel! Once right, once left. Or once up, once down.

Green's formula

$$\int\int_{\Omega}\left(\frac{\partial Q}{\partial x}(x,y)-\frac{\partial P}{\partial y}(x,y)\right)dxdy=\oint_C(P(x,y)dx+Q(x,y)dy)$$

This part is 0 if **F** = ∇ f.

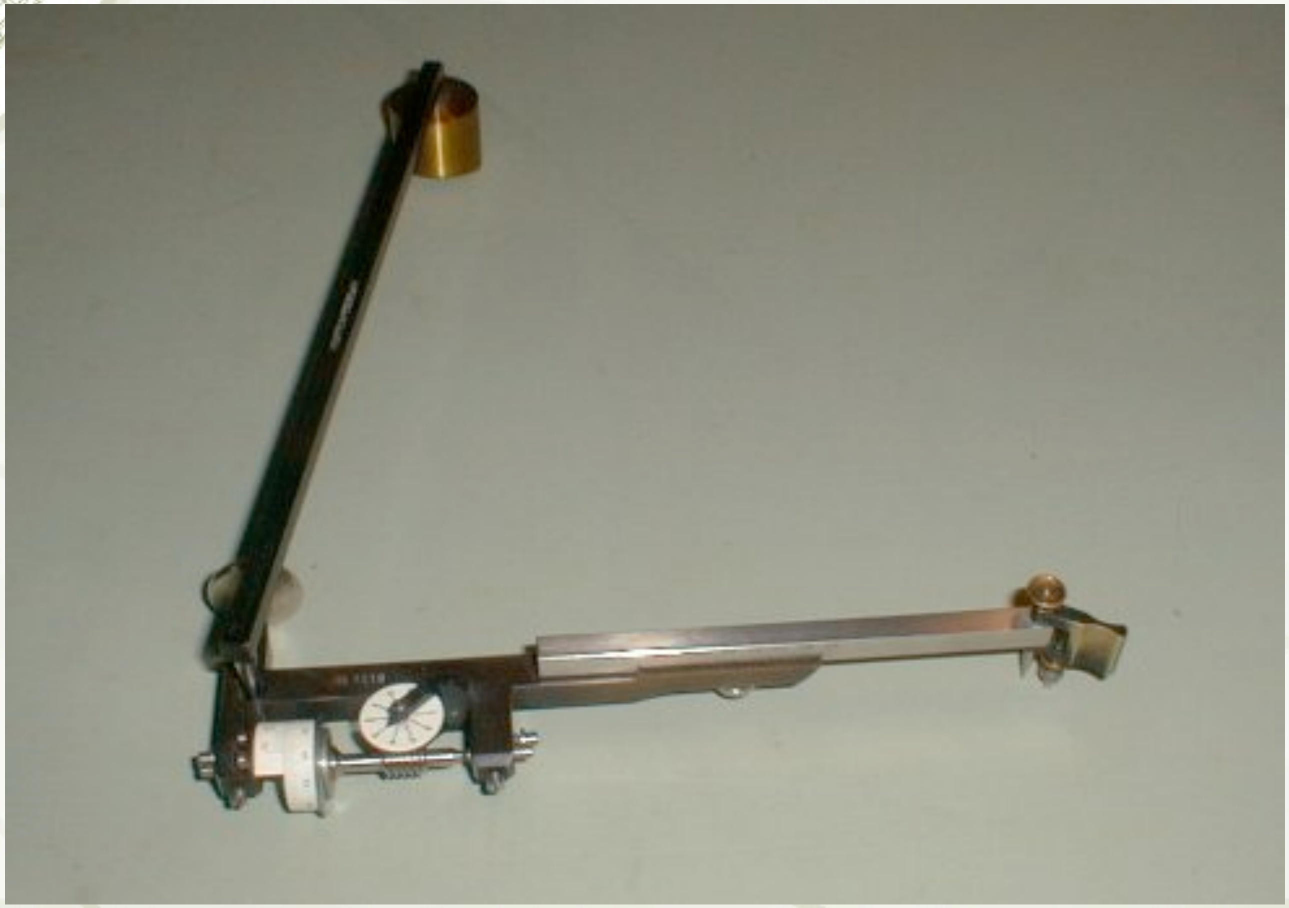
This part is $\oint_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$

Green's formula

$$\int_\Omega \int \left(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y) \right) dx dy = \oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

The spooky thing about Green's theorem is that you can find out something about the inside by integrating around the outside.

Application: The planimeter



Picture by Paul E. Kunkel, with his kind permission.

What does Green tell us when

1. $P = 0, Q = x$?

2. $P = -y, Q = 0$?

3. $P = -y/2, Q = x/2$?

Oh, there are the slides with the answers! Er, … I don't think I'll post them just now.

Green is a two-way street

- The line integral may be easier than the area integral.
- The area integral may be easier than the line integral.

Green is a two-way street

- The line integral may be easier than the area integral
  - Example: unit circle,
    - $P = -y\cos(\pi (x^2+y^2)^{7/3})$, $Q = x\cos(\pi (x^2+y^2)^{7/3})$

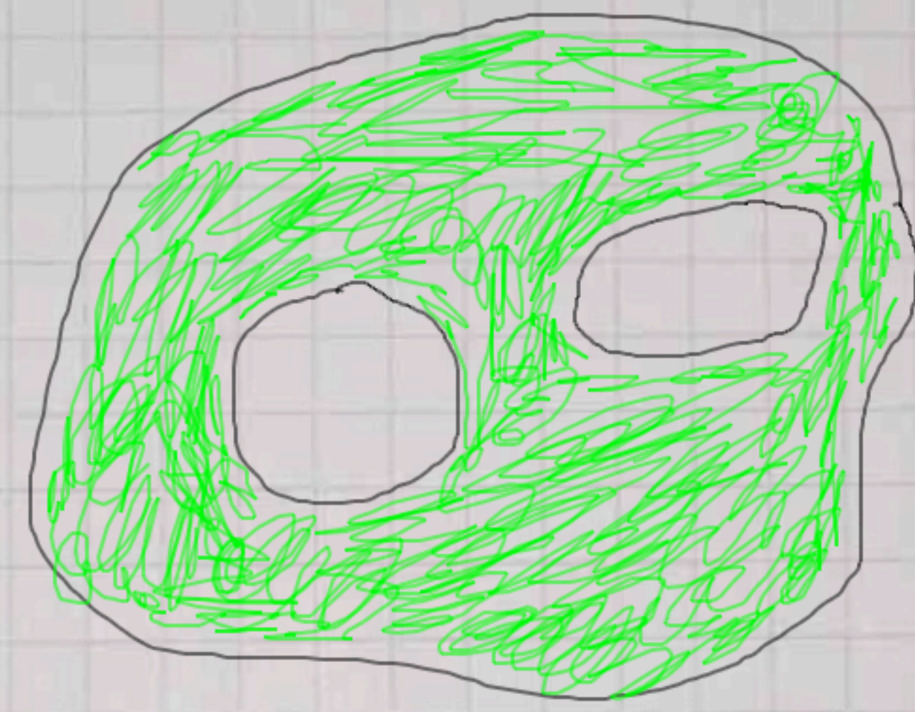
      $\partial Q/\partial x - \partial P/\partial y = 2\cos(\pi (x^2+y^2)^{7/3})$.
  - Not so nice inside the circle, but…

Green is a two-way street

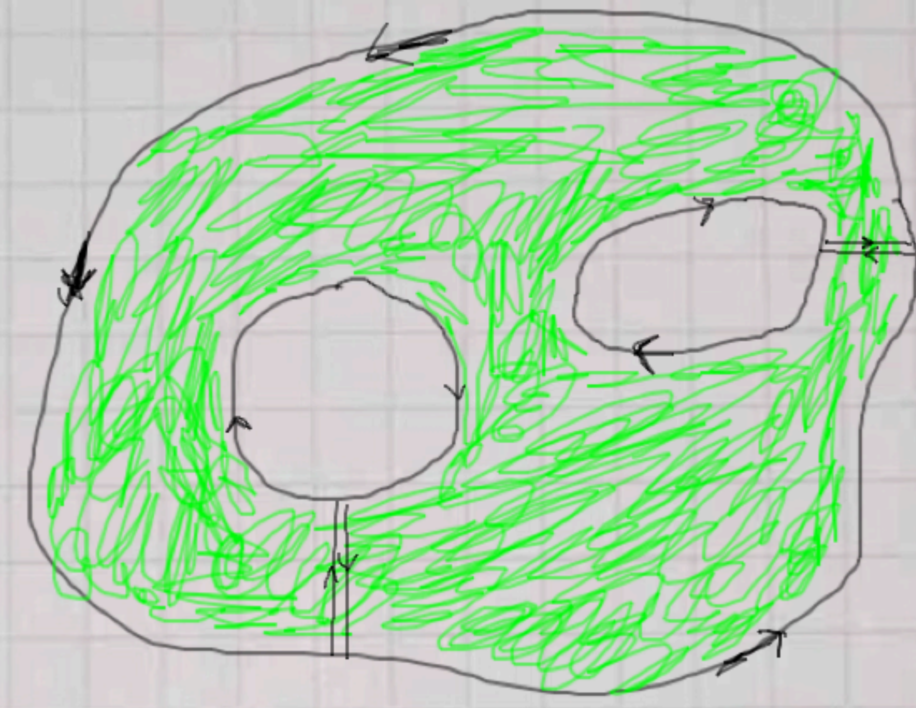
- The line integral may be easier than the area integral

- The area integral may be easier than the line integral
  - $\oint (e^x \cos(2\, y) + \sin(x^4) - 2\, y)\, dx + \oint (y^2 \sin(y^2) \sinh(y^4) - 2\, e^x \sin(2\, y))\, dy$

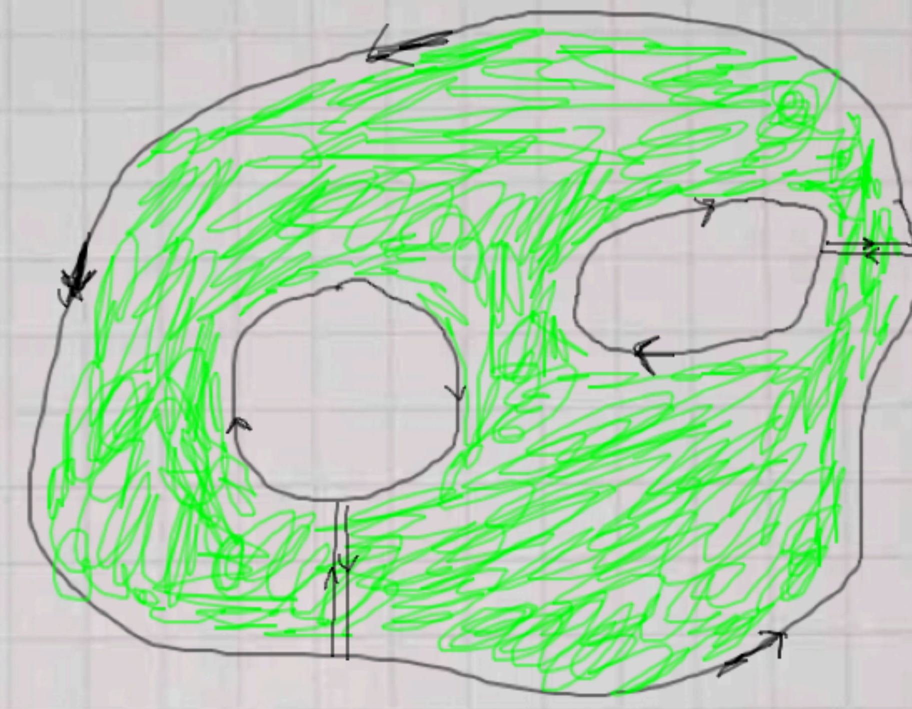
Holey Green regions



Holey Green regions



Holey Green regions



On the inside, counterclockwise is clockwise!

Holey Green regions