Getting beneath the surface

About that test…

- Median was 69
  - max = 98
  - 75^{th} percentile = 80
  - 25^{th} percentile = 57

About course surveys…

Good morning/afternoon/evening,

The institute has convened a Task Force to study a potential redesign of the CIOS (Course/Instructor Opinion Survey) instrument used to provide feedback to your instructors and program administrators. A research team has produced a separate PILOT survey, and as your instructor, I have volunteered this course to be a part of the research pool. *Please help Georgia Tech improve our survey by responding to the PILOT questions which can accessed at http://www.surveymonkey.com/s.aspx?sm=kK_2f_2bC3Mi3htyeXvdg9bDgA_3d_3d beginning Monday Nov 24. This will only take 5-15 minutes of your time (depending on how many comments you choose to add). The survey will be open until Dec 14.

About course surveys…

Note that Prof. H will have no information about who does or does not participate in this pilot survey – He will only receive a tabulated report with the anonymous results. Therefore your participation (or lack of participation) will have no effect on your grade in this class. Further, you will receive no compensation for participating in this project.

Please also note that you should still* also complete your regular CIOS*survey for all courses including this one. Responses on BOTH the PILOT and CIOS are needed in order to make good decisions about any potential changes.

About Wednesday.

Contest for the best math videos to show at that time.

About Dead Week...

WHEN:

8 p.m. on Tuesday, December 2, 2008

WHERE:

Physics Lecture Room L-1

Sponsors: Club Math, the SGA and the School of Mathematics

The performance is free, but we encourage people to make reservations because of limited seatings—www.math.gatech.edu

In our previous episode…

Integrating over surfaces

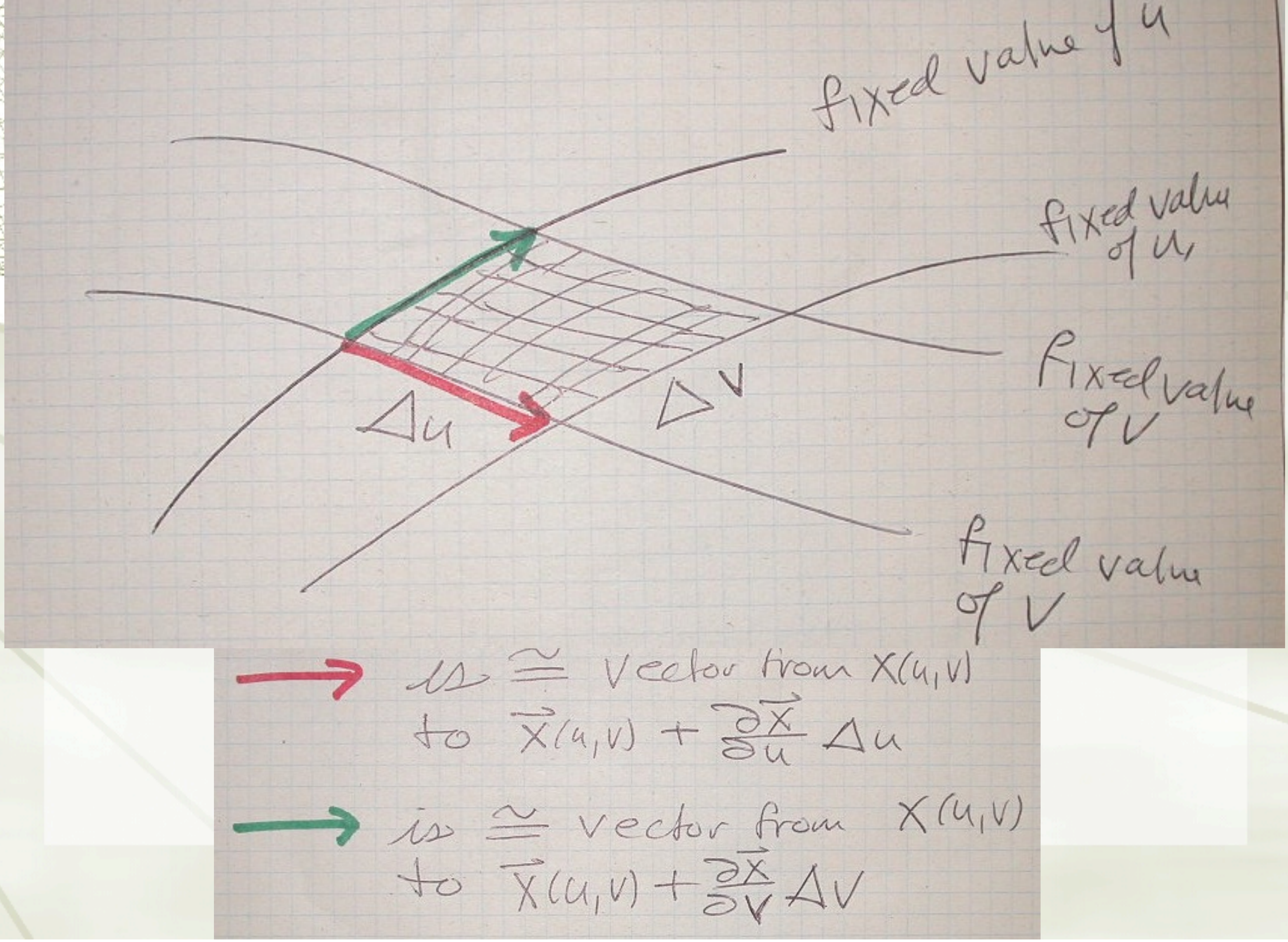
How large is the little parallelogram when we increase

θ to $\theta+\Delta\theta$

and

ϕ to $\phi+\Delta\phi$?

There is nothing flat about this picture!



The Fundamental Vector Product

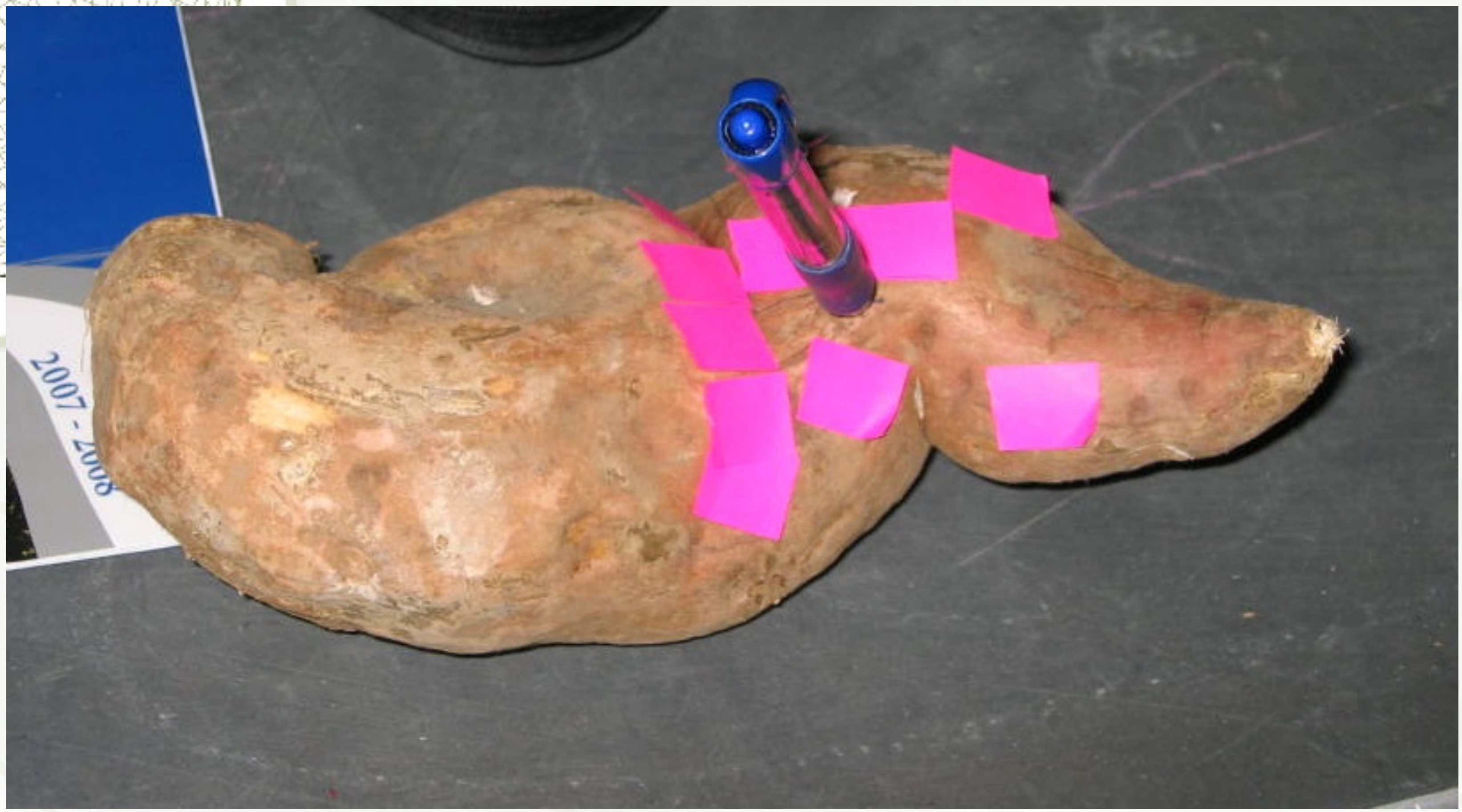
$$\mathbf{N}(u,v) = \frac{\partial \mathbf{r}}{\partial u}(u,v) \times \frac{\partial \mathbf{r}}{\partial v}(u,v)$$

$$\mathbf{N}(u,v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$\mathbf{N}(u,v) = \begin{pmatrix} \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \end{pmatrix}$$

What makes the FVP so cool?

- The area element dA = |**N**| du dv
  - So we use it whenever we integrate over the surface.


2007 - 2008

Simplifications when z=f(x,y)

- $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + f(x,y)\,\mathbf{k}$
- Identify u = x, v = y:
  - $\mathbf{N} = -f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}.$
  - $dA = (1 + |\nabla f|^2)^{1/2}$

What makes the FVP so cool?

- The area element dA = |**N**| du dv

- **N** points perpendicularly out from the surface.
  - So we use it when we figure out tangent planes.

Simplifications when z=f(x,y)

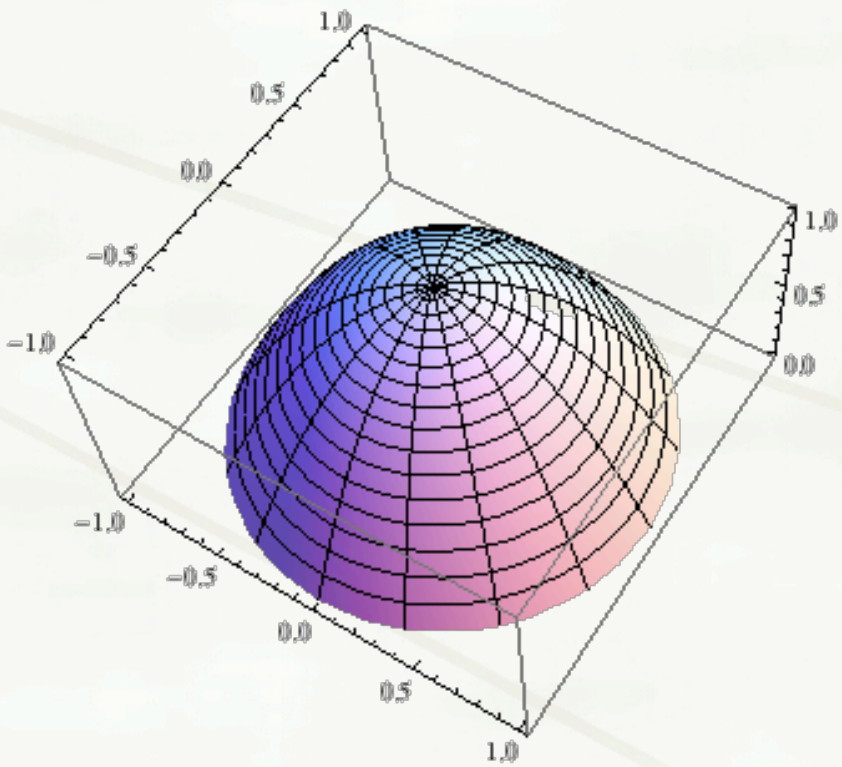
- **r** = x **i** + y **j** + f(x,y) **k**
- Identify u = x, v = y:
  - **N** = $-f_x$ **i** - f_y **j** + **k.**
    - *More upward than downward.*
    - *Not* a unit vector. $|N| \geq 1$
  - $dA = (1 + |\nabla f|^2)^{1/2}\, dx\, dy$

Paraboloid ($z = 1 - r^2$)

1. Paraboloid.

In[1]:=
```
ParametricPlot3D[{r Cos[th], r Sin[th], 1 - r^2}, {r, 0, 1},
  {th, -Pi, Pi}]
```

Out[1]=

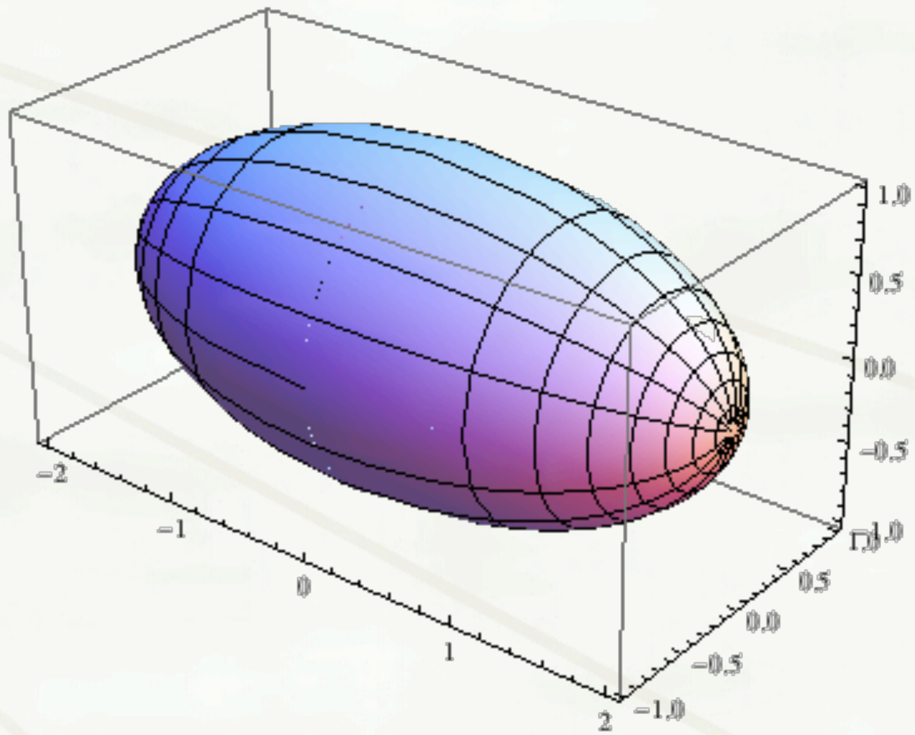


Ellipsoid $(x^2/a^2 + y^2/b^2 + z^2/c^2 = 1)$

2. Ellipsoid.

In[8]:=
```
ParametricPlot3D[{2 Sqrt[1 - r^2] r / Abs[r], r Cos[th], r Sin[th]},
  {r, -1, 1}, {th, -Pi, Pi}]
```

Out[8]=

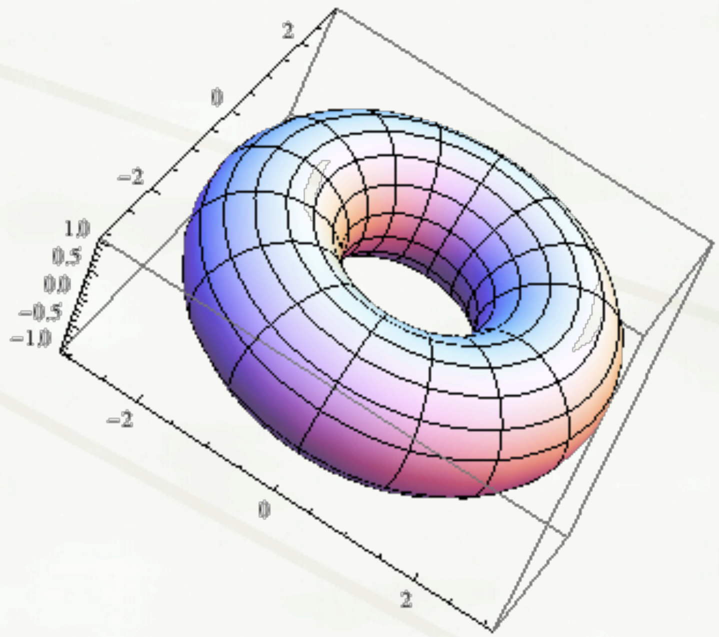


Torus - rotate a circle around an axis

3. Torus.

In[4]:=
```
ParametricPlot3D[{(2 + Cos[beta]) Cos[alpha],
  (2 + Cos[beta]) Sin[alpha], Sin[beta]}, {alpha, 0, 2 Pi},
 {beta, 0, 2 Pi}]
```

Out[4]=



Example. How many square cm. of chocolate are needed to glaze a torus?

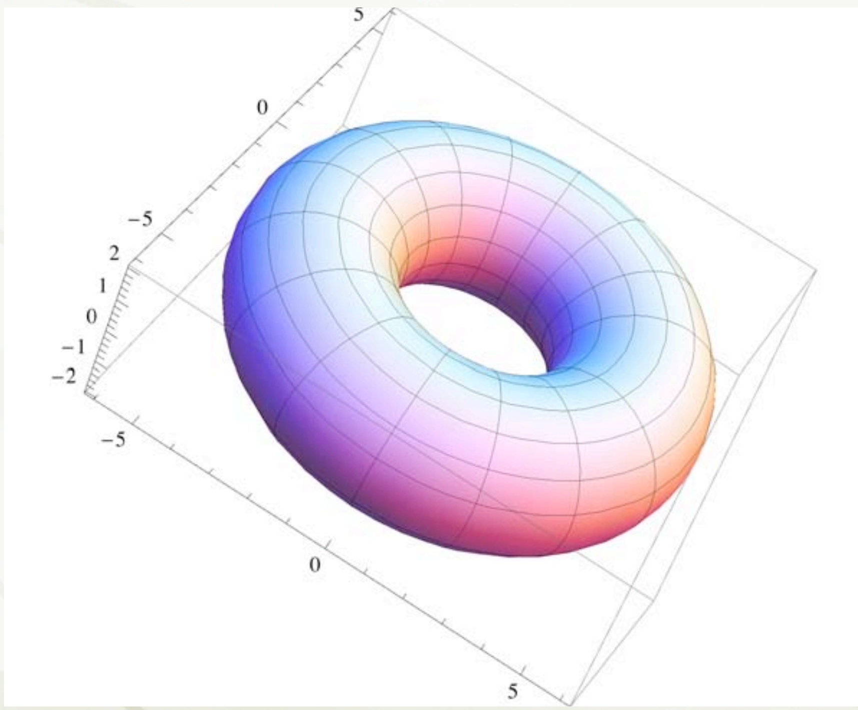
Step 1. Describe the torus. The shop says that the torus is 12 cm across and 4 cm high

In[1]:=
```
Torus[beta_, theta_] := {(4 + 2 Cos[beta]) Cos[theta], (4 + 2 Cos[beta]) Sin[theta], 2 Sin[beta]}
```

To produce the whole surface of the torus, beta and theta go from 0 to 2π.

In[2]:=
```
ParametricPlot3D[Torus[beta, theta], {theta, 0, 2 Pi}, {beta, 0, 2 Pi}]
```

Out[2]=



Step 2. Differentiate and calculate the Fun. Vector Product and Jacobian factor.

In[12]:= **D[Torus[beta, theta], beta]**

Out[12]= {-2 Cos[theta] Sin[beta], -2 Sin[beta] Sin[theta], 2 Cos[beta]}

In[13]:= **D[Torus[beta, theta], theta]**

Out[13]= {-(4 + 2 Cos[beta]) Sin[theta], (4 + 2 Cos[beta]) Cos[theta], 0}

In[14]:= **FunN[beta_, theta_] = Simplify[Cross[%%, %]]**

Out[14]= {-4 Cos[beta] (2 + Cos[beta]) Cos[theta],
-4 Cos[beta] (2 + Cos[beta]) Sin[theta], -4 (2 + Cos[beta]) Sin[beta]}

In[17]:= **Norm[FunN[beta, theta]]**

Out[17]= √(16 Abs[Cos[beta] (2 + Cos[beta]) Cos[theta]]^2 +
16 Abs[(2 + Cos[beta]) Sin[beta]]^2 + 16 Abs[Cos[beta] (2 + Cos[beta]) Sin[theta]]^2)

We help *Mathematica* along here by noticing that this simplifies to

$$4\,(2 + \cos(\beta)).$$

Step 3. Integrate

We see that the integral over theta just gives a factor of 2 Pi. As for the rest:

```
In[ ]:= SurfArea = 2 Pi Integrate[TorusJac[beta, theta], {beta, 0, 2 Pi}]

Out[ ]= 32 π²
```

Yummy!

$$2\pi \int_0^{2\pi} 4(2 + \cos\beta)d\beta$$

Example. A thimble in the shape of a paraboloid has a density proportional to the height.

Step 1. Describe the thimble. We'll make some choices and suppose that its shape is z = 1 - r^2, and that the mass density is 4 z (in grams per square cm).

The formula is not parametric, but we can use x and y as parameters and write z = F (x, y), where

```
In[20]:= F[x_, y_] := 1 - x^2 - y^2
```

Step 2. Differentiate and calculate the Fun. Vector Product and Jacobian factor. There is a simplification when z = F[x,y], so that

```
In[21]:= FunN[x_, y_] = {-D[F[x, y], x], -D[F[x, y], y], 1}

Out[21]= {2 x, 2 y, 1}
```

and

```
In[22]:= ThimbleJac[x_, y_] := Sqrt[1 + 4 x^2 + 4 y^2]
```

What is the total mass?

Step 3. Integrate

The integrand is z = 4 (1 - x^2 - y^2), and we must not forget the Jacobian factor Sqrt[1 + 4 x^2 + 4 y^2] .

But wait! Instead of integrating

4 (1 - x^2 - y^2)Sqrt[1 + 4 x^2 + 4 y^2] dx dy,

we can switch to polar and integrate 4 (1 - r^2) Sqrt[1 + 4 r^2] **r** dr dtheta:

```
In[26]:= Mass = 8 Pi Integrate[(1 - r^2) Sqrt[1 + 4 r^2] r , {r, 0, 1}]
```

Out[26]= $\frac{1}{15}\left(-11+25\sqrt{5}\right)\pi$

```
In[27]:= N[%]

Out[27]= 9.40419
```

Step 3. Integrate

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```
In[27]:= N[%]
```

Out[27]= 9.40419

integrand   Jacobian

Flux and flux integrals

- At what rate does a fluid pass through a membrane?
  - If the velocity is **v**, only the normal component **v**·**n** transports matter across the membrane.
  - Rate is also proportional to area
- ∴Flux is the integral of **v**·**n** dA.

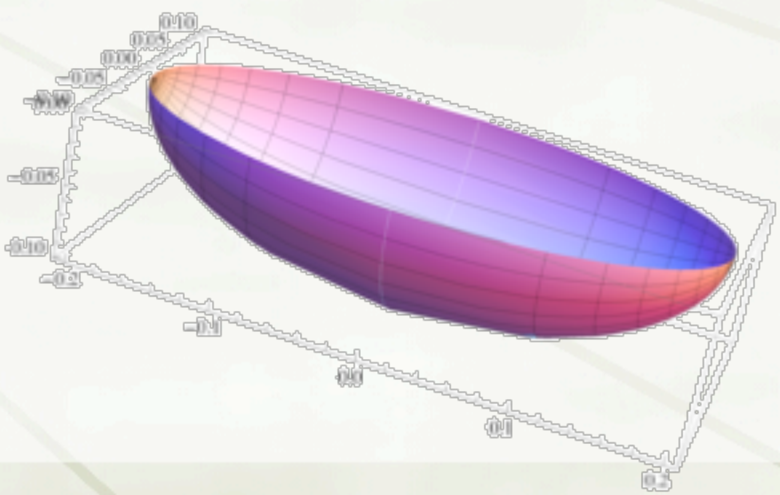
Example. What is the rate of flux of water that is seeping through the ellipsoidal bottom of a lake bed by moving vertically downward at constant speed?

Step 1. Describe the lake bottom. We'll make some choices and suppose that its shape is z = -(1/10) Sqrt[1 - (x/2)^2 - y^2] in km, and that the velocity field is -.00001 **k** km/hr.

The formula is not parametric, but we can use x and y as parameters and write z = F (x, y), where

```
F[x_, y_] := -.1 Sqrt[1 - x^2 - y^2]
```

```
ParametricPlot3D[
 (1 / 10) {2 Sqrt[1 - r^2] r / Abs[r], r Cos[th], Abs[r] Sin[th]}, {r, -1, 1}, {th, -Pi, 0}]
```



Step 2. Differentiate and calculate the Fun. Vector Product and *unit* normal. We can use symmetry by dividing the lake bottom into four equal pieces, and calculate only for x,y ≥ 0. Then

```
LakeBot[x_, y_] = -(1/10) Sqrt[1 - y^2 - (x/2)^2]
```

$$-\frac{1}{10}\sqrt{1-\frac{x^2}{4}-y^2}$$

```
FunN[x_, y_] = {-D[LakeBot[x, y], x], -D[LakeBot[x, y], y], 1}
```

$$\left\{-\frac{x}{40\sqrt{1-\frac{x^2}{4}-y^2}}, -\frac{y}{10\sqrt{1-\frac{x^2}{4}-y^2}}, 1\right\}$$

This is a normal vector, but it points up and has a length different from 1:

```
LengthN[x_, y_] := (1/10) Sqrt[1 + (x/4)^2 + y^2] / Sqrt[1 - (x/2)^2 - y^2]
```

(I scaled a 1/10 out of both numerator and denominator.) The more or less downward unit normal is:

```
Simplify[-FunN[x, y] / LengthN[x, y]]
```

$$\left\{\frac{x}{\sqrt{x^2+16\left(1+y^2\right)}}, \frac{4y}{\sqrt{x^2+16\left(1+y^2\right)}}, -\frac{2\sqrt{4-x^2-4y^2}}{\sqrt{x^2+16\left(1+y^2\right)}}\right\}$$

Step 3. Calculate the flux

The flux vector is the dot product of the unit normal with -.00001 **k** :

In[48]:= Flux[x_, y_] := $\frac{-.00002\sqrt{4 - x^2 - 4y^2}}{\sqrt{x^2 + 16(1 + y^2)}}$

Step 4. Integrate

When we integrate, don't forget the Jacobian factor. We can choose to use symmetry and integrate only where $x,y \geq 0$:

```
In[49]:= 4 Integrate[Flux[x, y] LengthN[x, y], {y, 0, Sqrt[1 - (x / 2) ^2]}]
```

Out[49]= $-0.00002\sqrt{4 - x^2}$

```
In[50]:= Integrate[%, {x, 0, 2}]
```

Out[50]= -0.0000628319

This may look small, but the units are km^3 /hr !

A tip for flux integrals

- Instead of normalizing **N** to a unit vector **n**, you can often short-cut the calculation as follows:
- The unit vector **n** = ± **N**/|**N**|and the area element is dA = |**N**|du dv, so…

A tip for flux integrals

- Instead of normalizing **N** to a unit vector **n**, you can often short-cut the calculation as follows:
- The unit vector **n** = ± **N**/ |**N**|and the area element is dA = |**N**|du dv, so…
  - **v•n** dA = **v•N** du dv .
  - Some times you can avoid calculating |**N**|.

Maxwell's equations

Wikipedia article on *Maxwell's Equations*:

General case

[edit]

The Equations are given in SI units. See below for CGS units.

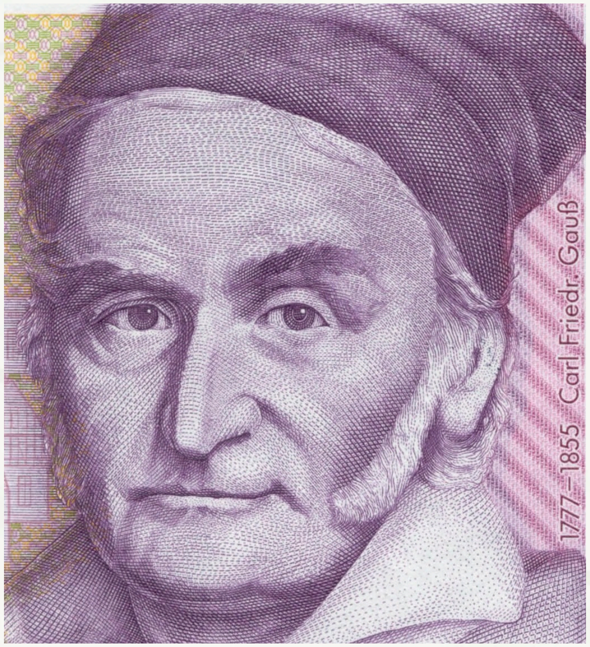
Name	Differential form	Integral form
Gauss's law:	$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q_S}{\epsilon_0}$
Gauss' law for magnetism (absence of magnetic monopoles):	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0$
Faraday's law of induction:	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{d\Phi_{B,S}}{dt}$
Ampère's Circuital Law (with Maxwell's correction):	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_S + \mu_0 \epsilon_0 \frac{d\Phi_{E,S}}{dt}$

The following table provides the meaning of each symbol and the SI unit of measure:

Symbol	Meaning (first term is the most common)	SI Unit of Measure
$\nabla \cdot$	the divergence operator	per meter (factor contributed by applying either operator)
$\nabla \times$	the curl operator	

Coming attractions: Grad, Curl, and Div

Green, Gauß, and Stokes



Grad, Curl, and Div

- Just for fun, think of ∇ as a vector "operator" with components
  - $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$.
- And do with it what you like to do with vectors.

This Thanksgiving, don't just eat the turkey and π ….

Remember your Green's!