MATH 2401 - Harrell

Tangent vectors, or

how to go straight when you are on a bender.

In our previous episode:

1. *Vector functions are curves. The algebraic side of the mathematian's brain thinks about vector functions. The geometric side sees curves.*

In our previous episode:

1. *Vector functions are curves.*

2. *Don't worry about the basic rules of calculus for vector functions. They are pretty much like the ones you know and love.*

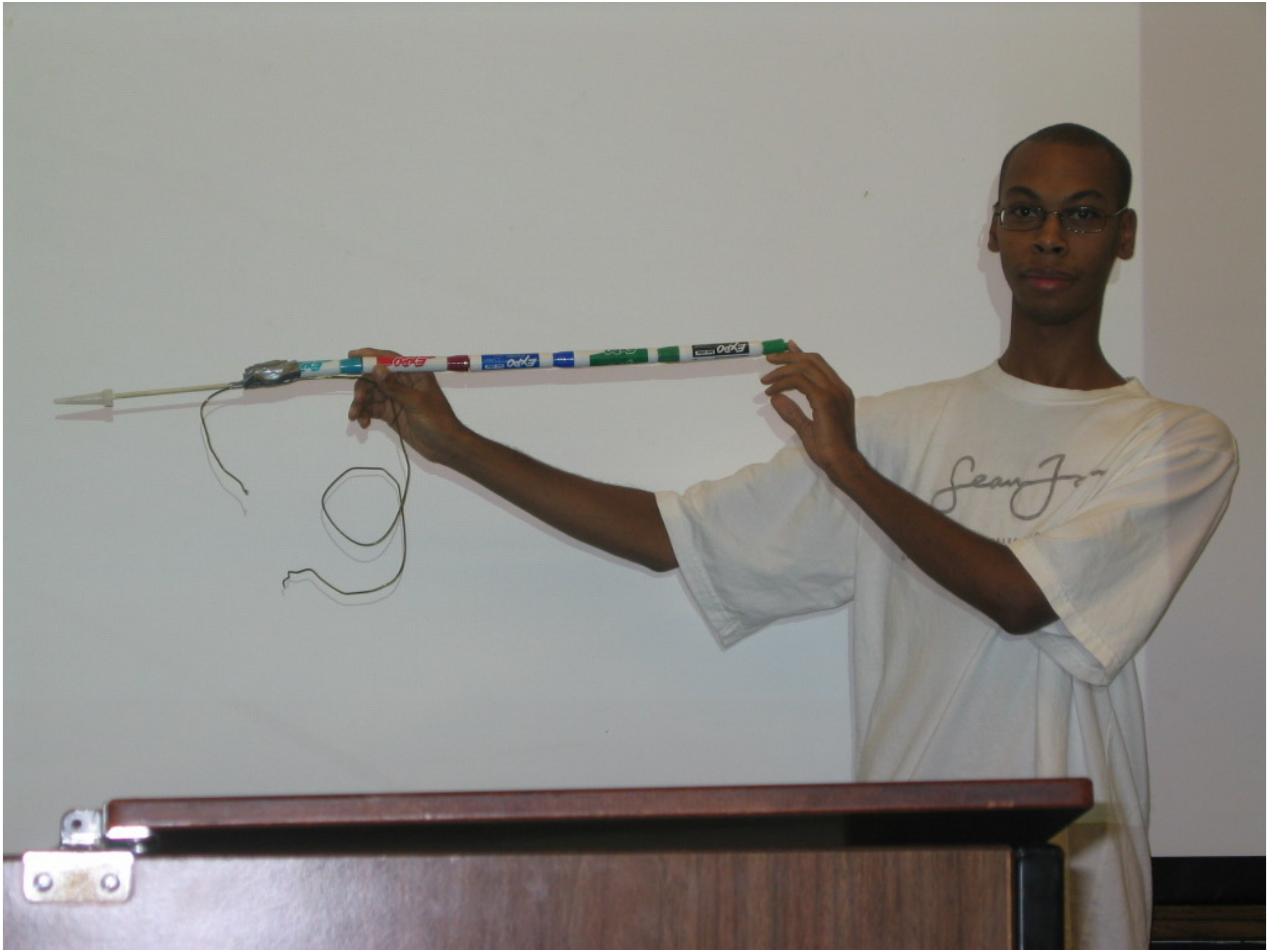
Tangent vectors – the derivative of a vector function

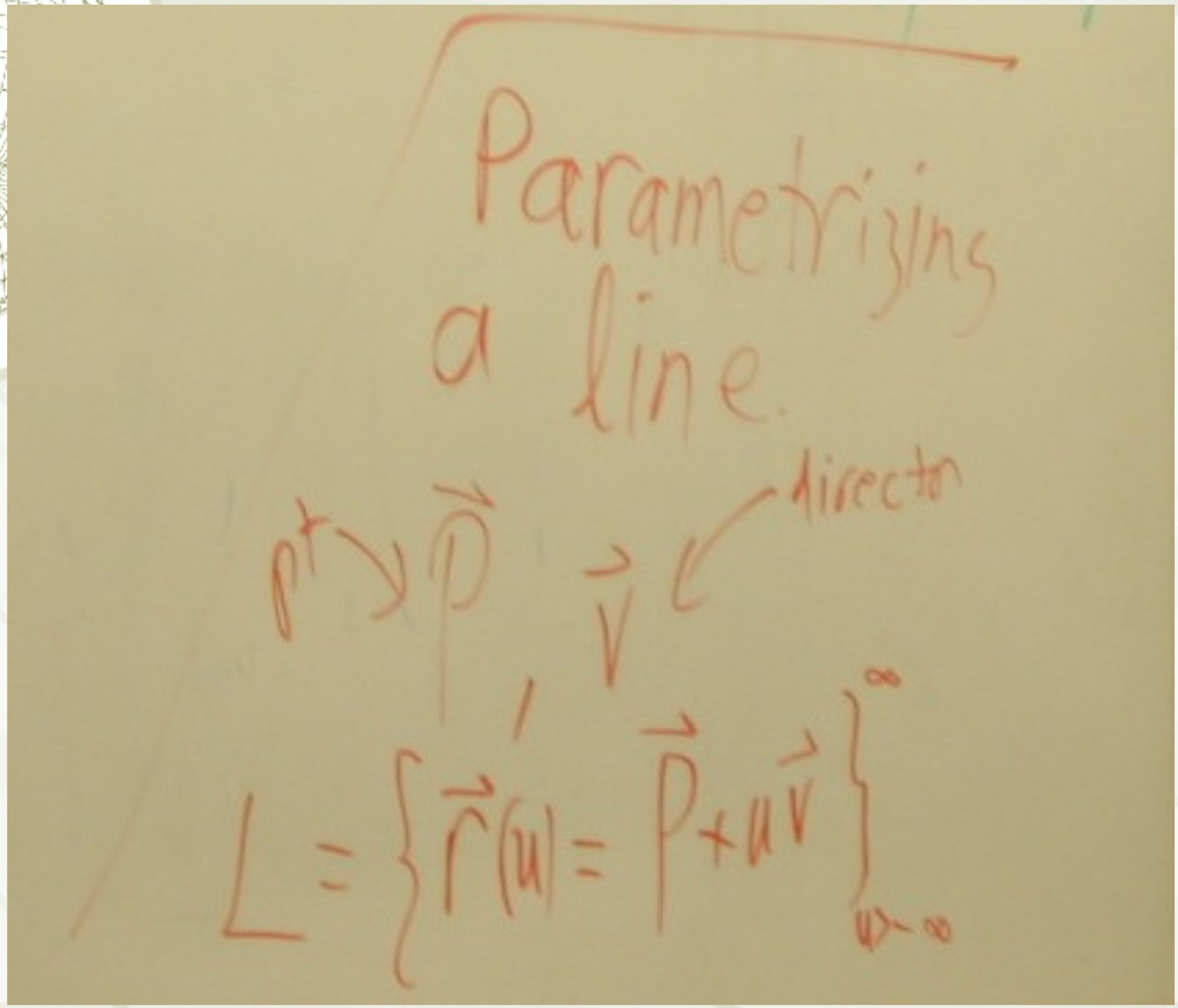
Tangent vectors

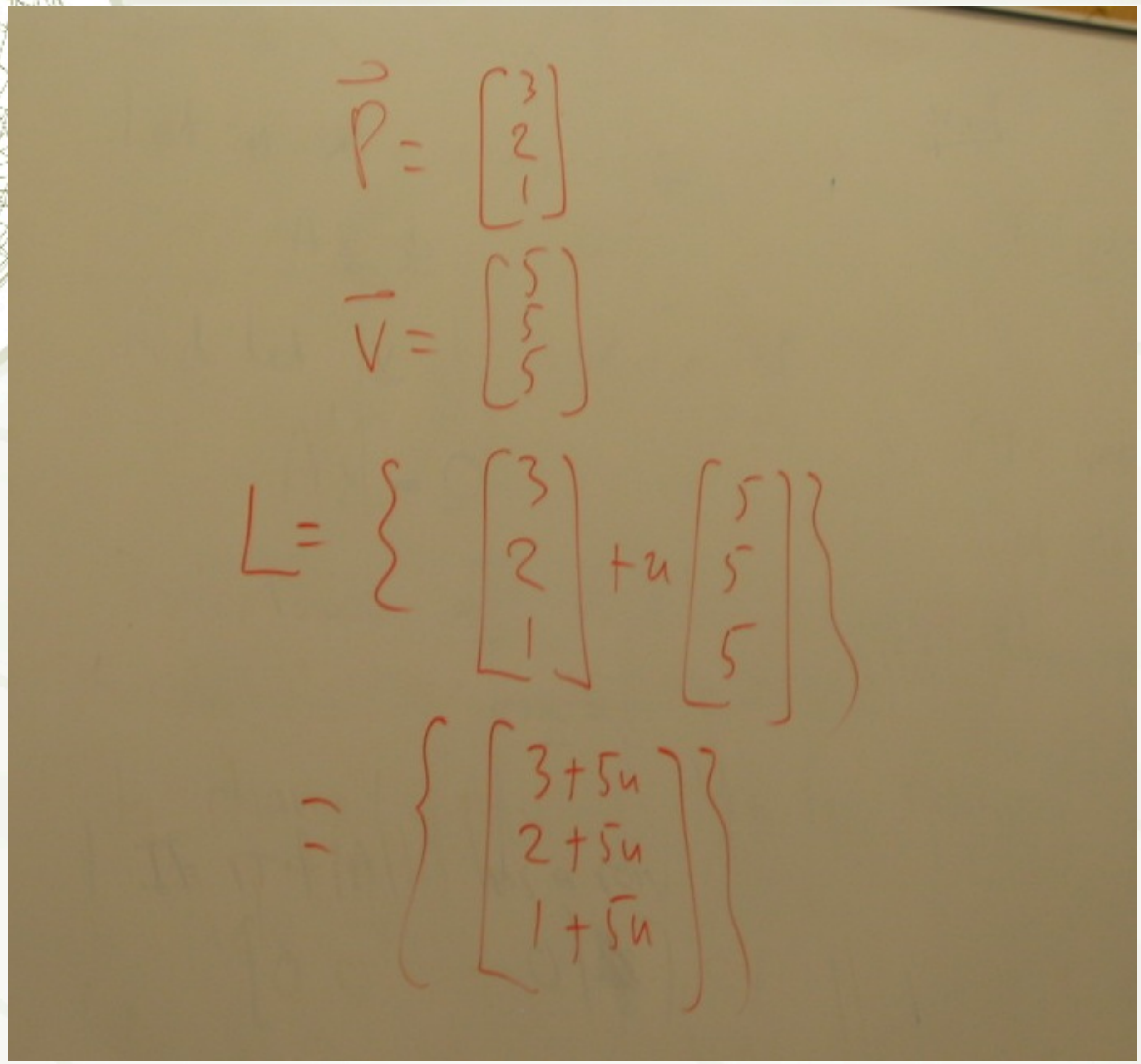
- Think velocity!
- Tangent lines
  - *Tell us more about these!*

Tangent vectors

- The velocity vector **v**(t) = **r**′(t) is tangent to the curve – points along it and not across it.




Parametrizing a line.
direction
pt → P, v
L = { r(u) = P + u v }



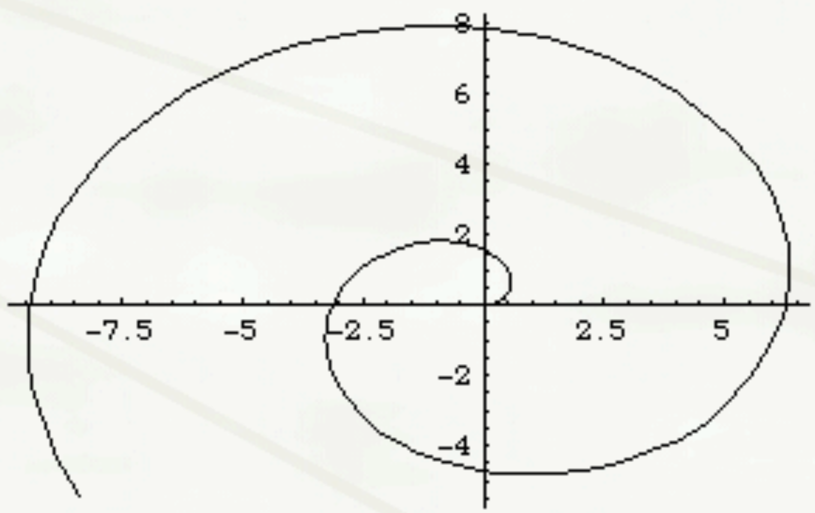
$\vec{P} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$
$\vec{V} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$
$L = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + u \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} \right\}$
$= \left\{ \begin{bmatrix} 3+5u \\ 2+5u \\ 1+5u \end{bmatrix} \right\}$

Example: spiral

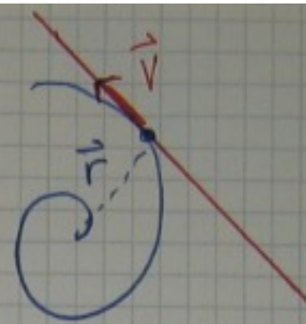
```
Spiral[t_] := {t Cos[t], t Sin[t]}

Spiral3D[t_] := {t Cos[t], t Sin[t], 0}

ParametricPlot[Spiral[t], {t, 0, 10}]
```



Spiral $\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t\cos t \\ t\sin t \end{bmatrix}$



Velocity $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos t - t\sin t \\ \sin t + t\cos t \end{bmatrix}$

This is a tangent vector $\vec{V}$ at pt. $\vec{r}$

$$\vec{L} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} = \vec{r}_0 + u\vec{V} \right\}$$

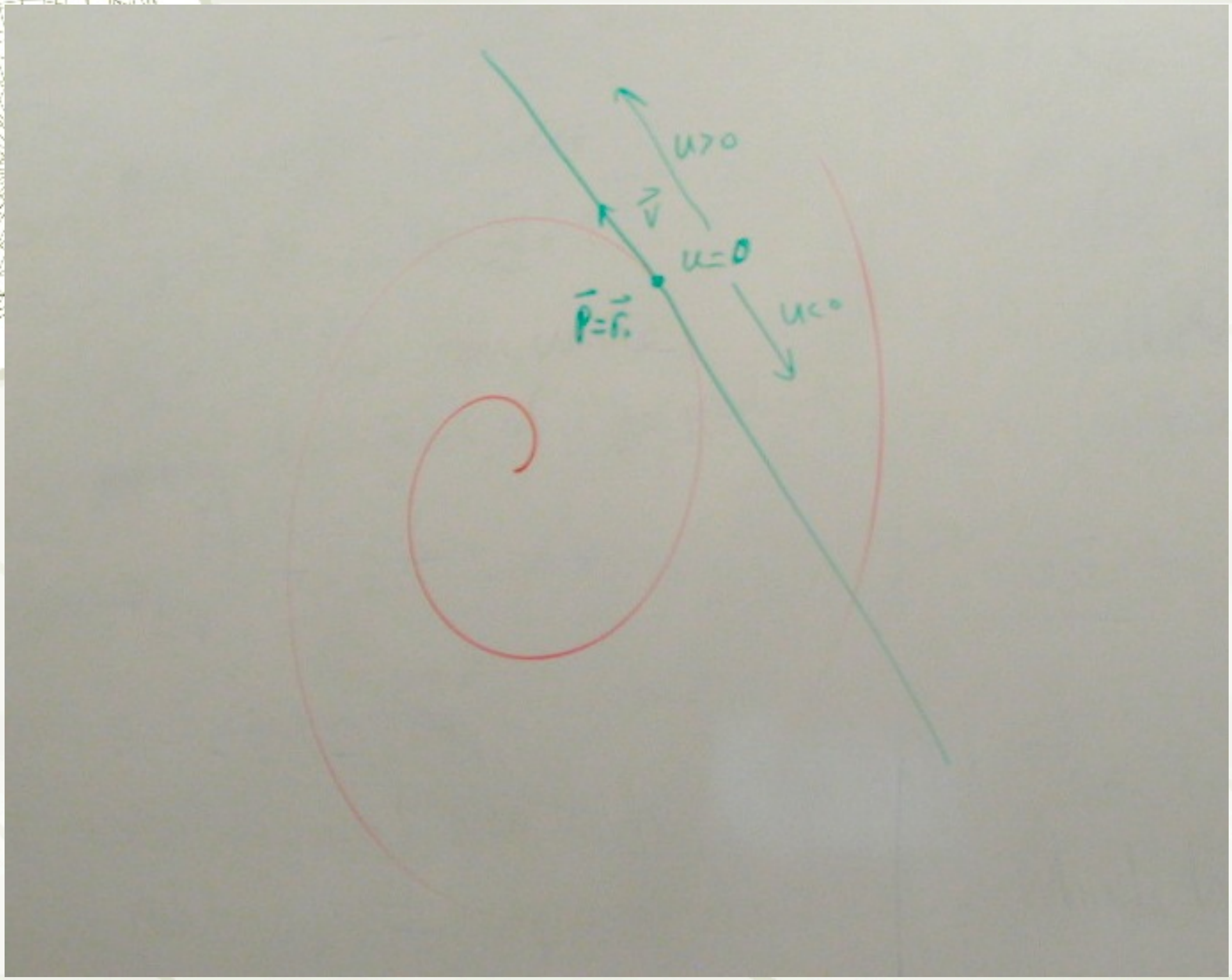
* parametric form!

Ex: $t = \frac{\pi}{4}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\pi}{4\sqrt{2}} \\ \frac{\pi}{4\sqrt{2}} \end{bmatrix}$$

$$\vec{V} = \begin{matrix} \frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}} \\ \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}} \end{matrix}$$

$$\vec{L} = \begin{bmatrix} \frac{\pi}{4\sqrt{2}} + u\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}\right) \\ \frac{\pi}{4\sqrt{2}} + u\left(\frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}\right) \end{bmatrix}$$



u>0
$\vec{v}$
u=0
$\vec{p}=\vec{r}_0$
u<0

Curves

- Position, velocity, tangent lines and all that: See Rogness applets (UMN) at

http://www.math.umn.edu/~rogness/multivar/calc2demos/curves.html

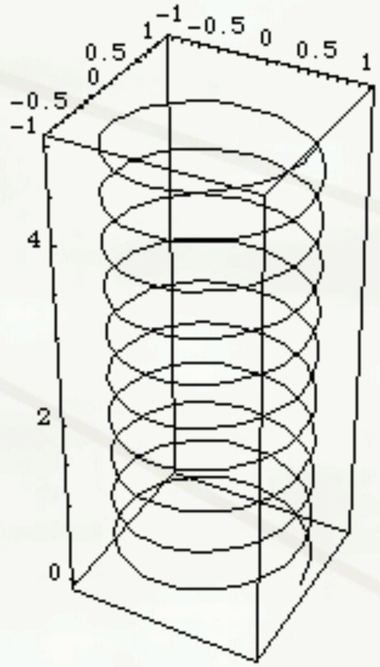
Tangent vectors

- Think velocity!
- Tangent lines
- Approximation and Taylor’s formula
- Numerical integration
- Maybe most importantly – **T** is our tool for taking curves apart and understanding their geometry.

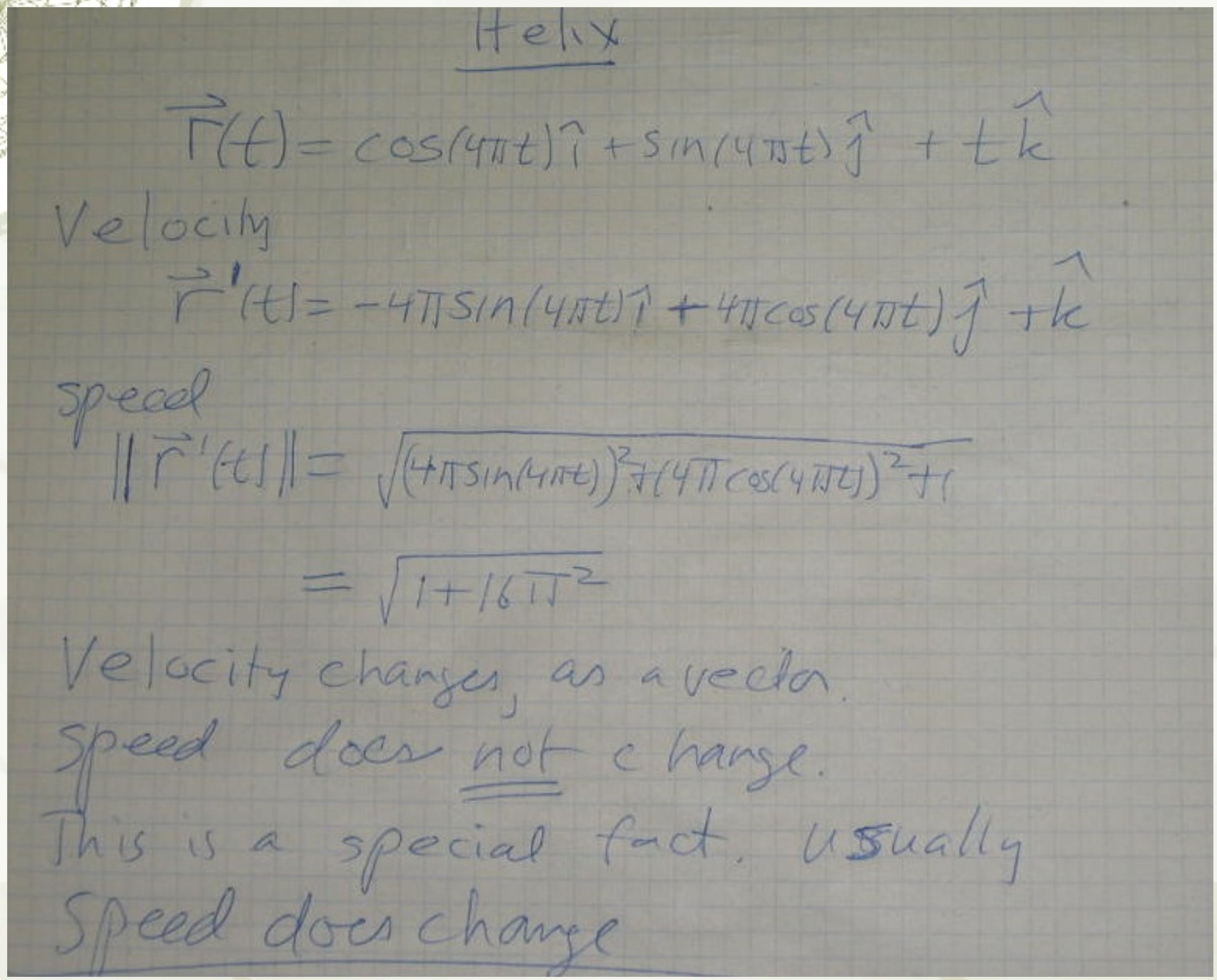
Example: helix

```
In[8]:= Helix[t_] := {Cos[4 Pi t], Sin[4 Pi t], t}

In[9]:= ParametricPlot3D[Helix[t], {t, 0, 5}, PlotPoints → 360]
```



Example: Helix



Helix

$\vec{r}(t) = \cos(4\pi t)\hat{i} + \sin(4\pi t)\hat{j} + t\hat{k}$

Velocity

$\vec{r}'(t) = -4\pi\sin(4\pi t)\hat{i} + 4\pi\cos(4\pi t)\hat{j} + \hat{k}$

speed

$\|\vec{r}'(t)\| = \sqrt{(4\pi\sin(4\pi t))^2 + (4\pi\cos(4\pi t))^2 + 1}$

$= \sqrt{1 + 16\pi^2}$

Velocity changes, as a vector.

Speed does not change.

This is a special fact. usually

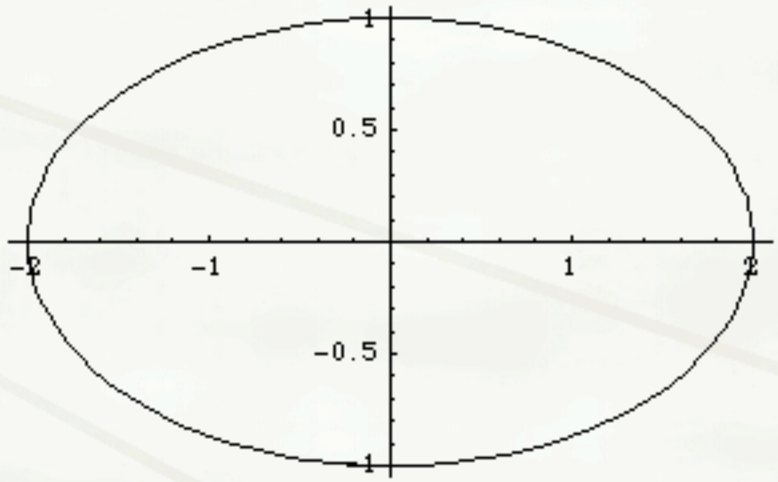
Speed does change

Unit tangent vectors

- Move on curve with speed 1.
- $\mathbf{T}(t) = \mathbf{r}'(t) / |\mathbf{r}'(t)|$
- Only 2 possibilities, $\pm$ **T**.

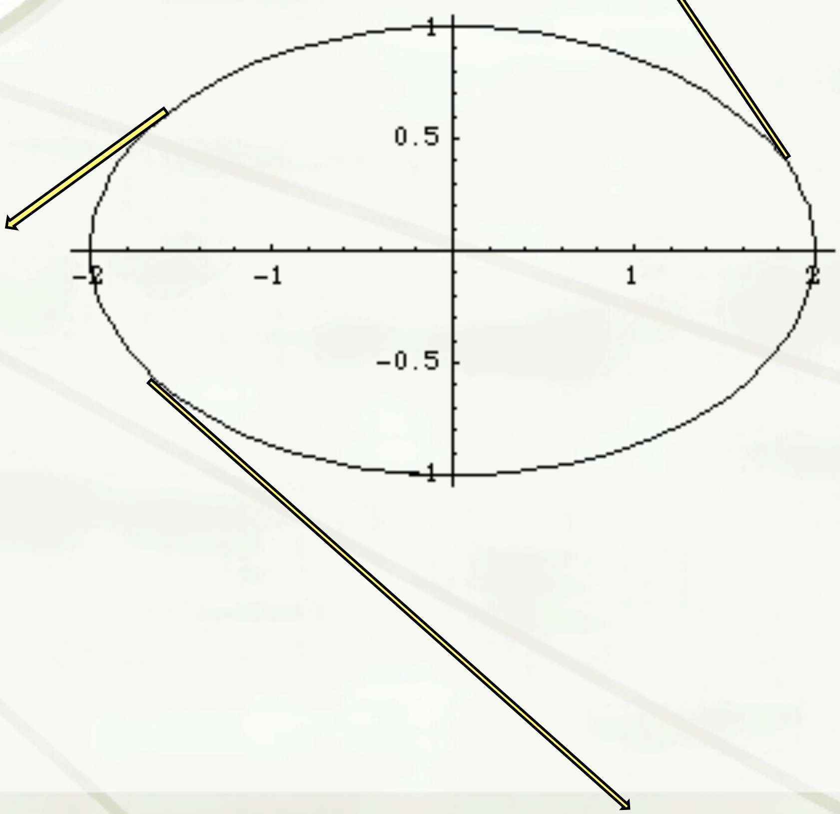
Example: ellipse

```
In[2]:= ParametricPlot[{2 Cos[t], Sin[t]}, {t, 0, 2 Pi}]
```



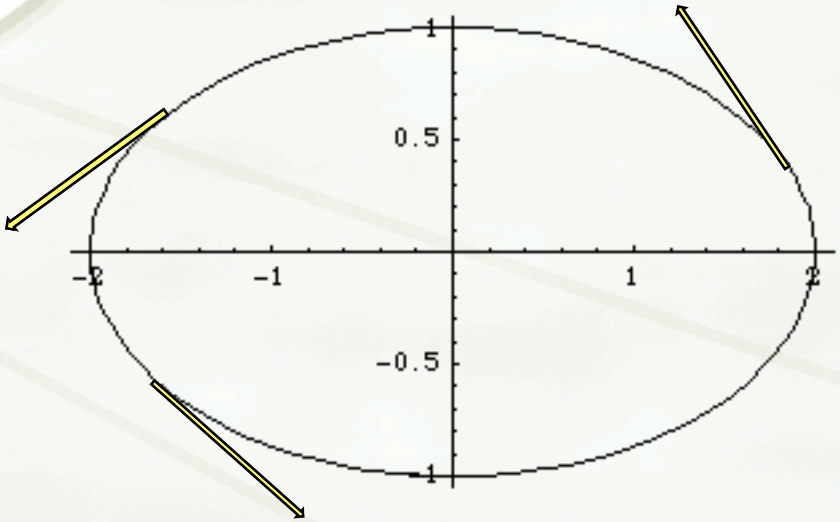
Example: ellipse

```
In[2]:= ParametricPlot[{2 Cos[t], Sin[t]}, {t, 0, 2 Pi}]
```



Example: ellipse

In[2]:= ParametricPlot[{2 Cos[t], Sin[t]}, {t, 0, 2 Pi}]

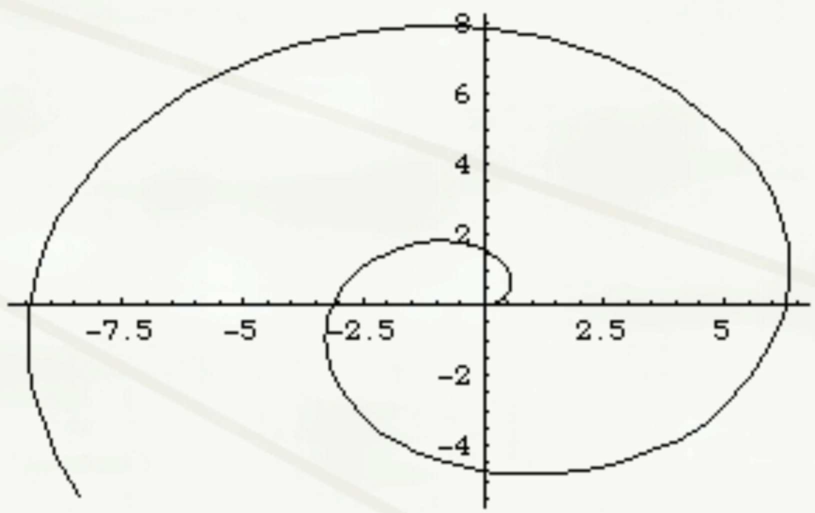


Example: spiral

```
Spiral[t_] := {t Cos[t], t Sin[t]}

Spiral3D[t_] := {t Cos[t], t Sin[t], 0}

ParametricPlot[Spiral[t], {t, 0, 10}]
```



Example: spiral

```
Speed[r_] := Sqrt[D[r[[1]], t]^2 + D[r[[2]], t]^2 + D[r[[3]], t]^2]

Tang[r_] := {D[r[[1]], t], D[r[[2]], t], D[r[[3]], t]}/Speed[r]

General::spell : Possible spelling error: new symbol name "Tang" is similar to existing symbols {Tan, Tanh}. More…

Tang[Spiral3D[t]]
```

$$\left\{\frac{\text{Cos}[t] - t\,\text{Sin}[t]}{\sqrt{(t\,\text{Cos}[t] + \text{Sin}[t])^2 + (\text{Cos}[t] - t\,\text{Sin}[t])^2}},\ \frac{t\,\text{Cos}[t] + \text{Sin}[t]}{\sqrt{(t\,\text{Cos}[t] + \text{Sin}[t])^2 + (\text{Cos}[t] - t\,\text{Sin}[t])^2}},\ 0\right\}$$

```
Simplify[%]
```

$$\left\{\frac{\text{Cos}[t] - t\,\text{Sin}[t]}{\sqrt{1 + t^2}},\ \frac{t\,\text{Cos}[t] + \text{Sin}[t]}{\sqrt{1 + t^2}},\ 0\right\}$$

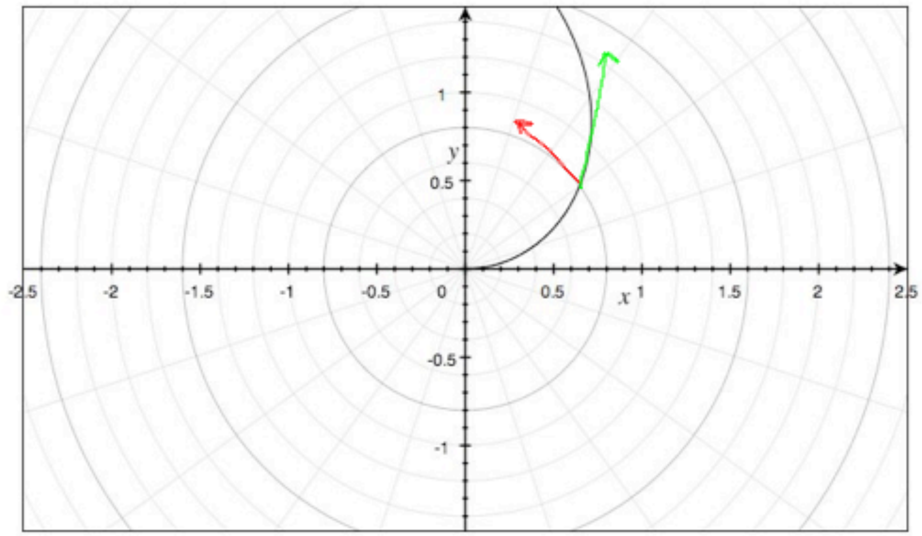
Example: Helix

$$\hat{T} = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{1+16\pi^2}} \begin{bmatrix} -4\pi \sin(4\pi t) \\ 4\pi \cos(4\pi t) \\ 1 \end{bmatrix}$$

(writing as column vector.)

Curves

- Angle of intersection
  - Spiral $r = 4\theta/\pi$ and circle



The angle looks less than $\frac{\pi}{2}$. About 1 rad?

    - $x = 4\ t \cos \pi t$, $y = 4\ t \sin \pi t$
    - $x = \cos(s)$, $y = \sin(s)$
    - intersect at $t=1/4$, $s=\pi/4$

tangent vectors

$$\vec{r}_1{}'(t) = \begin{bmatrix} 4\cos \pi t - 4\pi t \sin \pi t \\ 4 \sin \pi t + 4\pi t \cos \pi t \end{bmatrix}$$

$$r_2{}'(s) = \begin{bmatrix} -\sin(s) \\ \cos(s) \end{bmatrix}$$

$$\vec{r}_1'\left(\tfrac{1}{4}\right) = \begin{bmatrix} 2\sqrt{2} - \pi/\sqrt{2} \\ 2\sqrt{2} + \pi/\sqrt{2} \end{bmatrix} \qquad \left|r_1'\left(\tfrac{1}{4}\right)\right| = \sqrt{16+\pi^2}$$

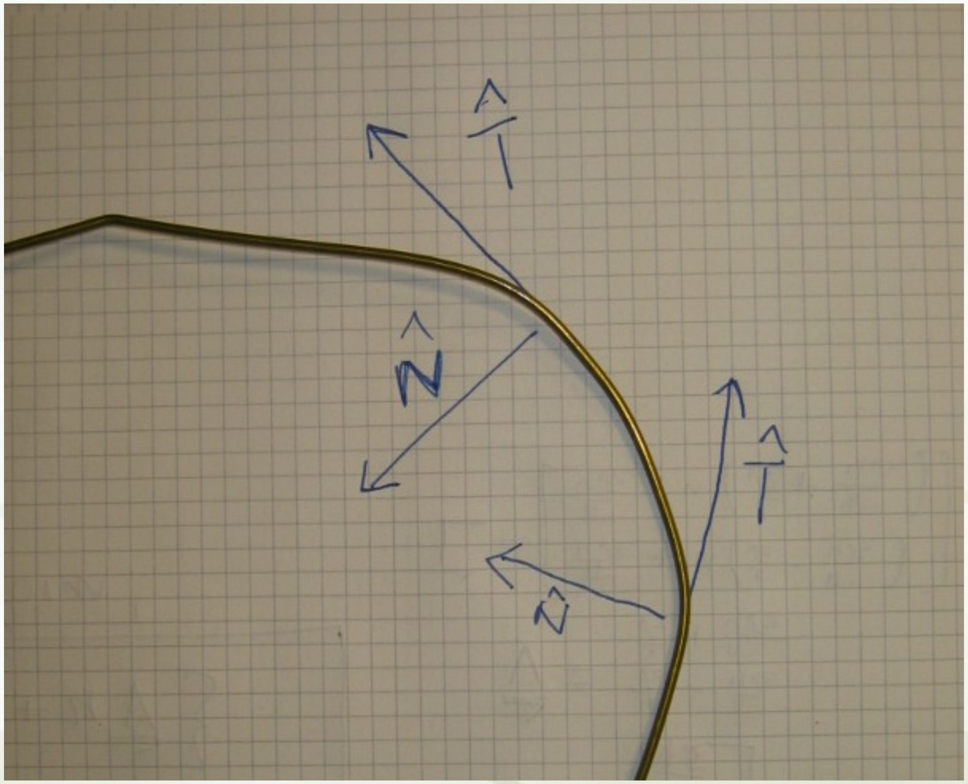
$$r_2'\left(\tfrac{\pi}{4}\right) = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \left|r_2'\left(\tfrac{\pi}{4}\right)\right| = 1$$

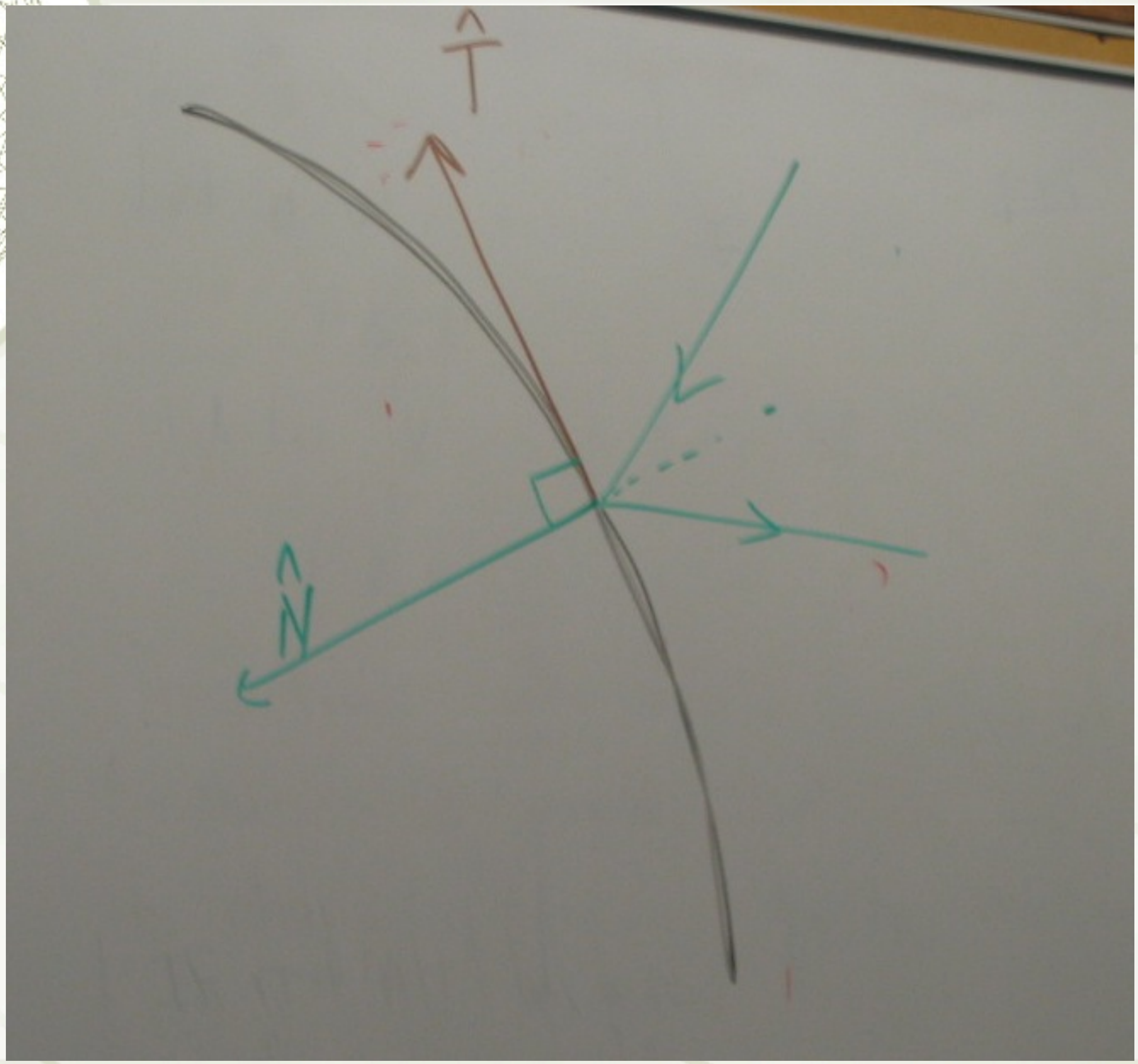
$$\vec{r}_1'\left(\tfrac{1}{4}\right)\cdot r_2'\left(\tfrac{\pi}{4}\right) = \pi = \sqrt{16+\pi^2}\cdot 1 \cdot \cos\theta$$

Therefore $\cos\theta = \frac{\pi}{\sqrt{16+\pi^2}}$, $\theta \doteq 0.905$ rad

Curves

- Even if you are twisted, you have a normal!





$\hat{T}$
$\hat{N}$

Normal vectors

- A normal vector points in the direction the curve is bending.
- It is always perpendicular to **T**.
- What's the formula?……………

Normal vectors

- $\mathbf{N} = \mathbf{T}' / ||\mathbf{T}'||.$
- Unless the curve is straight at position P, by this definition **N** is a unit vector perpendicular to **T**. *Why?*

Tangents, Normals, and Arc Length

Some nice curves

How to look at the curves from the inside

Example: spiral in 3D

```
Speed[r_] := Sqrt[D[r[[1]], t]^2 + D[r[[2]], t]^2 + D[r[[3]], t]^2]

Tang[r_] := {D[r[[1]], t], D[r[[2]], t], D[r[[3]], t]} / Speed[r]

General::spell : Possible spelling error: new symbol name "Tang" is similar to existing symbols {Tan, Tanh}. More…

Tang[Spiral3D[t]]
```

$$\left\{\frac{\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])^2+(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])^2}},\ \frac{t\,\mathrm{Cos}[t]+\mathrm{Sin}[t]}{\sqrt{(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])^2+(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])^2}},\ 0\right\}$$

```
Simplify[%]
```

$$\left\{\frac{\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ \frac{t\,\mathrm{Cos}[t]+\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ 0\right\}$$

The normal vector is then given by the derivative of this, scaled to length 1:

$$\mathrm{D}\left[\left\{\frac{\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ \frac{t\,\mathrm{Cos}[t]+\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ 0\right\},\ t\right]$$

$$\left\{\frac{-t\,\mathrm{Cos}[t]-2\,\mathrm{Sin}[t]}{\sqrt{1+t^2}}-\frac{t\,(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])}{(1+t^2)^{3/2}},\ -\frac{t\,(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])}{(1+t^2)^{3/2}}+\frac{2\,\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ 0\right\}$$

```
Len[v_] := Sqrt[v[[1]]^2 + v[[2]]^2 + v[[3]]^2]
```

$$\mathrm{Len}\left[\left\{\frac{-t\,\mathrm{Cos}[t]-2\,\mathrm{Sin}[t]}{\sqrt{1+t^2}}-\frac{t\,(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])}{(1+t^2)^{3/2}},\ -\frac{t\,(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])}{(1+t^2)^{3/2}}+\frac{2\,\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ 0\right\}\right]$$

$$\sqrt{\left(\frac{-t\,\mathrm{Cos}[t]-2\,\mathrm{Sin}[t]}{\sqrt{1+t^2}}-\frac{t\,(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])}{(1+t^2)^{3/2}}\right)^2+\left(-\frac{t\,(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])}{(1+t^2)^{3/2}}+\frac{2\,\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}}\right)^2}$$

Example: spiral in 3D

$$\sqrt{\left(\frac{-t\,\text{Cos}[t]-2\,\text{Sin}[t]}{\sqrt{1+t^2}}-\frac{t\,(\text{Cos}[t]-t\,\text{Sin}[t])}{(1+t^2)^{3/2}}\right)^2+\left(-\frac{t\,(t\,\text{Cos}[t]+\text{Sin}[t])}{(1+t^2)^{3/2}}+\frac{2\,\text{Cos}[t]-t\,\text{Sin}[t]}{\sqrt{1+t^2}}\right)^2}$$

Simplify[%]

$$\sqrt{\frac{(2+t^2)^2}{(1+t^2)^2}}$$

NormVec[t] = $$\left\{\frac{-t\,\text{Cos}[t]-2\,\text{Sin}[t]}{\sqrt{1+t^2}}-\frac{t\,(\text{Cos}[t]-t\,\text{Sin}[t])}{(1+t^2)^{3/2}},\ -\frac{t\,(t\,\text{Cos}[t]+\text{Sin}[t])}{(1+t^2)^{3/2}}+\frac{2\,\text{Cos}[t]-t\,\text{Sin}[t]}{\sqrt{1+t^2}},\ 0\right\}\Big/\sqrt{\frac{(2+t^2)^2}{(1+t^2)^2}}$$

$$\left\{\frac{\frac{-t\,\text{Cos}[t]-2\,\text{Sin}[t]}{\sqrt{1+t^2}}-\frac{t\,(\text{Cos}[t]-t\,\text{Sin}[t])}{(1+t^2)^{3/2}}}{\sqrt{\frac{(2+t^2)^2}{(1+t^2)^2}}},\ \frac{-\frac{t\,(t\,\text{Cos}[t]+\text{Sin}[t])}{(1+t^2)^{3/2}}+\frac{2\,\text{Cos}[t]-t\,\text{Sin}[t]}{\sqrt{1+t^2}}}{\sqrt{\frac{(2+t^2)^2}{(1+t^2)^2}}},\ 0\right\}$$

Thus the normal vector reduces to

Simplify[%, Assumptions → {t > 0}]

$$\left\{-\frac{t\,\text{Cos}[t]+\text{Sin}[t]}{\sqrt{1+t^2}},\ \frac{\text{Cos}[t]-t\,\text{Sin}[t]}{\sqrt{1+t^2}},\ 0\right\}$$

Example: spiral in 3D

Interpretation. Notice that the normal vector

$$\left\{-\frac{t\,\mathrm{Cos}[t]+\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ \frac{\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ 0\right\}$$

has no vertical component. This is because the spiral lies completely in the x-y plane, so an object moving on it is not accelerated vertically.

Example: Helix

■ **Example: Helix**

In[5]:= `Tang[Helix[t]]`

Out[5]= $\left\{-\frac{4\pi\,\mathrm{Sin}[4\pi t]}{\sqrt{1+16\pi^2\,\mathrm{Cos}[4\pi t]^2+16\pi^2\,\mathrm{Sin}[4\pi t]^2}},\ \frac{4\pi\,\mathrm{Cos}[4\pi t]}{\sqrt{1+16\pi^2\,\mathrm{Cos}[4\pi t]^2+16\pi^2\,\mathrm{Sin}[4\pi t]^2}},\ \frac{1}{\sqrt{1+16\pi^2\,\mathrm{Cos}[4\pi t]^2+16\pi^2\,\mathrm{Sin}[4\pi t]^2}}\right\}$

In[6]:= `Simplify[%]`

Out[6]= $\left\{-\frac{4\pi\,\mathrm{Sin}[4\pi t]}{\sqrt{1+16\pi^2}},\ \frac{4\pi\,\mathrm{Cos}[4\pi t]}{\sqrt{1+16\pi^2}},\ \frac{1}{\sqrt{1+16\pi^2}}\right\}$

The normal vector is then given by the derivative of this, scaled to length 1:

In[7]:= $\mathbf{D}\left[\left\{-\frac{4\pi\,\mathrm{Sin}[4\pi t]}{\sqrt{1+16\pi^2}},\ \frac{4\pi\,\mathrm{Cos}[4\pi t]}{\sqrt{1+16\pi^2}},\ \frac{1}{\sqrt{1+16\pi^2}}\right\},\ t\right]$

Out[7]= $\left\{-\frac{16\pi^2\,\mathrm{Cos}[4\pi t]}{\sqrt{1+16\pi^2}},\ -\frac{16\pi^2\,\mathrm{Sin}[4\pi t]}{\sqrt{1+16\pi^2}},\ 0\right\}$

In[8]:= `Len[v_] := Sqrt[v[[1]]^2 + v[[2]]^2 + v[[3]]^2]`

In[9]:= $\mathrm{Len}\left[\left\{-\frac{16\pi^2\,\mathrm{Cos}[4\pi t]}{\sqrt{1+16\pi^2}},\ -\frac{16\pi^2\,\mathrm{Sin}[4\pi t]}{\sqrt{1+16\pi^2}},\ 0\right\}\right]$

Out[9]= $\sqrt{\frac{256\pi^4\,\mathrm{Cos}[4\pi t]^2}{1+16\pi^2}+\frac{256\pi^4\,\mathrm{Sin}[4\pi t]^2}{1+16\pi^2}}$

Example: Helix

In[9]:= $\text{Len}\left[\left\{-\frac{16\pi^2 \text{Cos}[4\pi t]}{\sqrt{1+16\pi^2}}, -\frac{16\pi^2 \text{Sin}[4\pi t]}{\sqrt{1+16\pi^2}}, 0\right\}\right]$

Out[9]= $\sqrt{\frac{256\pi^4 \text{Cos}[4\pi t]^2}{1+16\pi^2}+\frac{256\pi^4 \text{Sin}[4\pi t]^2}{1+16\pi^2}}$

In[10]:= Simplify[%]

Out[10]= $\frac{16\pi^2}{\sqrt{1+16\pi^2}}$

Remarkably, it does not depend on time. This sort of simplification is often encountered when a curve is simple or has symmetries.

In[11]:= $\text{NormVec}[t] = \text{Simplify}\left[\left\{-\frac{16\pi^2 \text{Cos}[4\pi t]}{\sqrt{1+16\pi^2}}, -\frac{16\pi^2 \text{Sin}[4\pi t]}{\sqrt{1+16\pi^2}}, 0\right\} \Big/ \left(\frac{16\pi^2}{\sqrt{1+16\pi^2}}\right)\right]$

Out[11]= $\{-\text{Cos}[4\pi t], -\text{Sin}[4\pi t], 0\}$

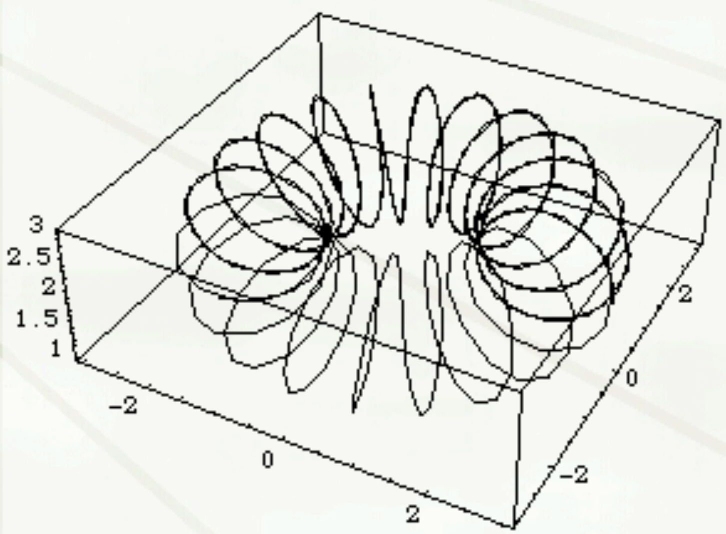
Thus the normal vector to the 3D spiral traces out a circle in 2D

Example: Helix

Interpretation. Notice that the normal vector again has no vertical component. If a particle rises in a standard helical path, it does not accelerate upwards of downwards. The acceleration points inwards in the x-y plane. It points towards the central axis of the helix.

Example: solenoid

```
Solenoid[t_, r_, R_, w_] := {(R + r Cos[w t]) Cos[t], (R + r Cos[w t]) Sin[t], R + r Sin[w t]}
ParametricPlot3D[Solenoid[t, 1, 2, 20], {t, 0, 10}, PlotPoints → 360]
```



Example: solenoid

Scary, but it might be fun to work it out!

Arc length

- If an ant crawls at 1 cm/sec along a curve, the time it takes from a to b is the arc length from a to b.