Last but not least

Final office period

- Monday, 2:30-4:00, Skiles 218D.

New theorems from old.

Green: $\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint \vec{F} \cdot d\vec{r} = \oint \vec{F} \cdot \hat{t}\, ds$

where $\vec{F} = P\hat{i} + Q\hat{j}$.

But what if we choose

$\vec{H} = Q\hat{i} - P\hat{j}$? Left $= \iint_\Omega \nabla \cdot H \, dA$.

Right is $\oint (P t_x + Q t_y)\, ds$

$$= \oint \begin{bmatrix} Q \\ -P \end{bmatrix} \cdot \begin{bmatrix} t_y \\ -t_x \end{bmatrix} ds$$

$$= \oint \left(\vec{H} \cdot \hat{n}_{out}\right) ds$$

$$\boxed{\iint_\Omega \nabla \cdot \vec{H}\, da = \oint (\vec{H} \cdot \hat{n})\, ds}$$ "Divergence theorem."

Conservation of mass and the continuity equation

- $\partial\rho/dr = -(\nabla \bullet \rho\mathbf{v})$ says that the mass in a small region changes in time in proportion (negatively) to the flux out of the region.

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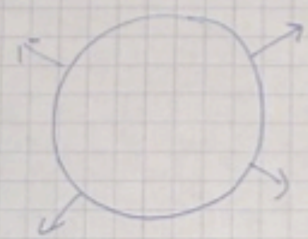
$$\frac{\partial M(R)}{\partial t} = \frac{\partial}{\partial t}\iiint_R \rho\, dV = -\iiint_R \nabla\cdot(\rho\vec{v})\, dV$$

$$\iiint_R \left(\rho_t + \nabla\cdot(\rho\vec{v})\right) dV = 0$$

Suppose a charge density λ is distributed inside a sphere independently of angle.
What is the electric field $\vec{D}$ at the boundary?



(In free space **E** and **D** are the same.)

S1. Symmetry indicates that it will (a) point radially and (b) have a constant magnitude

Therefore $\iint_{\partial B} \vec{E} \cdot \hat{n}\, d\sigma = |E| \cdot 4\pi R^2$ ← area of sphere.

By Gauß, $\int_{\partial B} \vec{E} \cdot \hat{n}\, da = \int_{B} \nabla \cdot E\, d^3x$

$$= \frac{1}{\varepsilon_0} \iiint \lambda\, d^3x = \frac{Q}{\varepsilon_0}$$

Conclusion: $|E| = \frac{Q}{4\pi \varepsilon_0 R^2}$ $\quad \vec{E} = \frac{Q\, \vec{r}}{4\pi \varepsilon_0 |\vec{r}|^3}$

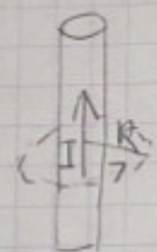
Stokes's Theorem

Let $\mathcal{C}$ be a curve along the edge of a surface Ω (not assumed flat).

Let $\mathcal{C}$ be a curve along the edge of a (smooth, orientable, connected) surface Ω. Let the unit normal to Ω point according to the right hand rule with respect to the orientation of $\mathcal{C}$. Then for any smooth vector field F,

$$\iint_{\Omega} (\nabla \times F) \cdot \hat{n}\, da = \oint_{\mathcal{C}} \vec{F} \cdot d\vec{r}$$

Magnetic field near a wire



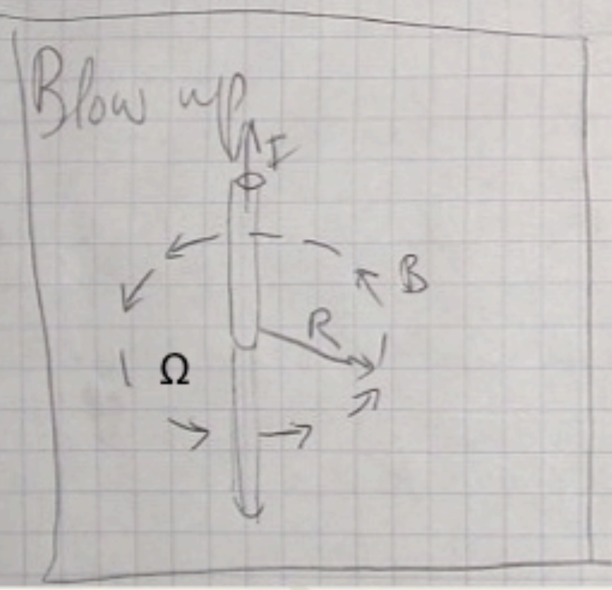
Ampère's law

$\nabla \times B = \mu_0 J$

$$\Rightarrow \iint_{\Omega} (\nabla \times B)\cdot n\, ds = \mu_0 \iint_{\Omega} J \cdot \hat{n}\, ds = \mu_0 I$$

also $= \oint_{\partial\Omega} \vec{B}\cdot d\vec{r} = 2\pi R\,|B|$

so ... $|B| = \dfrac{\mu_0 I}{2\pi R}$



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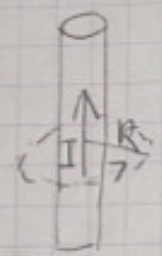

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$|B|\,ds$

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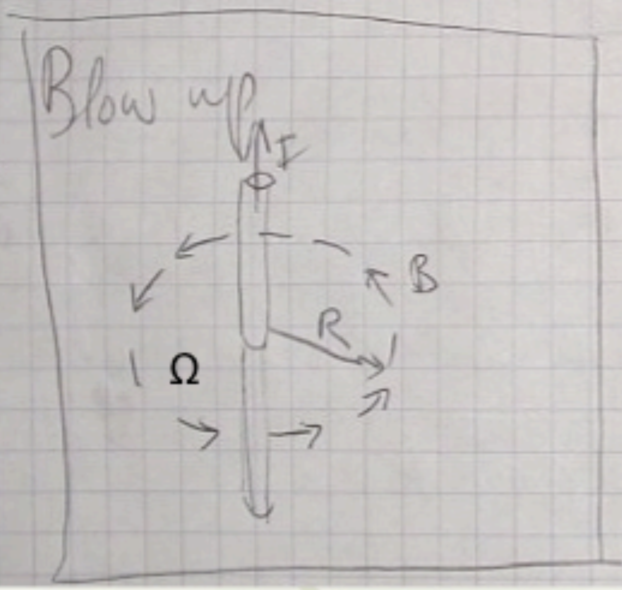


PROF. H's ABSOLUTE TOP TEN LIST

10

The PDEs using grad, curl, and div

9 ....The Laplacian

- Δ or $\nabla^2 = \nabla \bullet \nabla = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$
- If $\Delta u = 0$, "Laplace's equation," then u is "harmonic."
  - $x - y$
  - $x^2 - y^2$
  - $e^x \cos y$
  - $x^2 + z^2 - y^2$ **(NOT THIS ONE!)**

Laplacian

- Laplace's equation for "harmonic fns":
  - $\Delta u = 0$.
    - Equilibrium membrane, electric potential
- Heat or diffusion equation
  - $u_t = k \, \Delta u$.
    - Temperature, density of dye
- Wave equation
  - $u_{tt} = c^2 \, \Delta u$.
    - Sound, light

8, 7, 6…

Green, Gauß, and Stokes

- You can swap out integrals with curls and divs for integrals of a lower dimension
- or vice versa.

5,4…

Curl and Divergence

- (We’ll get to Grad later)
- Think of $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ as a “vector operator.”
- Div: vector in, scalar out $\nabla \cdot \mathbf{v}$
- Curl: vector in, vector out $\nabla \times \mathbf{v}$
- Sometines simplify with Green, Gauß or Stokes.

3….

- Triple integrals
  - Related to volume.
  - Do ‘em as 3 kindergarten integrals in a row.
  - The tricky part is the limits.

2....

- Double integrals
  - Related to area.
  - Do ‘em as 2 kindergarten integrals in a row.
  - The tricky part is the limits.

I .... *The Fundamental VP*

- $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$.
  - It is normal (perpendicular) to the surface, as defined by **r**(u,v), with two parameters.
  - If $z = f(x,y)$, $\mathbf{N} = -f_x\,\mathbf{i} - -f_y\,\mathbf{j} + \mathbf{k}$
  - $d\sigma = |\mathbf{N}|\ du\ dv$

O…. *Line integrals*

- *Sometimes* you can evaluate them as
  f(**b**) - f(**a**) …..

- Otherwise you have to *parametrize*.
  - (Or use Green or Stokes)

-1.... Jacobians

- *Curvilinear coordinates*
  - My favorite is paraboloidal! Mine is bipolar!

- Change to convenient coordinates to do an integral.

-2…. *Cylindrical*

- Cylindrical is just polar with z.
- x = r cos θ, y = r sin θ, z = z
- Remember r dr dθ dz

3…. *Spherical*

- Spherical is colatitude (like polar, but from north pole down) and longitude
- $r = \rho \sin \phi$, $\theta = \theta$, $z = \rho \cos \phi$
- $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \theta$
- Remember $\rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$

We pause here for one of the great integrals

Newton's theorem. $\lambda(\vec{r})$ = mass density.

$$\phi(\vec{r}) = -Gm \iiint \frac{\lambda(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r'$$

If λ is spherically symmetric (indep of θ, ϕ), and $|r|$ > radius of object, then

$$\phi(\vec{r}) = -\frac{GmM}{|r|} \quad \text{(as if all mass at origin).}$$

Proof. We maws. $\vec{r} = a\hat{k}$, and use spherical coords. for $\vec{r}\,'$

$$|\vec{r}-\vec{r}\,'| = \sqrt{(a-\rho\cos\phi)^2 + (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2}$$

$$= \sqrt{a^2+\rho^2-2a\rho\cos\phi}$$

$$\phi(\vec{r}) = -Gm \int_0^R \int_0^\pi \int_0^{2\pi} \frac{\lambda(\rho)\cdot\rho^2\sin\phi}{\sqrt{a^2+\rho^2-2a\rho\cos\phi}}\, d\theta\, d\phi\, d\rho$$

$$= -2\pi Gm \int_0^R \lambda(\rho)\rho^2 \left(\int_{\cos\phi=1}^{\cos\phi=-1} (a^2+\rho^2-2a\rho\cos\phi)^{-\frac{1}{2}}\, d(\cos\phi) \right) d\rho$$

$$\phi(\vec{r}) = -2\pi G m \int_0^R \lambda(\rho)\rho^2 \cdot \frac{2}{(-2a\rho)} \left(a^2+\rho^2-2a\rho\cos\phi\right)^{\frac{1}{2}} \Big|_{\cos\phi=1}^{\cos\phi=-1} d\rho$$

$$= -\frac{2\pi G m}{a} \int_0^R \lambda(\rho)\cdot\rho\cdot\left(\left(a^2+\rho^2+2a\rho\right)^{\frac{1}{2}} - \left(a^2+\rho^2-2a\rho\right)^{\frac{1}{2}}\right) d\rho$$

$$= -\frac{2\pi G m}{a} \int_0^R \lambda(\rho)\cdot\rho\left((a+\rho)-(a-\rho)\right) d\rho$$

$$= -\frac{Gm}{a}\cdot 4\pi \cdot \int_0^R \lambda(\rho)\rho^2 \, d\rho$$

Recall $\int_0^{\pi}\int_0^{2\pi} \sin\phi \, d\theta \, d\phi = 4\pi$

-4....

- Reconstructing a function from its gradient
  - Just when is a vector field the gradient of a function?
    - xi + yj
    - xi - yj
    - yi - xj (NOT THIS ONE!)
    - yi + xj

-5...Lagrange

- Max/Min with constraints
  - $\nabla F = \lambda \nabla G$
  - n equations in n+1 unknowns

-6: *local max/min*

- $\nabla F = 0$ … or ∇F not defined.

- Two questions: *where*, and *how big*?

- The ***Hessian***.

∇…*GRAD*

- The basic vector derivative.
- Direction uphill
- Tangent planes
- Approximation
  - Seems kind of kindergarteny now that we know div and curl, doesn't it?

-8... arc length

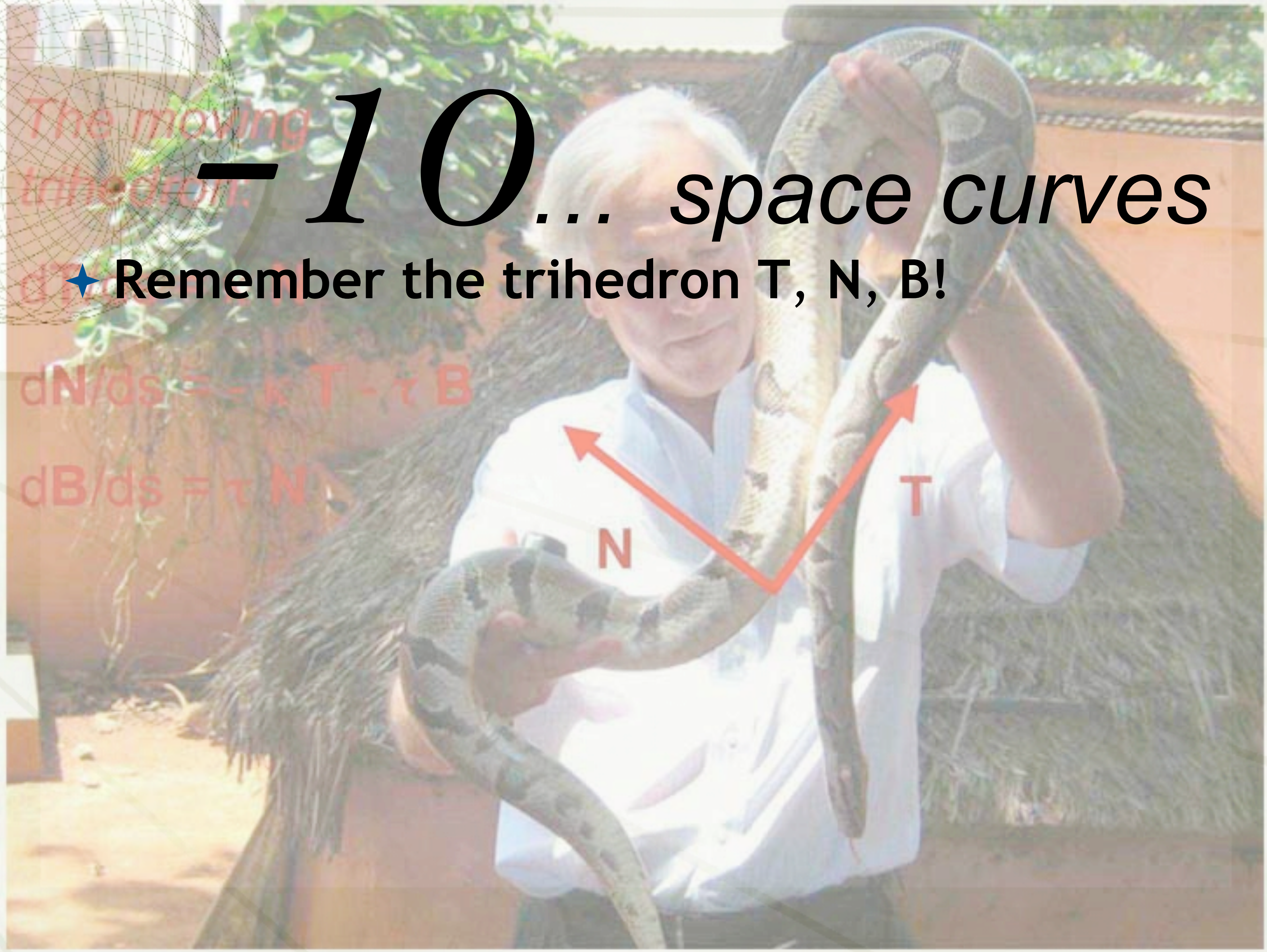
- $|d\mathbf{r}|$
- $|\mathbf{v}|\ dt = |\ d\mathbf{r}/dt\ |\ dt$

-9... *curvature*

- A kind of second derivative
- There's also torsion!

–10… *space curves*

- **Remember the trihedron T, N, B!**



The End