MATH 2401 - Harrell

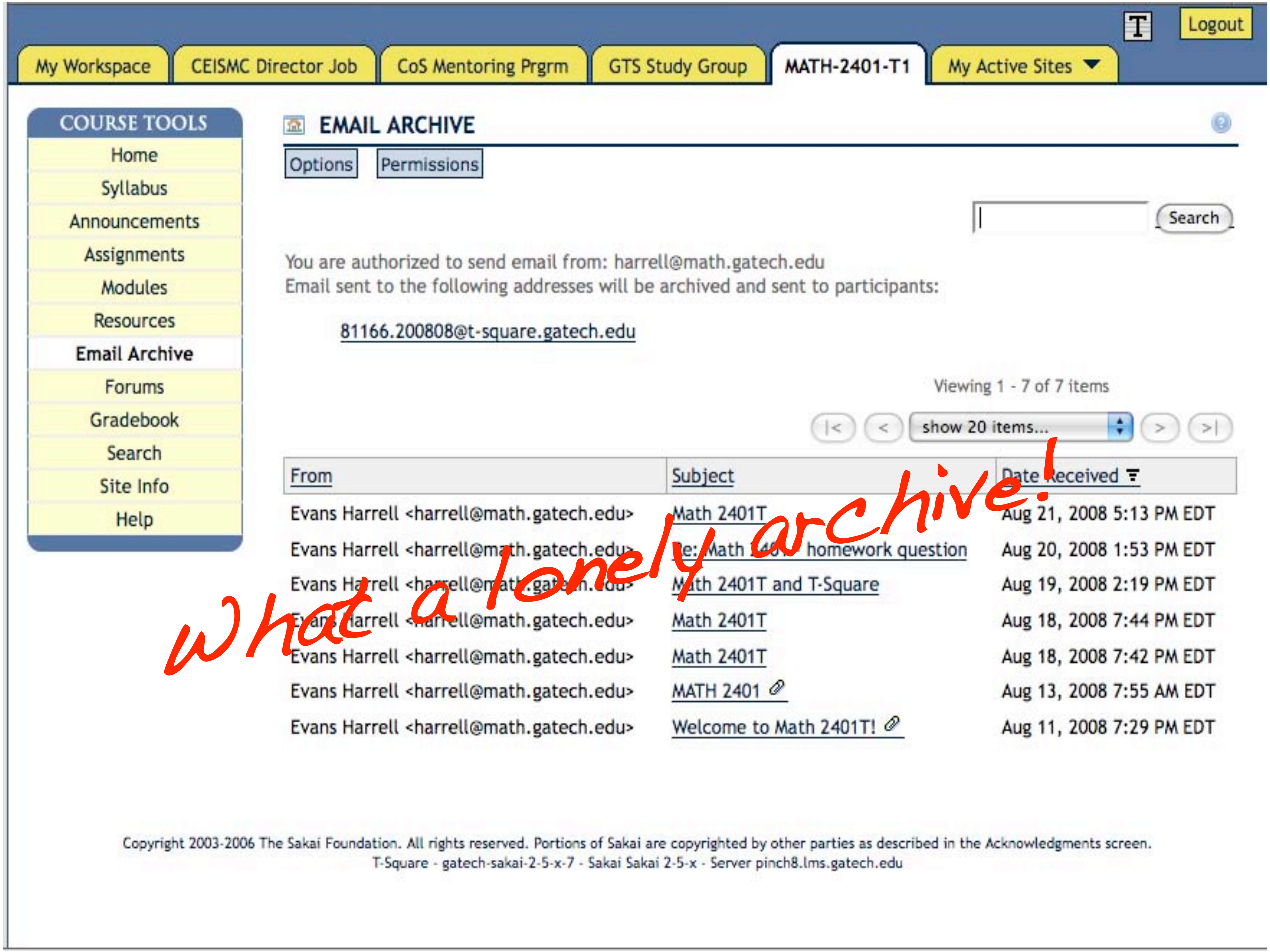
Curves - A lengthy story

Lecture 4

Reminder…

No class on Monday, but ….

There's a test on Thursday!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What a lonely archive!

Exercises:

- SHE, Section 14.2, # 4-8,16,18,20,21,22,26,32,34 (Hand in at least 5 even-numbered ones.)
- SHE, Section 14.3, # 4-8,9,14-17,22-24,31-34,38,40 (Hand in at least 5 even-numbered ones.)
- SHE, Section 14.4, #2,4,10 (Hand in 2 of these.)

- Due Thursday, 4 September:

  No assignments will be collected on this day. Instead, **there will be an hour-long test.** 

  Reading:
  - SHE, Sections 14.4-14.6
  - Optional, but helpful, see Chapter 4 of Cain and Herod's on-line vector-calculus text
  - Review the lecture of 28 August (as of that evening)
  - Review the lecture of 3 September (as of that evening)

Current contests

Note about contest entries. These must be entirely your own work and not, for example, copid from the Web, even in modified form (which would be an honor code violation.

1. Due 4 September. One point for the most creative and interesting curve drawn with software, such as Mathematica's ParametricPlot command. Submit both a graphic file (pdf, jpg, pict, png) and the code used by e-mail.
2. Due 4 September. One point for the most informative on-paragraph explanation expanding on material from the lectures connected with curves. This should be submitted to the T-Square archive with a copy to Prof. H.

Past homework assignments

- Due Monday, 18 August:

  Reading:
  - As necessary, review basic vectors, for example reading SHE (Salas, Hille, and Etgen), Chapter 13.
  - SHE, Sections 14.1-14.2

Who in the cast of characters might show up on the test?

- Curves **r**(t), velocity **v**(t).
- Tangent and normal lines.
- Angles at which curves cross.
- **T**,**N**, **B**, and the curvature κ.
- **The arc length s.**
- The osculating plane.

Velocity vs. speed

- The velocity $\mathbf{v}(t) = d\mathbf{r}/dt$ is a vector function.

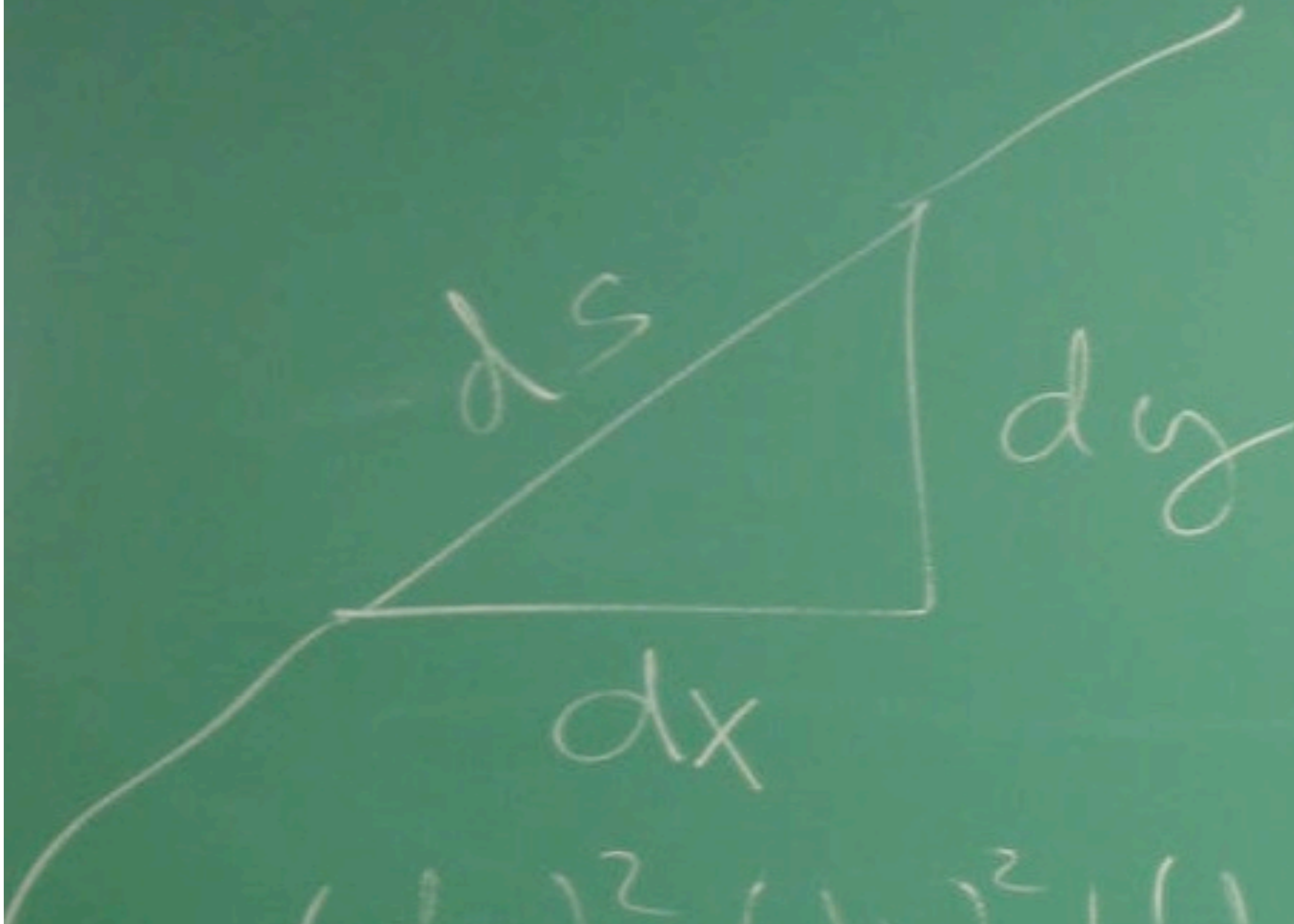
- The speed $|\mathbf{v}(t)|$ is a scalar function.
  $|\mathbf{v}(t)| \geq 0$.

Arc length

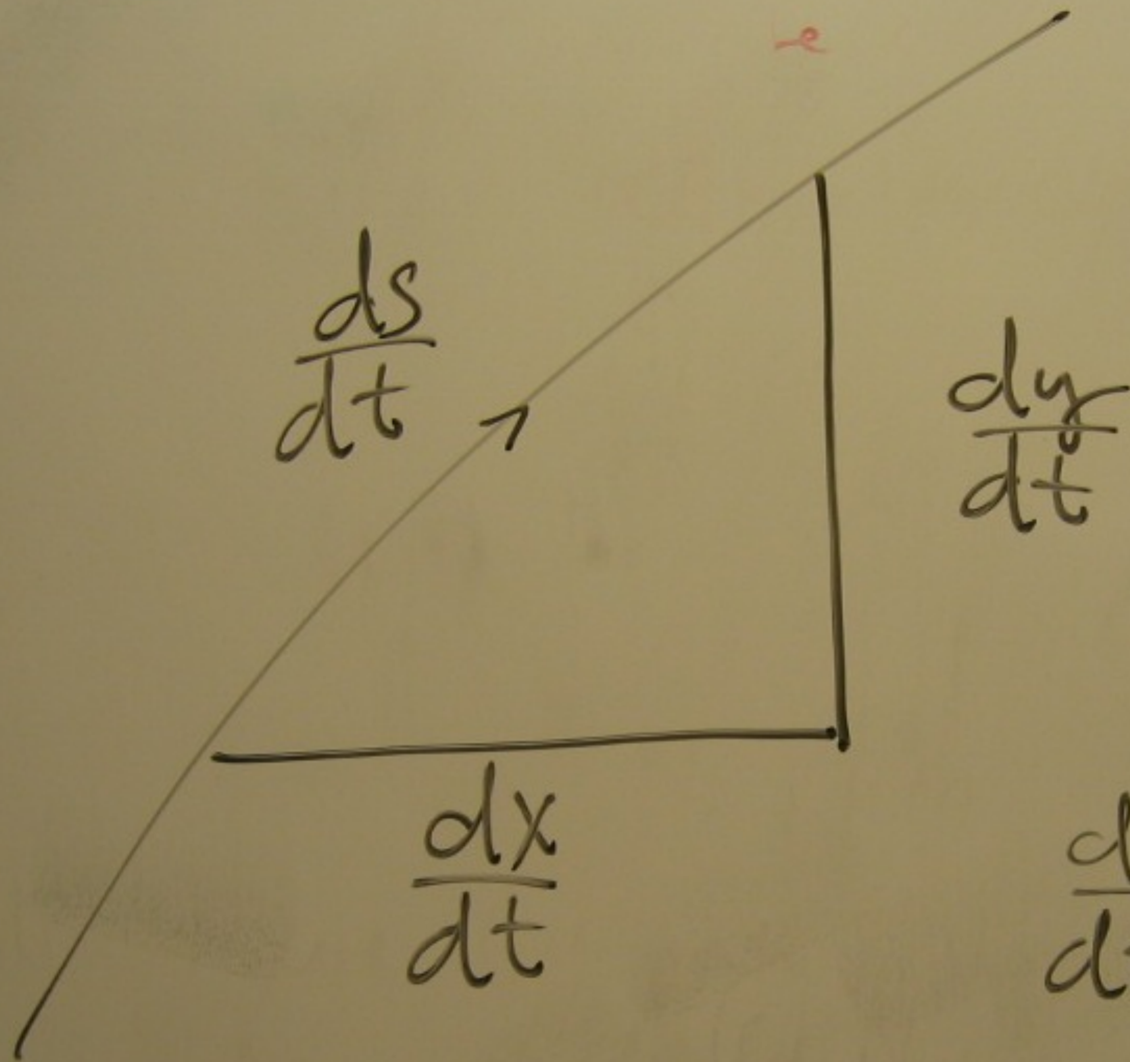
- If an ant crawls at 1 cm/sec along a curve, the time it takes from a to b is the arc length from a to b.
- More generally, $ds = |\mathbf{v}(t)|\, dt$
- In 2-D $ds = (1 + y'^2)^{1/2}\, dx$, or

$$ds^2 = dx^2 + dy^2$$

or…



$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$



$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$ds = |\vec{V}(t)|\, dt$$

Arc length

$$ds = \frac{ds}{dt}dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

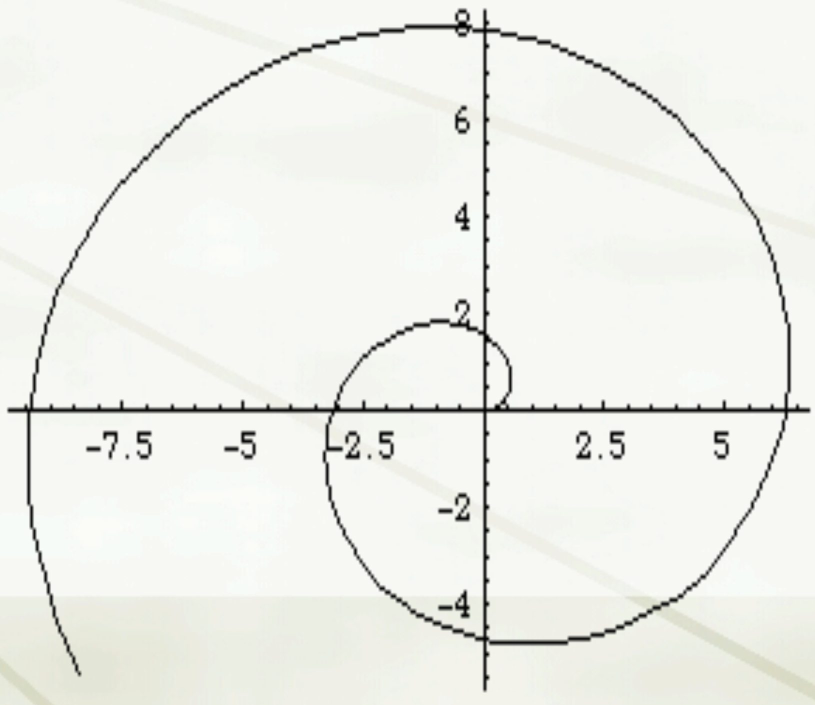
$$L(C) = \int_C ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Example: spiral

```
Spiral[t_] := {t Cos[t], t Sin[t]}

Spiral3D[t_] := {t Cos[t], t Sin[t], 0}

ParametricPlot[Spiral[t], {t, 0, 10}]
```



Examples

$$|\vec{r}'(t)| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}$$

$$= \sqrt{\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t}$$

$$= \sqrt{1 + t^2}$$

$$ds = \frac{ds}{dt}\,dt = \sqrt{1+t^2}\,dt$$

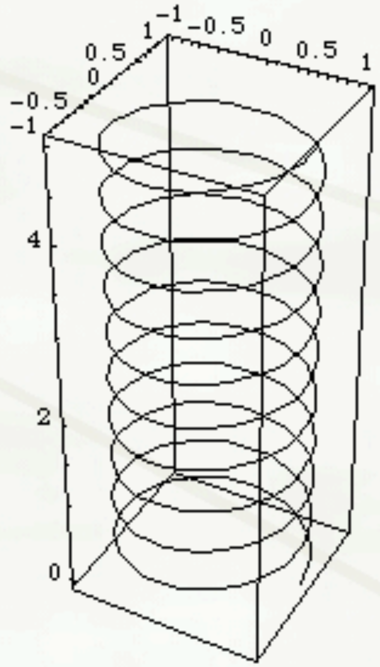
```
In[1]:= Integrate[Sqrt[1 + t^2], {t, 0, 4 Pi}]
```

Out[1]= $\frac{1}{2}\left(4\pi\sqrt{1+16\pi^2} + \mathrm{ArcSinh}[4\pi]\right)$

Example: helix

```
In[8]:= Helix[t_] := {Cos[4 Pi t], Sin[4 Pi t], t}

In[9]:= ParametricPlot3D[Helix[t], {t, 0, 5}, PlotPoints → 360]
```



Examples

$$|\vec{r}\,'(t)| = \sqrt{(-4\pi \sin(4\pi t))^2 + (4\pi \cos(4\pi t))^2 + 1}$$

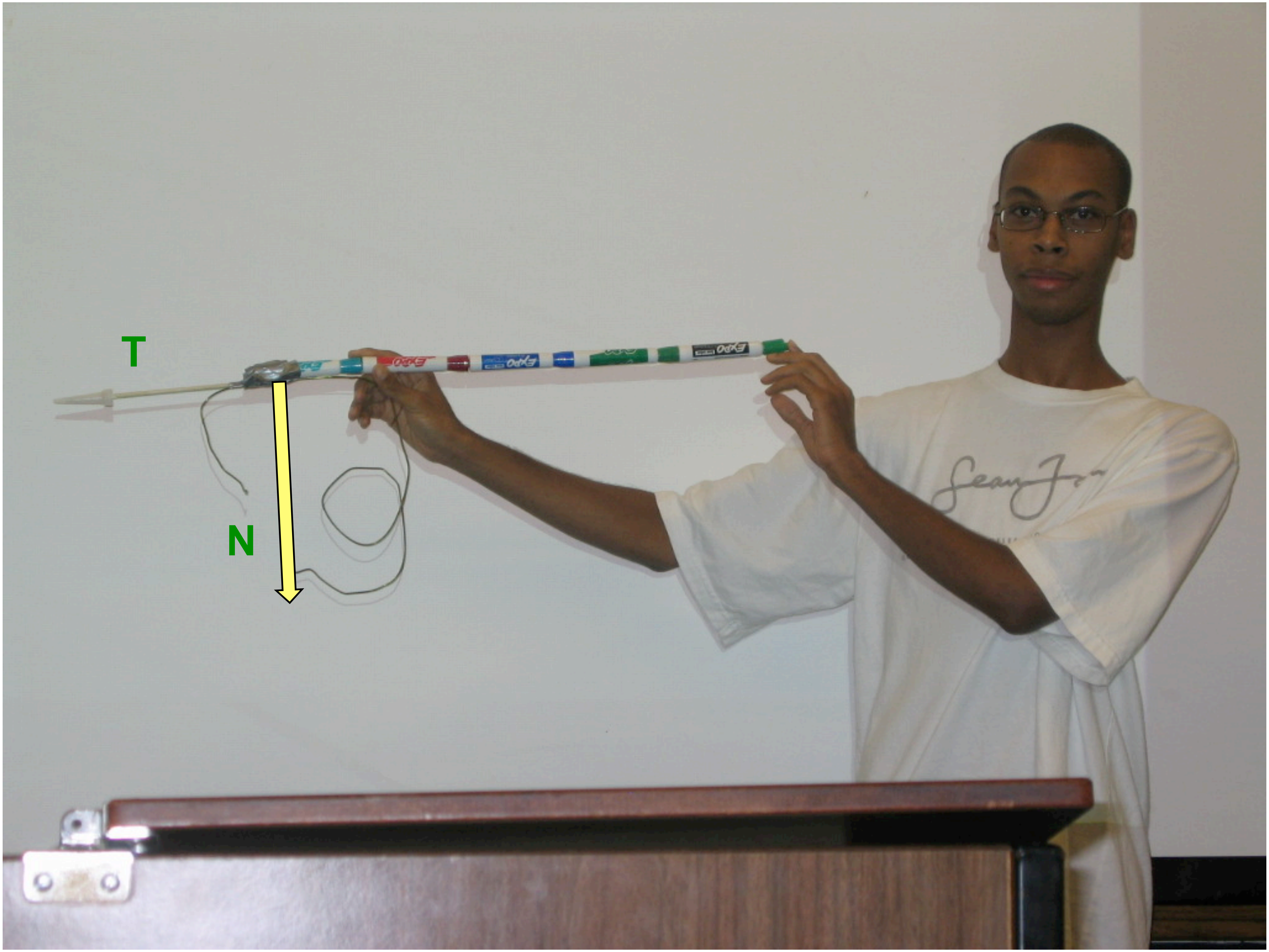
$$= \sqrt{1+16\pi^2} \quad \text{(a constant)}$$

$$ds = \sqrt{1+16\pi^2}\ dt$$

Miraculously - don't expect this in other examples - the speed does not depend on t. The arclength in 2 coils, t from 0 to 1, is the integral of $|r'|$ over this integral, i.e., $(1+ 16\ \pi^2)^{1/2}$.

Unit tangent vectors

- Not only useful for arc length, also for understanding the curve ‘from the inside.’
- Move on curve with speed 1.
- $\mathbf{T}(t) = \mathbf{r}'(t)/\ |\mathbf{r}'(t)|$

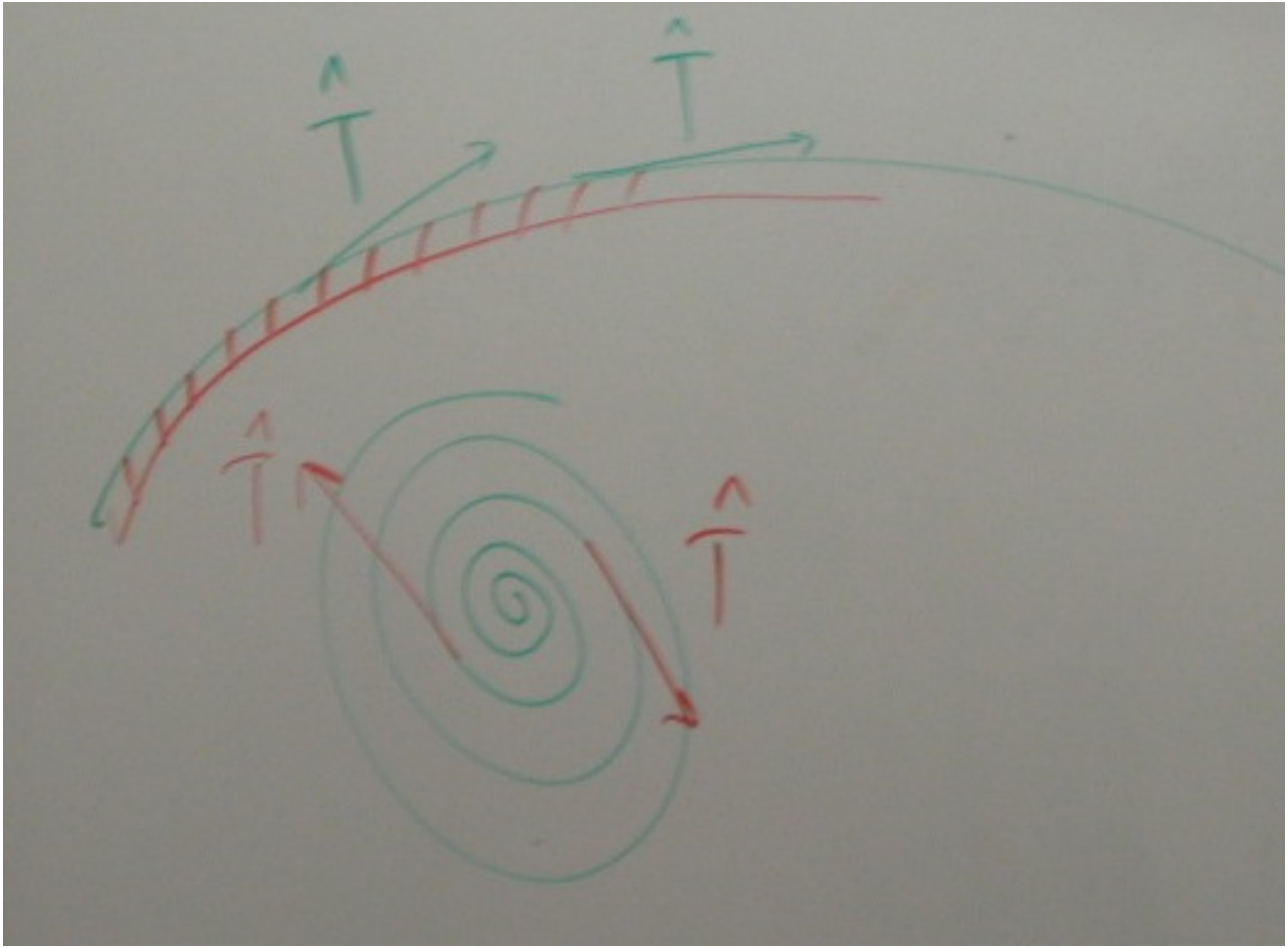

T
N

Normal vectors

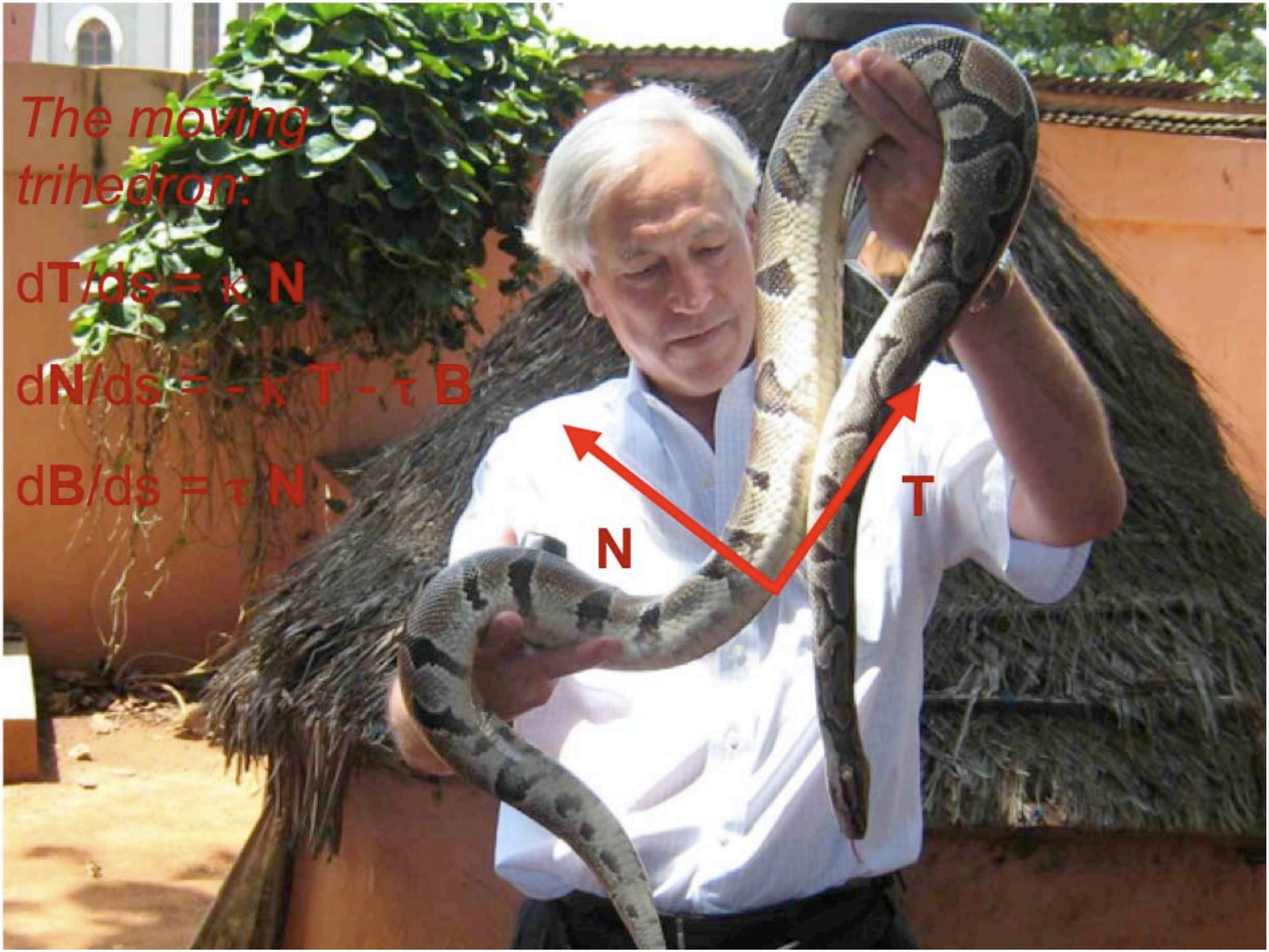
- $$\mathbf{N} = \mathbf{T}' / |\mathbf{T}'|.$$
- Unless the curve is straight at position P, **N** is defined as a unit vector perpendicular to **T**.

Tangent and normal vectors, and arc length.

- If you "parametrize with arc length, what does that mean for **T** and **N**?
  - **T** = dr(s)/ds - No denominator!
  - **N** = **T**′/|**T**′| - You still have to "normalize"
    - Next week we'll use |**T**′| to quantify curvature.



$\hat{T}$
$\hat{T}$
$\hat{I}$
$\hat{I}$


The moving trihedron:
d**T**/ds = κ **N**
d**N**/ds = - κ **T** - τ **B**
d**B**/ds = τ **N**
N
T

Admittedly….

You can really get tangled up in these calculations!

Tangent and normal lines:

- Recall the helix:

$$\vec{r}(t) = \cos(4\pi t)\hat{i} + \sin(4\pi t)\hat{j} + t\hat{k}$$

Velocity

$$\vec{r}\,'(t) = -4\pi\sin(4\pi t)\hat{i} + 4\pi\cos(4\pi t)\hat{j} + \hat{k}$$

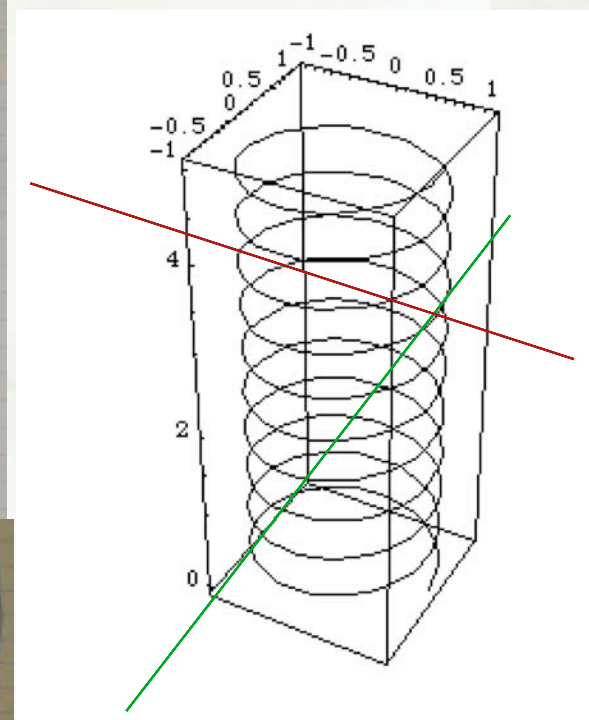
speed

$$\|\vec{r}\,'(t)\| = \sqrt{(4\pi\sin(4\pi t))^2 + (4\pi\cos(4\pi t))^2 + 1}$$

$$= \sqrt{1+16\pi^2}$$

$$\hat{T} = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{1+16\pi^2}} \begin{bmatrix} -4\pi\sin(4\pi t) \\ 4\pi\cos(4\pi t) \\ 1 \end{bmatrix}$$

(writing as column vector.)



Tangent and normal lines:

Ways to describe a line:

slope-intercept $y = m x + b$

2 points, point-slope

These are not so useful in 3-D.

Better:

parametric form: $\mathbf{r}(t) = \mathbf{r}_0 + u\,\mathbf{v}$ (call parameter something other than t)

Tangent and normal lines:

The essential facts about the helix:

$$\mathbf{r}(t) = \cos(4\pi t)\,\mathbf{i} + \sin(4\pi t)\,\mathbf{j} + t\,\mathbf{k}$$

$$\mathbf{T}(t) = (1/(1+16\pi^2)^{1/2})\,(-4\pi \sin(4\pi t)\,\mathbf{i} + 4\pi \cos(4\pi t)\,\mathbf{j} + \mathbf{k})$$

$$\mathbf{N}(t) = -\cos(4\pi t)\,\mathbf{i} - \sin(4\pi t)\,\mathbf{j}$$

Example: Tangent and normal lines at (1,0,1)

Tangent and normal lines:

Example: Tangent and normal lines at (1,0,1)

$\mathbf{r}(t) = (\cos(4\pi t),\sin(4\pi t),t)$

$= (1,0,1)$ when $t = 1$.

$\mathbf{T}(1) = (1/(1+16\pi^2)^{1/2})\ (-4\pi \sin(4\pi)\,\mathbf{i} + 4\pi \cos(4\pi)\,\mathbf{j} + \mathbf{k})$

$= (1/(1+16\pi^2)^{1/2})\ (4\pi\ \mathbf{j} + \mathbf{k})$.

Line: $(1,0,1) + u\ (4\pi\ \mathbf{j} + \mathbf{k})$

Hey! What in &#$ happened to the* $(1/(1+16\pi^2)^{1/2})$?

Tangent and normal lines:

Example: Tangent and normal lines at (1,0,1)

$\mathbf{r}(t) = (\cos(4\pi t), \sin(4\pi t), t)$

$= (1,0,1)$ when $t = 1$.

$\mathbf{N}(1) = -\cos(4\pi)\,\mathbf{i} - \sin(4\pi)\,\mathbf{j}$

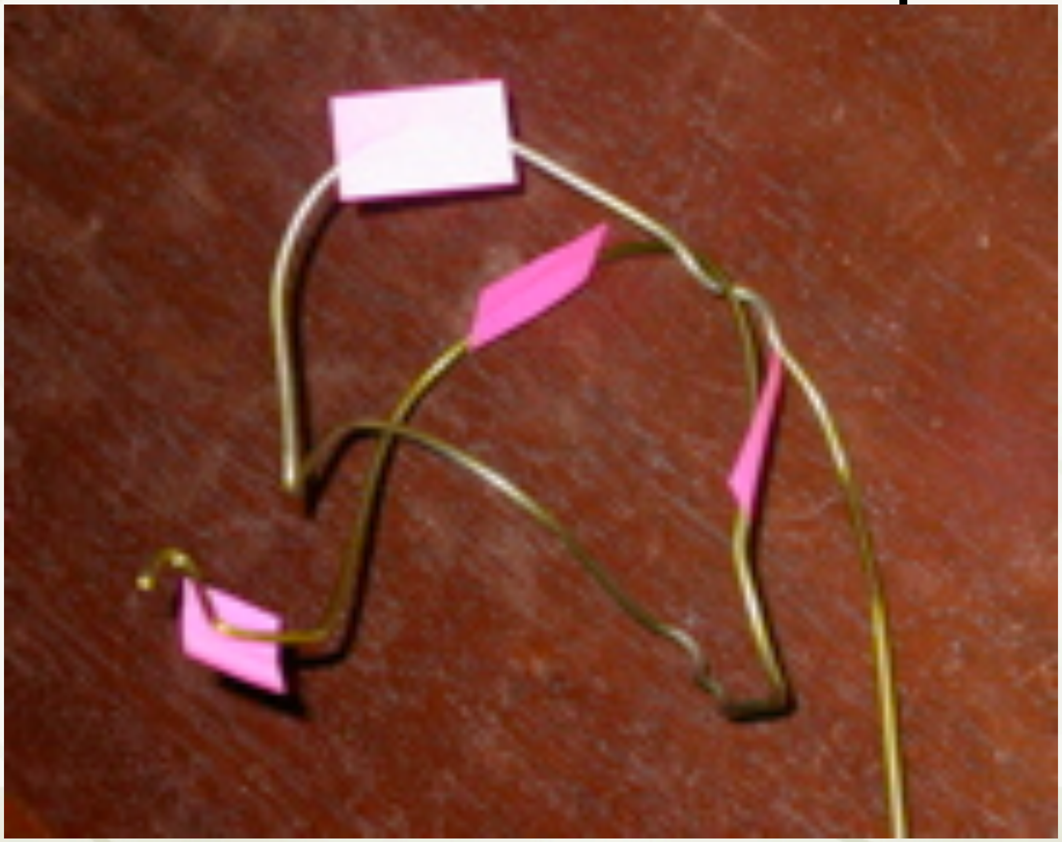
$= -\mathbf{i}$.

Line: $(1,0,1) + u\,\mathbf{i}$.

Wait a minute! What about the sign ?

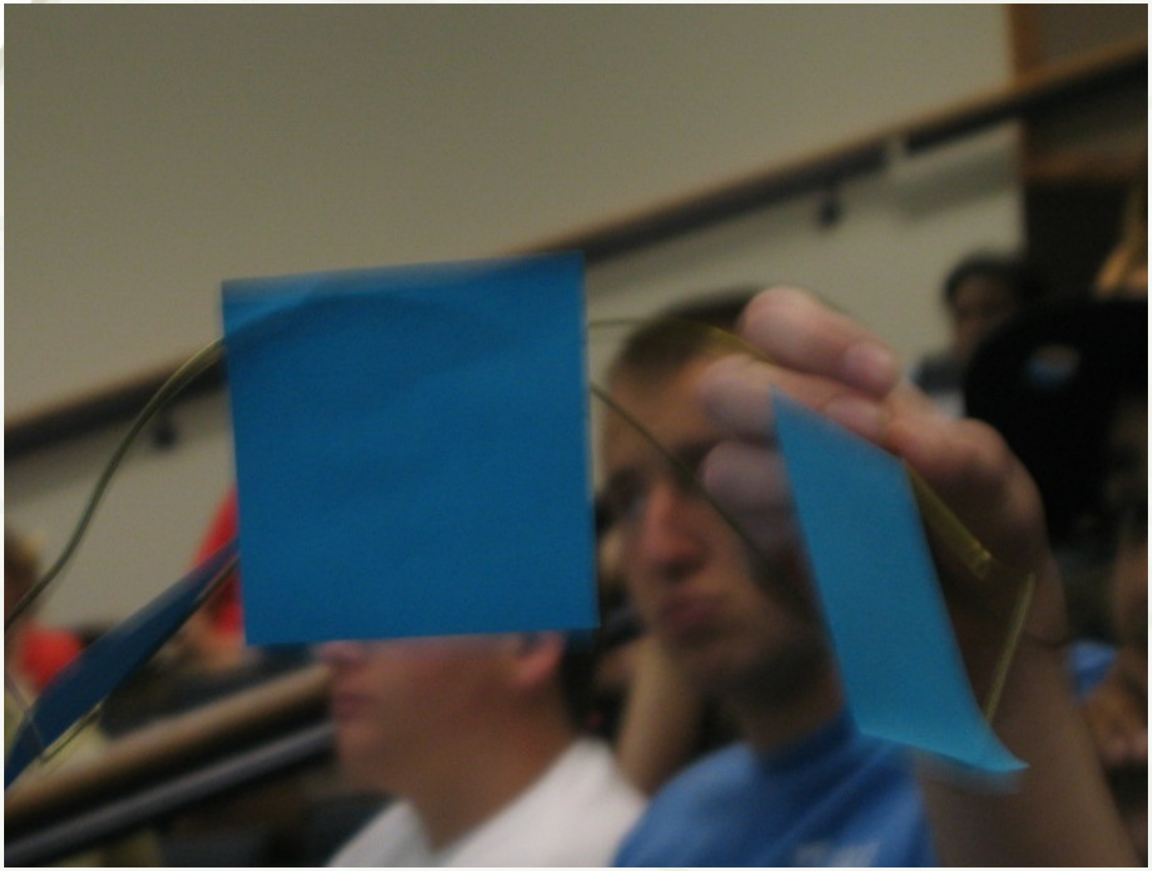
The osculating plane

- Bits of curve have a “best plane.”



stickies on wire.

Each stickie contains **T** *and* **N**.



The osculating plane

- Bits of curve have a “best plane.”
- One exception - a straight line lies in infinitely many planes.

The osculating plane

- What’s the formula, for example for the helix?

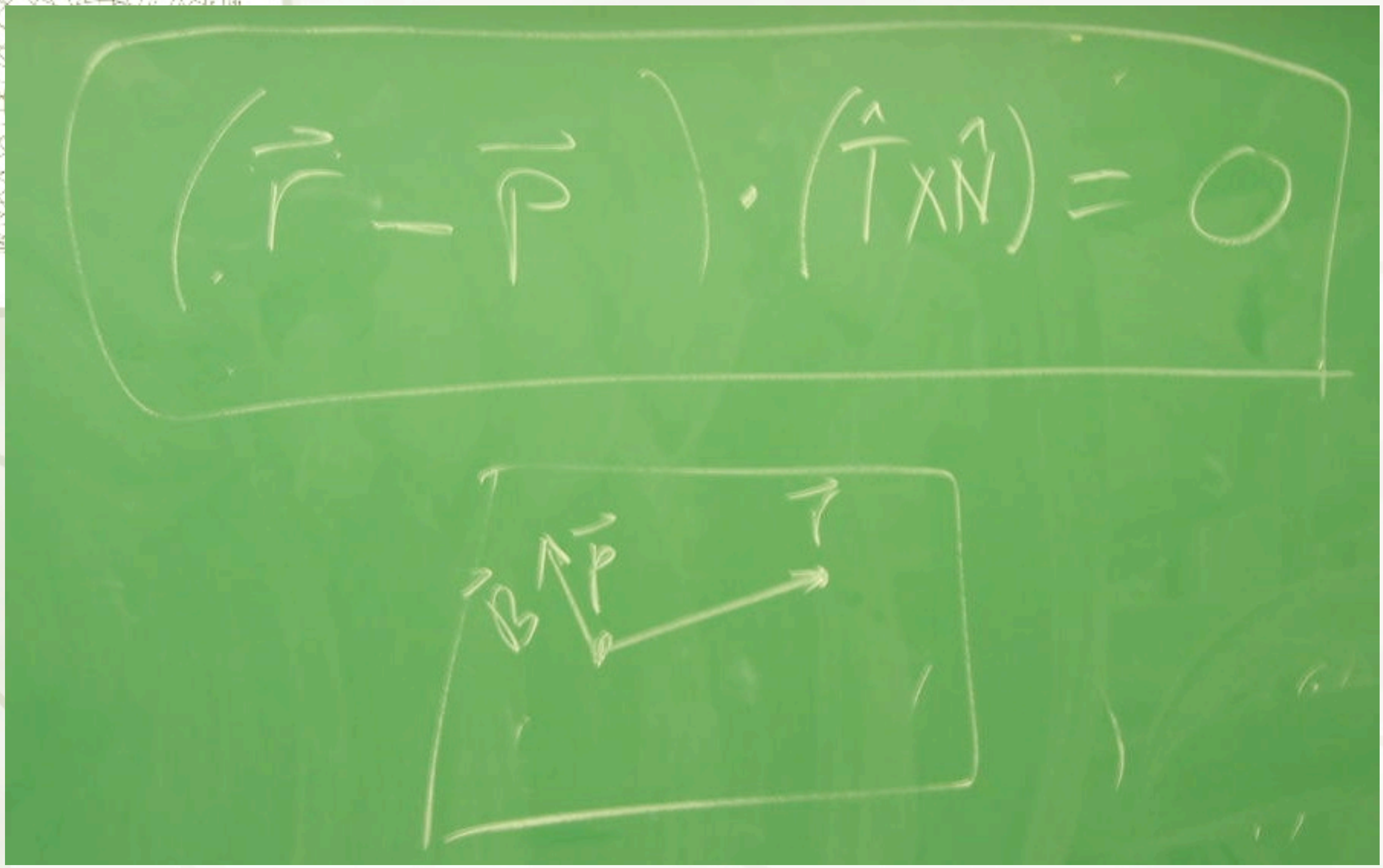
1. Parametric form

2. Single equation

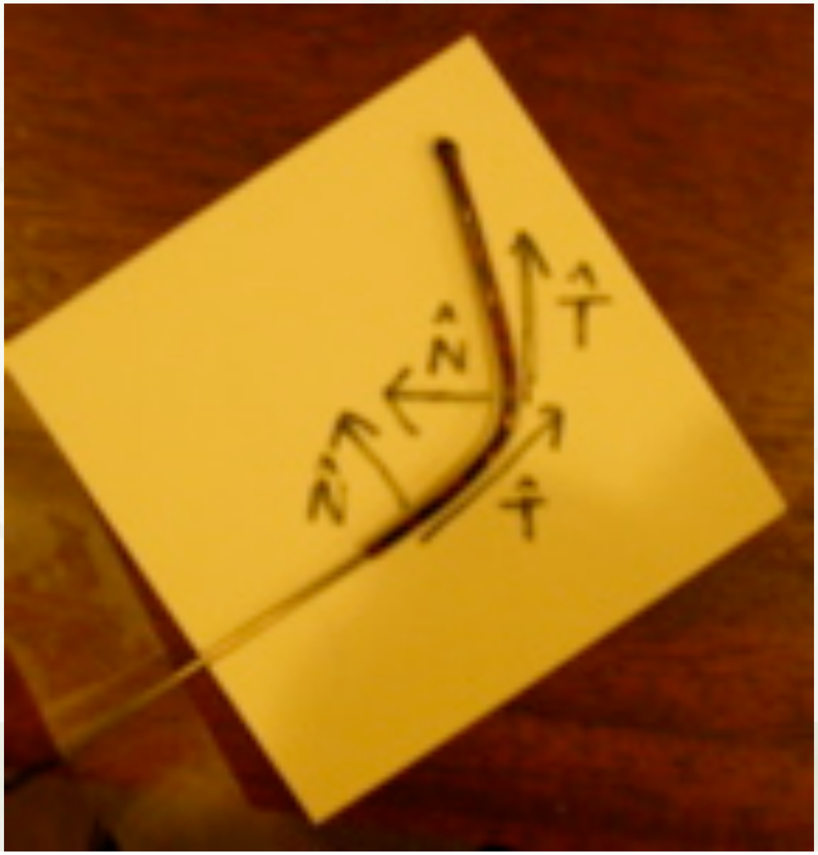
The binormal ***B***

- The normal vector to a plane is *not* the same as the normal to a curve in the plane. It has to be $\perp$ to all the curves and vectors that lie within the plane.
- Since the osculating plane contains **T** and **N**, a normal to the plane is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$


$(\vec{r} - \vec{p}) \cdot (\hat{T} \times \hat{N}) = 0$
$\vec{B}$
$\vec{p}$
$\vec{r}$

Close-up



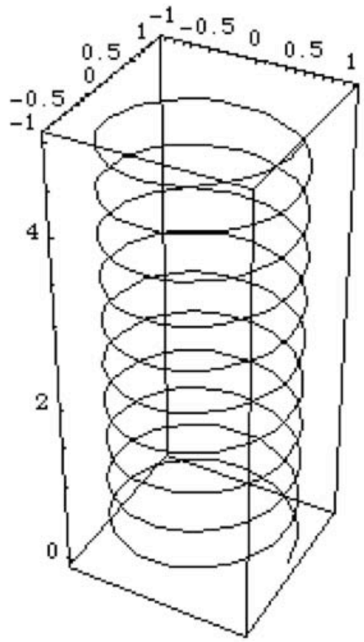
The osculating plane

- What’s the formula, for example for the helix?

1. Parametric form

2. Single equation

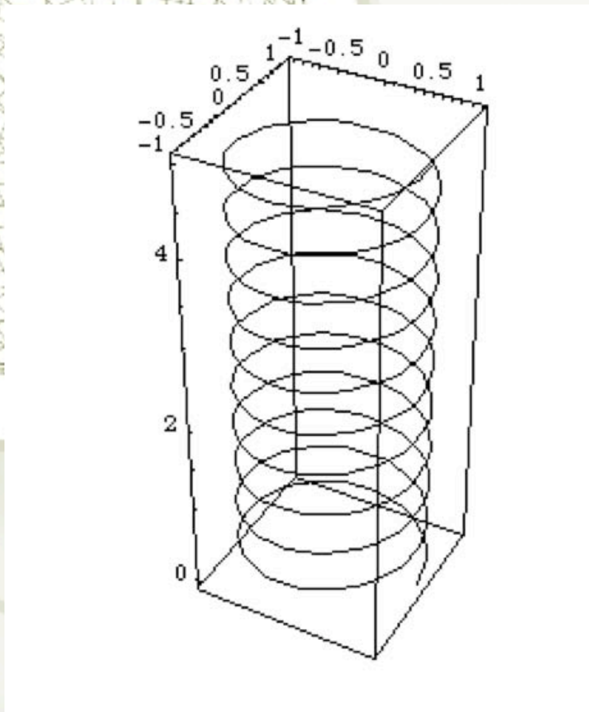
Example: The helix



$\mathbf{r}(t) = \cos(4\pi t)\,\mathbf{i} + \sin(4\pi t)\,\mathbf{j} + t\,\mathbf{k}$

$\mathbf{T}(t) = (-4\pi \sin(4\pi t)\,\mathbf{i} + 4\pi \cos(4\pi t)\,\mathbf{j} + \mathbf{k})/(1+16\pi^2)^{1/2}$

$\mathbf{N}(t) = -\cos(4\pi t)\,\mathbf{i} - \sin(4\pi t)\,\mathbf{j}$



Example: The helix

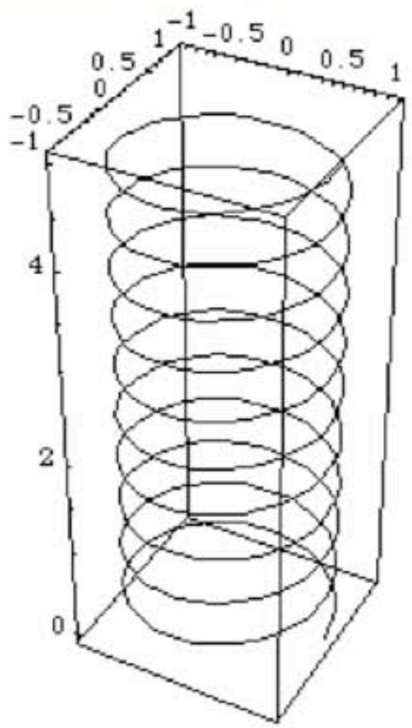
$\mathbf{r}(t) = \cos(4\pi t)\,\mathbf{i} + \sin(4\pi t)\,\mathbf{j} + t\,\mathbf{k}$

$\mathbf{T}(t) \times \mathbf{N}(t) = (-\sin(4\pi t)\,\mathbf{i} + \cos(4\pi t)\,\mathbf{j} - 4\pi\,\mathbf{k})/(1+16\pi^2)^{1/2}$

Osculating plane at (1,0,1): Calculate at t=1.

$(\mathbf{r}_{osc} - (\mathbf{i} + \mathbf{k})) \cdot (1\,\mathbf{j} - 4\pi\,\mathbf{k}) = 0$

(The factor $(1+16\pi^2)^{1/2}$ can be dropped.)



Example: The helix

In coordinates,

$x_{helix}(t) = \cos(4\,\pi\, t)$,

$y_{helix}(t) = \sin(4\,\pi\, t)$

$z_{helix}(t) = t$

And

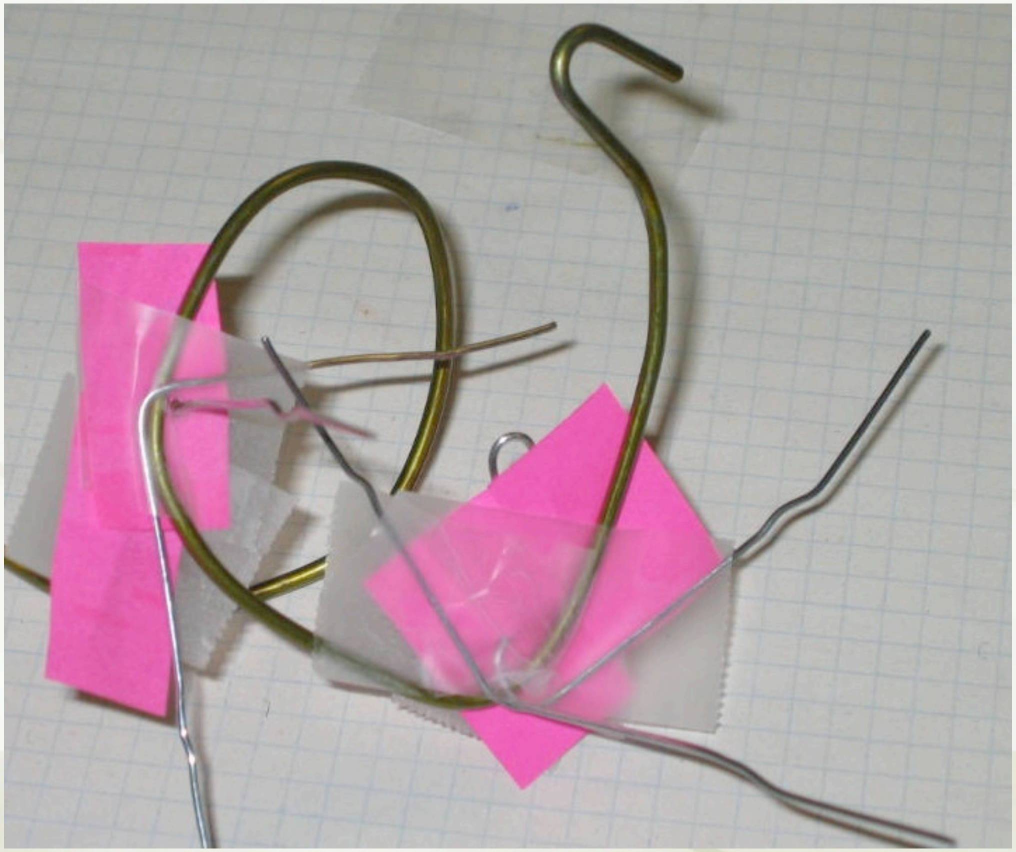
$(x_{osc} - 1)\cdot 0 + (y_{osc} - 0)\cdot 1 + (z_{osc} - 1)\cdot(-4\pi) = 0$,

Which simplifies to:

$$y_{osc} - 4\,\pi\, z_{osc} = -\,4\,\pi$$

The moving trihedron

- The curve's preferred coordinate system is oriented along (**T**,**N**,**B**), *not* some Cartesian system (**i**,**j**,**k**) in the sky.



The moving trihedron

- A vehicle can rotate around any of these axes. A rotation around **T** is known as *roll*. If the vehicle has wings (or a hull) it may prefer a second direction over **N**. For example, the wing direction may correlate with **N** when the airplane turns without raising or lowering the nose. Such an acceleration is called *yaw*.

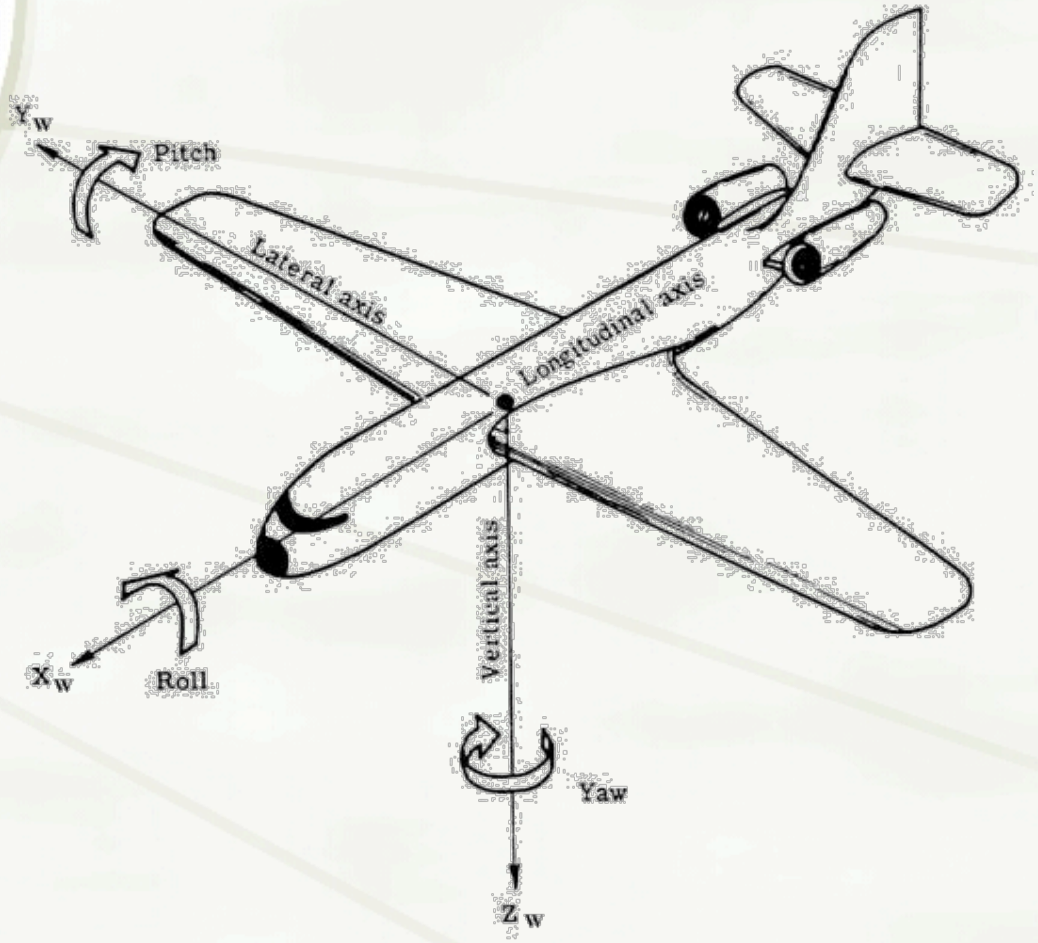
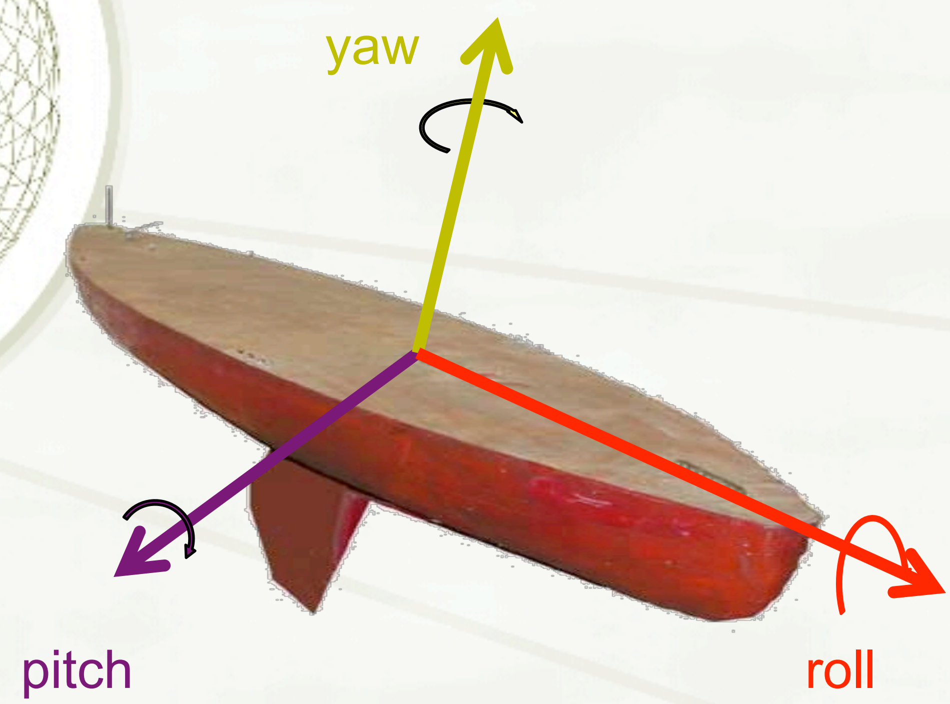


Figure from JPL/NASA

The moving trihedron

- However, when the aircraft soars or dives (this kind of acceleration is called *pitch*), the normal vector **N** is perpendicular to the wing axis, which in this case correlates with the binormal **B**.

- An aircraft can accelerate, roll, yaw, and pitch all at once. *Fasten your seatbelt!*

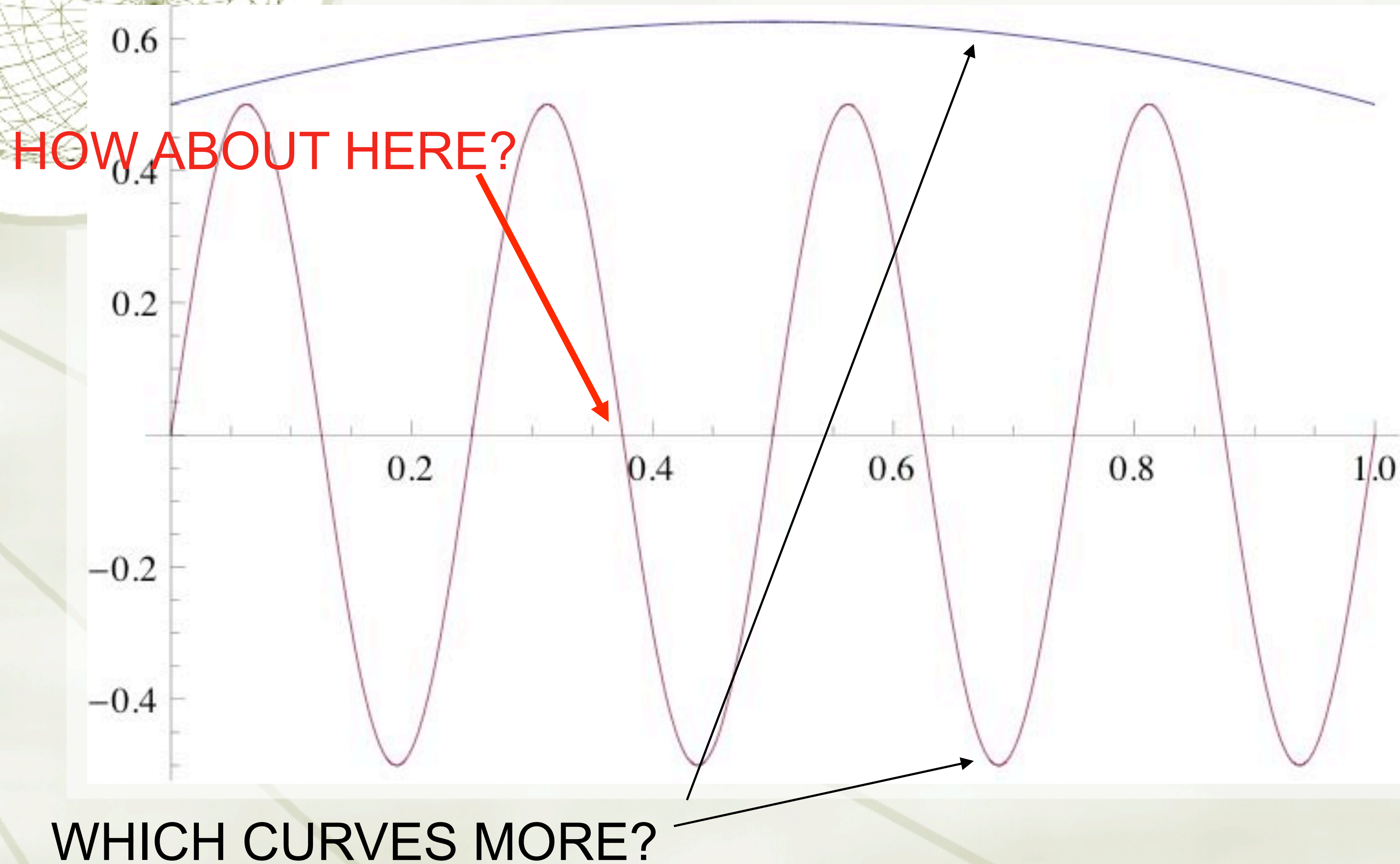


Watercraft have the same kinds of accelerations as aircraft. The rudder controls yaw. The boat is usually designed to minimize pitch and roll.

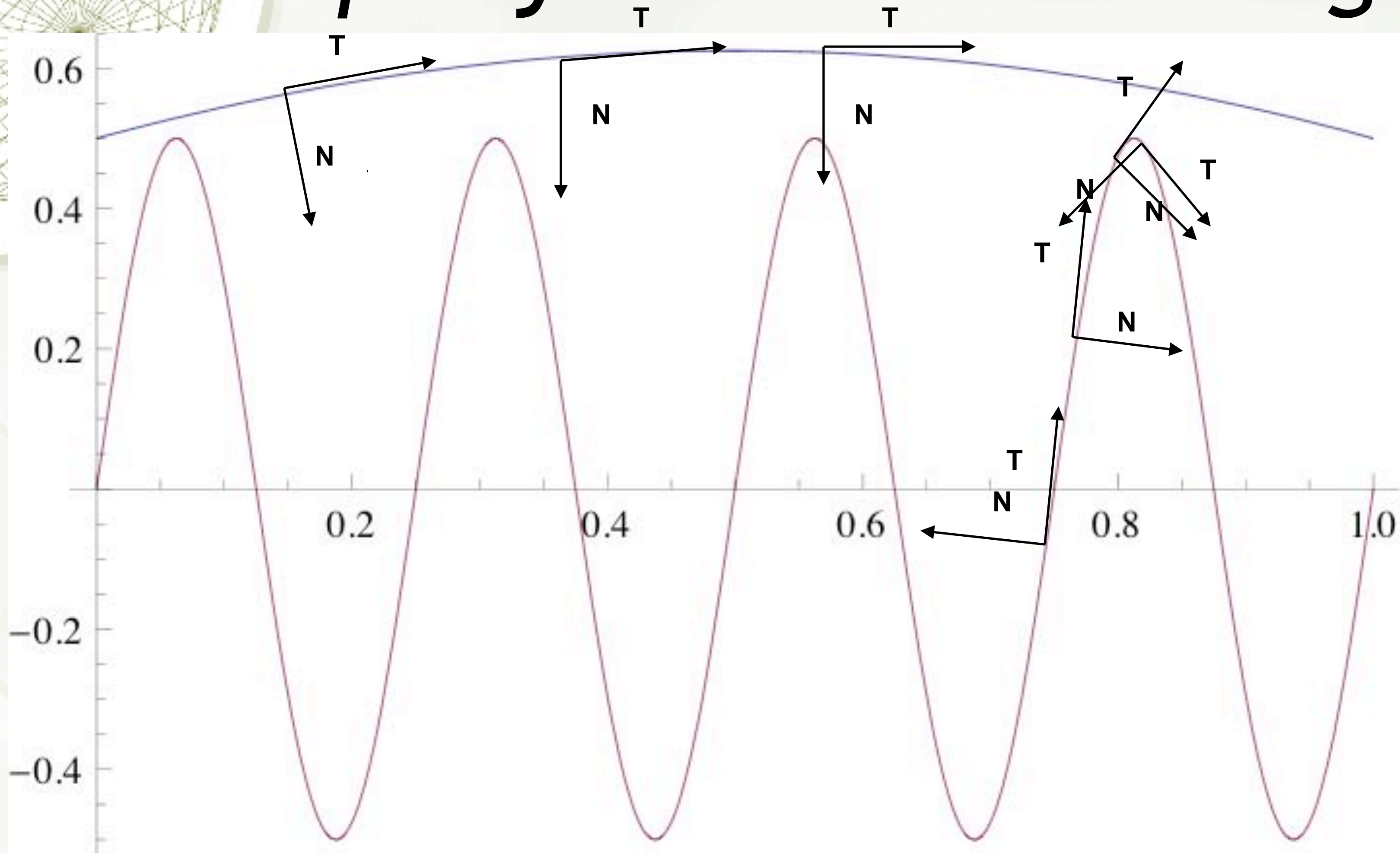
Just what is *curvature?*

- How do you know a curve is curving? And how much?
- The answer should depend just on the shape of the curve, not on the speed at which it is drawn. *So it connects with arclength s, not with a time-parameter t.*

Just what is *curvature?*



How rapidly do T and N change?



Just what *is* curvature?

- And let's be quantitative about it!
  - 2D: How about $|d\phi/ds|$, where ϕ is the direction of **T** with respect to the x-axis?
  - To get started, notice that the direction of **T** is the same as that of the tangent line. That is,

$$\tan \phi = dy/dx = (dy/ds)/(dx/ds)$$

(fasten seatbelts for the next slide!)

A tricky calculation of $K = \frac{d\phi}{ds}$

$$\tan\phi = \frac{dy}{dx} = (dy/ds)\big/(dx/ds)$$

$$\boxed{\frac{d}{ds}\tan\phi = \frac{y''x' - y'x''}{(x')^2}}$$ by quotient rule

$$= \sec^2\phi \frac{d\phi}{ds} = (1+\tan^2\phi)\frac{d\phi}{ds}$$

$$= \left(1 + \left(y'/x'\right)^2\right) \underbrace{d\phi/ds}_{K} = \frac{\overset{=1}{\cancel{(x')^2+(y')^2}}}{(x')^2}$$

K (at least up to $\pm$)

Solving:

$$K = \frac{y''x' - y'x''}{\underset{=1}{\cancel{(x')^2+(y')^2}}}$$

It's our old friend the chain rule, used in a creative way!

Different expressions for κ

- $\kappa = |d\phi/ds|$
- $\kappa = |(d\phi/dt)/(ds/dt)|$
- $\kappa = |x'(s)\, y''(s) - y'(s)\, x''(s)|$
- $\kappa = \dfrac{|x'(t)\, y''(t) - y'(t)\, x''(t)|}{|(x'(t))^2 + (y'(t))^2|^{3/2}}$

Huh??

Example

- Circle of radius 5.
  - *No calculus needed!*
  - *If you move distance Δs along the perimeter, the change in angle is $\Delta s/5$. So $\Delta\phi/\Delta s = 1/5$.* The general rule for a circle is that the curvature is the reciprocal of the radius.

Example

- Spiral: The formula for curvature is complicated, but the spiral is simple, so the curvature should be simple.
- Still, we'll be lazy and use Mathematica:

Example

```
In[5]:= Speed[r_] := Sqrt[D[r[[1]], t]^2 + D[r[[2]], t]^2]

In[22]:= NumeratorOfCurvature[r_] :=
          D[r[[2]], {t, 2}] D[r[[1]], t] - D[r[[1]], {t, 2}] D[r[[2]], t]

         Curvature[r_] := D[r[[2]], {t, 2}] D[r[[1]], t] -
           D[r[[1]], {t, 2}] D[r[[2]], t] /Speed[r]^3

In[11]:= Spiral := {t Cos[t], t Sin[t]}

In[12]:= Speed[Spiral]
```

Out[12]= $\sqrt{(t\,\mathrm{Cos}[t] + \mathrm{Sin}[t])^2 + (\mathrm{Cos}[t] - t\,\mathrm{Sin}[t])^2}$

```
In[19]:= Curvature[Spiral]
```

Out[19]= $(\mathrm{Cos}[t] - t\,\mathrm{Sin}[t])\,(2\,\mathrm{Cos}[t] - t\,\mathrm{Sin}[t]) - \dfrac{(-t\,\mathrm{Cos}[t] - 2\,\mathrm{Sin}[t])\,(t\,\mathrm{Cos}[t] + \mathrm{Sin}[t])}{\left((t\,\mathrm{Cos}[t] + \mathrm{Sin}[t])^2 + (\mathrm{Cos}[t] - t\,\mathrm{Sin}[t])^2\right)^{3/2}}$

```
In[23]:= Simplify[NumeratorOfCurvature[Spiral]]
```

Out[23]= $2 + t^2$

```
In[24]:= Simplify[Speed[Spiral]]
```

Out[24]= $\sqrt{1 + t^2}$

```
In[25]:= %% / %^3
```

Out[25]= $\dfrac{2 + t^2}{(1 + t^2)^{3/2}}$

The moving trihedron

A spaceship doesn't see a big Cartesian grid in the sky. Looked at from the inside, a better basis for vectors will use the unit tangent **T**, *the principal normal* **N**., *and the binormal* **B**.

Dimensional analysis

- What units do you use to measure curvature?

Dimensional analysis

- What units do you use to measure curvature?
  - Hint: angles are considered dimensionless, since radian measure is a ratio of arclength (cm) to radius (also cm)

Dimensional analysis

- What units do you use to measure curvature?
  - Answer: 1/distance, for instance 1/cm.

$1/\kappa$ is known as the *radius of curvature*. It's the radius of the circle that best matches the curve at a given contact point.