MATH 2401 - Harrell

Curves from the inside

Lecture 5

Who in the cast of characters might show up on the test?

- Curves **r**(t), velocity **v**(t).
- Tangent and normal lines.
- Angles at which curves cross.
- **T**,**N**,**B**.
- The arc length s.
- The curvature κ.
- The osculating plane.

In our previous episode:

1. *Tangent and normal lines.*
2. *Curvature as the rate the direction of **T** changes.*
3. *The best plane ("osculating" plane).*
4. *A spaceship doesn't see a big Cartesian grid in the sky. Looked at from the inside, a better basis for vectors will use the unit tangent **T**, the principal normal **N**., and the binormal **B**.*

Different 2D expressions for κ

- $\kappa = |d\phi/ds|$
- $\kappa = |(d\phi/dt)/(ds/dt)|$
- $\kappa = |x'(s)\, y''(s) - y'(s)\, x''(s)|$
- $\kappa = \dfrac{|x'(t)\, y''(t) - y'(t)\, x''(t)|}{|(x'(t))^2 + (y'(t))^2|^{3/2}}$

Huh??

Example

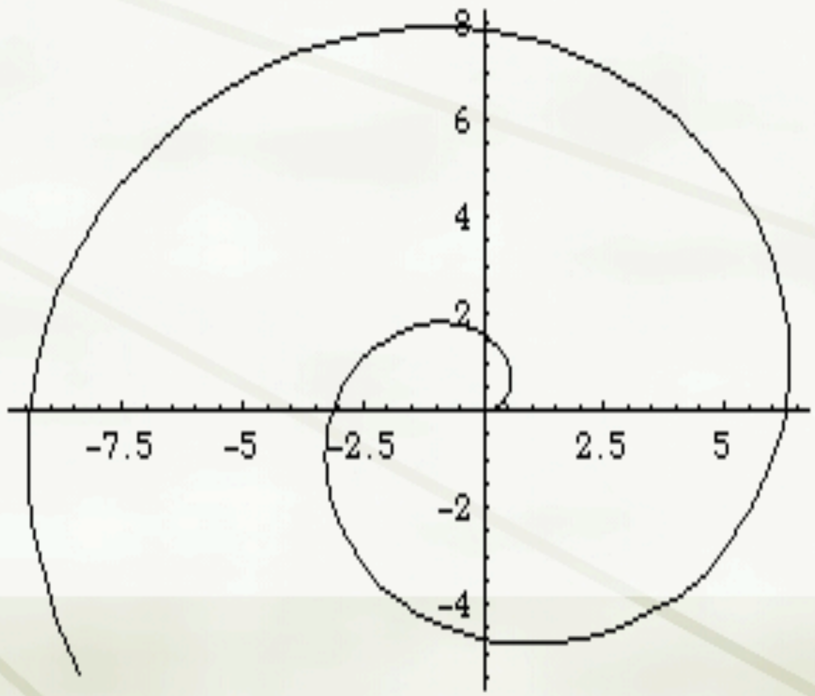
- Spiral: The formula for curvature is complicated, but the spiral is simple, so the curvature should be simple.
- Still, we'll be lazy and use Mathematica:

Example: spiral

```
Spiral[t_] := {t Cos[t], t Sin[t]}

Spiral3D[t_] := {t Cos[t], t Sin[t], 0}

ParametricPlot[Spiral[t], {t, 0, 10}]
```



Example

```
In[5]:= Speed[r_] := Sqrt[D[r[[1]], t]^2 + D[r[[2]], t]^2]

In[22]:= NumeratorOfCurvature[r_] :=
          D[r[[2]], {t, 2}] D[r[[1]], t] - D[r[[1]], {t, 2}] D[r[[2]], t]

         Curvature[r_] := D[r[[2]], {t, 2}] D[r[[1]], t] -
           D[r[[1]], {t, 2}] D[r[[2]], t] / Speed[r]^3

In[11]:= Spiral := {t Cos[t], t Sin[t]}

In[12]:= Speed[Spiral]
```

Out[12]= $\sqrt{(t\,\text{Cos}[t] + \text{Sin}[t])^2 + (\text{Cos}[t] - t\,\text{Sin}[t])^2}$

```
In[19]:= Curvature[Spiral]
```

Out[19]= $(\text{Cos}[t] - t\,\text{Sin}[t])\,(2\,\text{Cos}[t] - t\,\text{Sin}[t]) - \dfrac{(-t\,\text{Cos}[t] - 2\,\text{Sin}[t])\,(t\,\text{Cos}[t] + \text{Sin}[t])}{\left((t\,\text{Cos}[t] + \text{Sin}[t])^2 + (\text{Cos}[t] - t\,\text{Sin}[t])^2\right)^{3/2}}$

```
In[23]:= Simplify[NumeratorOfCurvature[Spiral]]
```

Out[23]= $2 + t^2$

```
In[24]:= Simplify[Speed[Spiral]]
```

Out[24]= $\sqrt{1 + t^2}$

```
In[25]:= %% / % ^ 3
```

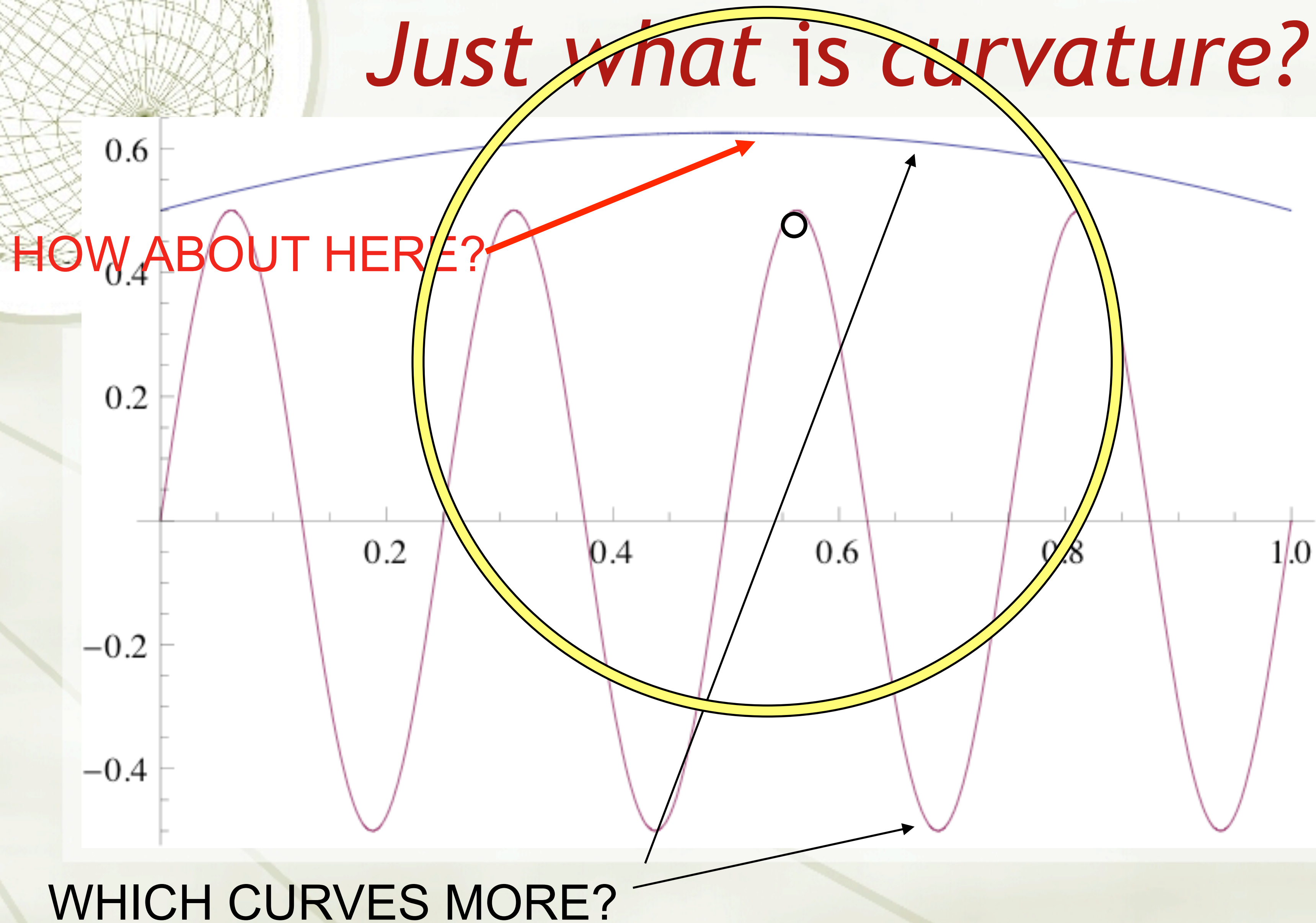
Out[25]= $\dfrac{2 + t^2}{(1 + t^2)^{3/2}}$

Dimensional analysis

- What units do you use to measure curvature?
  - Answer: 1/distance, for instance 1/cm.

1/κ is known as the *radius of curvature*. It's the radius of the circle that best matches the curve at a given contact point.

Just what is curvature?



3 D

- It seems as though 3D would be more complicated, but there is a sneaky mathematician trick: *Write what you know about a special case without referring explicitly to what makes it special*. The angle is special to 2D. Vectors **r**, **T**, **N**, and the arclength s are not.

2D or 3D

In 2D,

$$\hat{T} = \begin{bmatrix} \cos\phi(s) \\ \sin\phi(s) \end{bmatrix} \qquad \hat{N} = \pm \begin{bmatrix} -\sin\phi(s) \\ \cos\phi(s) \end{bmatrix}$$

Also

$$\frac{d\hat{T}}{ds} = \phi'(s) \begin{bmatrix} -\sin\phi(s) \\ \cos\phi(s) \end{bmatrix}, \text{ so}$$

$$\boxed{K = |\phi'(s)| = \left\| \frac{d\hat{T}}{ds} \right\|}$$

This formula does not refer in any way to two dimensions!

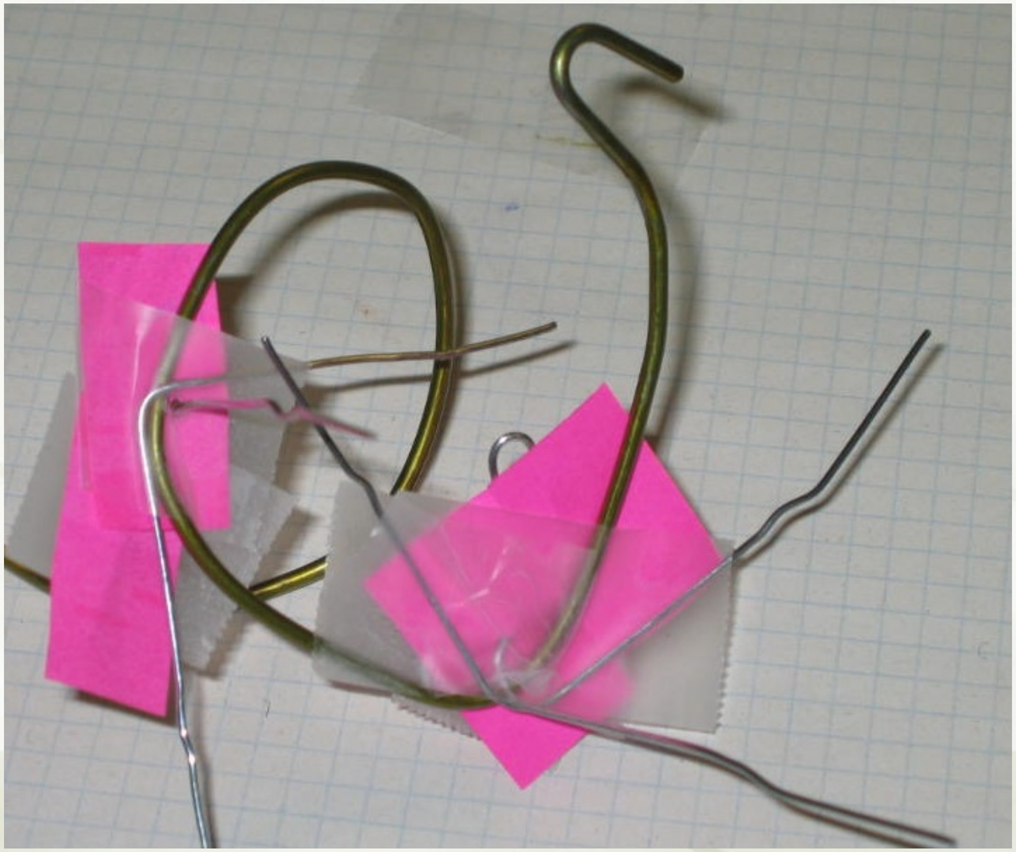
2D or 3D

- Another way to write this formula is

$$*** \quad d\mathbf{T}/ds = \kappa \, \mathbf{N} \quad ***$$

The moving trihedron

- The curve's preferred coordinate system is oriented along (**T**,**N**,**B**), *not* some Cartesian system (**i**,**j**,**k**) in the sky.



The trihedron of unit vectors

1. $\mathbf{T}(t) = \mathbf{r}'(t)/\ |\mathbf{r}'(t)|$ or just

.... $d\mathbf{r}/ds$.

- Because of the chain rule, since the speed $|\mathbf{r}'(t)|$ is ds/dt.

The trihedron of unit vectors

1. **T**(s) = d**r**/ds.

2. **N**(s) = (d**T**/ds)/κ, where κ = |d**T**/ds| is our definition of the curvature in 3D.

3. **B**(s) = **T**(s) $\times$ **N**(s).

The *trihedron* (**T**,**N**,**B**) is the basis for 3-space that the curve cares about.

The osculating plane

The equations for motion on a curve – “from the inside”

To keep life simple, we first work things out when moving at constant speed 1. Velocity is then a unit vector, our friend **T**:

$$\mathbf{v}(s) = \mathbf{T}(s).$$

The length of **T** is fixed, but not its direction.

The curve equations of Frenet and Serret

- The first of these is
  - $d\mathbf{T}/ds = \kappa\ \mathbf{N}$
- So… what is $d\mathbf{N}/ds$?

If $\vec{V} = a\hat{T} + b\hat{N} + c\hat{B}$,

Then $a = \vec{V}\cdot\hat{T}$, $b = \vec{V}\cdot\hat{N}$, $c = \vec{V}\cdot\hat{B}$

As for $\frac{d\hat{N}}{ds}$, what is its $\hat{N}$ component?

$\frac{d\hat{N}}{ds}\cdot\hat{N} =$ 0, because **N • N** = 1.

What is

$\frac{d\hat{N}}{ds}\cdot\hat{T} =$ ______________

If $\vec{V} = a\hat{T} + b\hat{N} + c\hat{B}$,

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What is

$\frac{d\hat{N}}{ds}\cdot\hat{T} =$ ______________

Well, **N** • **T** = 0, so **N′** • **T**.+ **N** • **T′** = 0.

Therefore **N′** • **T**.= - **N** • **T′** = - **N** • κ**N′** = - κ.

N′ = _- κ_ **T** + ___0___ **N** + ___τ___ **B** .

The curve equations of Frenet and Serret

- The first of these is
  - d**T**/ds = κ **N**
- Next:
  - d**N**/ds = -κ **T** - τ **B**
  - *What does torsion tell us?*

The curve equations of Frenet and Serret

- The first of these is
  - $d\mathbf{T}/ds = \kappa\,\mathbf{N}$
- Next:
  - $d\mathbf{N}/ds = -\kappa\,\mathbf{T} - \tau\,\mathbf{B}$
- Finally,
  - $d\mathbf{B}/ds = \tau\,\mathbf{N}$

Motion in 3 D

- Remember that a curve's favorite coordinate system is based on the *moving trihedron* (**T**,**N**,**B**).
- *What happens to a moving particle in this moving frame*?
  - velocity
  - acceleration

Motion in 3 D

- **v** = |**v**| **T** + 0 **N** + 0 **B**.
  - *Therefore* **v** = |**v**| **T** + 0 **N** + 0 **B**
- So… what's the acceleration in the local frame?

Motion in 3 D

A physicist or engineer would probably write

$s(t) :=$ distance along the curve

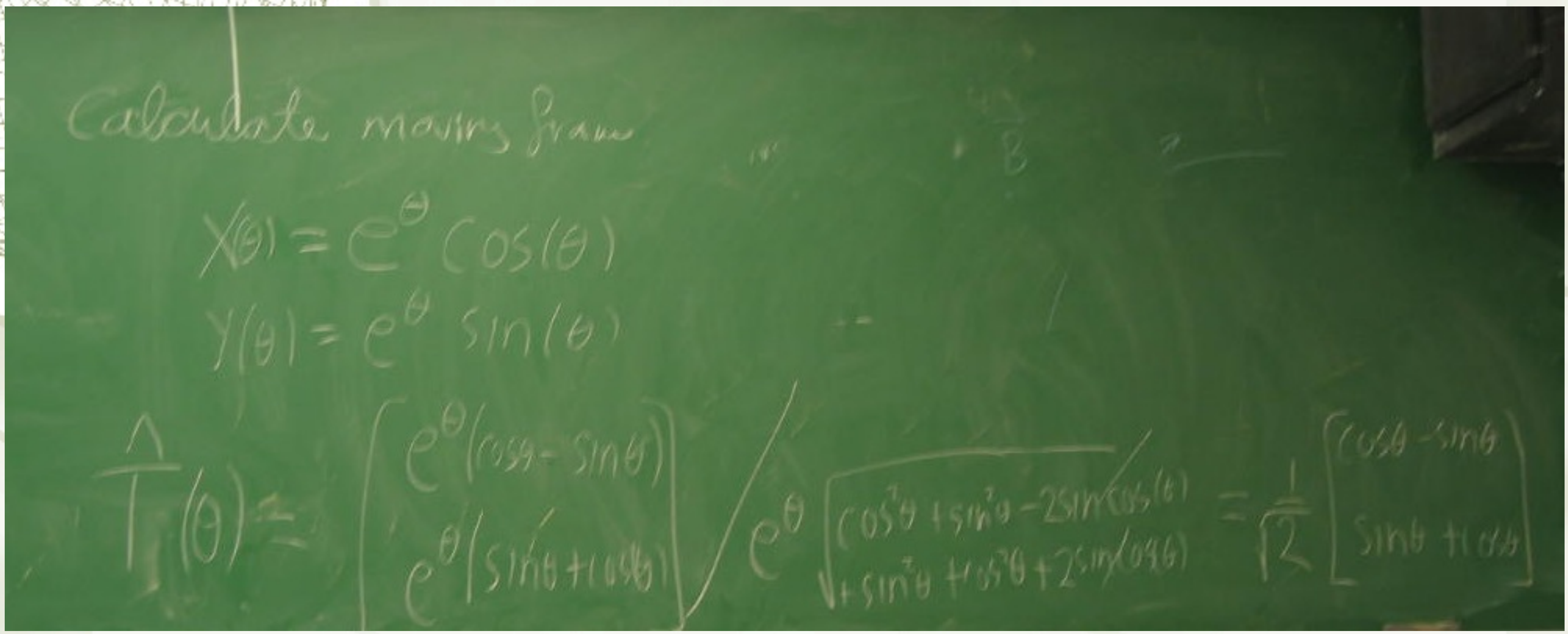
$\dot{s}(t) =$ speed. Dot $= \frac{d}{dt}$.

$$\vec{V}(t) = \dot{s}(t)\,\hat{T}(t)$$

$$\vec{a} = \frac{dV}{dt} = \ddot{s}\,\hat{T} + \dot{s}\,\frac{d}{dt}\hat{T}(t) \quad \left(\text{product rule}\right)$$

$$= \ddot{s}\,\hat{T} + \dot{s}\,\frac{ds}{dt}\frac{d}{ds}\hat{T} \quad \left(\text{chain rule}\right)$$

$$\boxed{\vec{a} = \ddot{\vec{r}} = \ddot{s}\,\hat{T} + \dot{s}^2 \kappa \hat{N}} + 0\,\mathbf{B}$$



Notation: The thing on the right is a column vector. We could as easily have written

$(\cos\theta - \sin\theta)\,\mathbf{i} + (\sin\theta + \cos\theta)\,\mathbf{j}$

To get κ, $\hat{N}$, we need

$$\frac{d\hat{T}}{ds} = \frac{d\theta}{ds}\frac{d\hat{T}}{d\theta} = \underbrace{\frac{d\theta}{ds}}_{\kappa}\ \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} -\sin\theta - \cos\theta \\ \cos\theta - \sin\theta \end{bmatrix}}_{\hat{N}}$$

We also need to calculate

$$\frac{ds}{d\theta} = \|V\| = \left\| e^{\theta}\begin{bmatrix} \cos\theta - \sin\theta \\ \sin\theta + \cos\theta \end{bmatrix}\right\| = \sqrt{2}\,e^{\theta}$$

So $\kappa = \frac{1}{\sqrt{2}e^{\theta}} = \frac{e^{-\theta}}{\sqrt{2}}.$

WARNING! It was somewhat coincidental that $\frac{d\theta}{ds} = \frac{d\phi}{ds}$, θ being the polar angle and ϕ being the angle of $\hat{T}(\theta)$. We knew $\frac{d\theta}{ds} = \kappa$ only because $\left\|\frac{1}{\sqrt{2}}\begin{bmatrix} -\sin\theta - \cos\theta \\ \cos\theta - \sin\theta \end{bmatrix}\right\| = 1.$

Selected applications of vector calculus to physics

Angular momentum,

$$\mathbf{L} := \mathbf{r} \times \mathbf{p} = \mathbf{r} \times \mathrm{m}\, \mathbf{r}'.$$

How does this change in time? (This is called the *torque*.)

Selected applications of vector calculus to physics

Angular momentum,

$\mathbf{L} := \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\,\mathbf{r}'$.

How does this change in time? (This is called the *torque*.)

$d\mathbf{L}/dt = \mathbf{r} \times m\,\mathbf{r}'' = \mathbf{r} \times \mathbf{f}$ (The other term given by the chain rule is 0 because cross prod of vector with self.)

Selected applications of vector calculus to physics

Magnetic motion,

F := (q/c) **v** × $\mathcal{B}$, *Lorentz force law*.

Funny font because the magnetic field is not the same as the binormal. Suppose for now that $\mathcal{B}$ is a constant vector.

Selected applications to physics

Magnetic motion,

$$\mathbf{F} = (q/c)\, \mathbf{v} \times \mathcal{B}$$

Suppose for now that $\mathcal{B}$ is a constant vector.

$$\mathbf{r}'' = (q/cm)\, \mathbf{r}' \times \mathcal{B}$$

Selected applications to physics

But if $\mathbf{r}'' = (q/cm)\, \mathbf{r}' \times \mathcal{B}$ and the initial velocity

$\mathbf{r}'(t)$ happens to be perpendicular to $\mathcal{B}$,

then $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ will both remain perpendicular to the magnetic field $\mathcal{B}$. *The entire trajectory is therefore in a plane perpendicular to* $\mathcal{B}$, which must be parallel to **B** after all! Moreover, $\mathbf{r}''$ and $\mathbf{r}'$ are perpendicular, so $||\mathbf{r}'||$ is constant, as we have seen. The velocities must be of the form

$$\mathbf{r}'(t) = A \cos(qt/cm - \phi)\, \mathbf{i} \pm A \sin(qt/cm - \phi)\, \mathbf{j}$$

Selected applications to physics

Finally, by integrating,

$$\mathbf{r}(t) - \mathbf{c} = A\sin(qt/cm - \phi)\,\mathbf{i} \mp A\cos(qt/cm - \phi)\,\mathbf{j}$$

….. which is a circle.