MATH 2401 - Harrell

Partial to partials

Lecture 8

Recollections of last week

- Cool shapes,
  like Möbius strips

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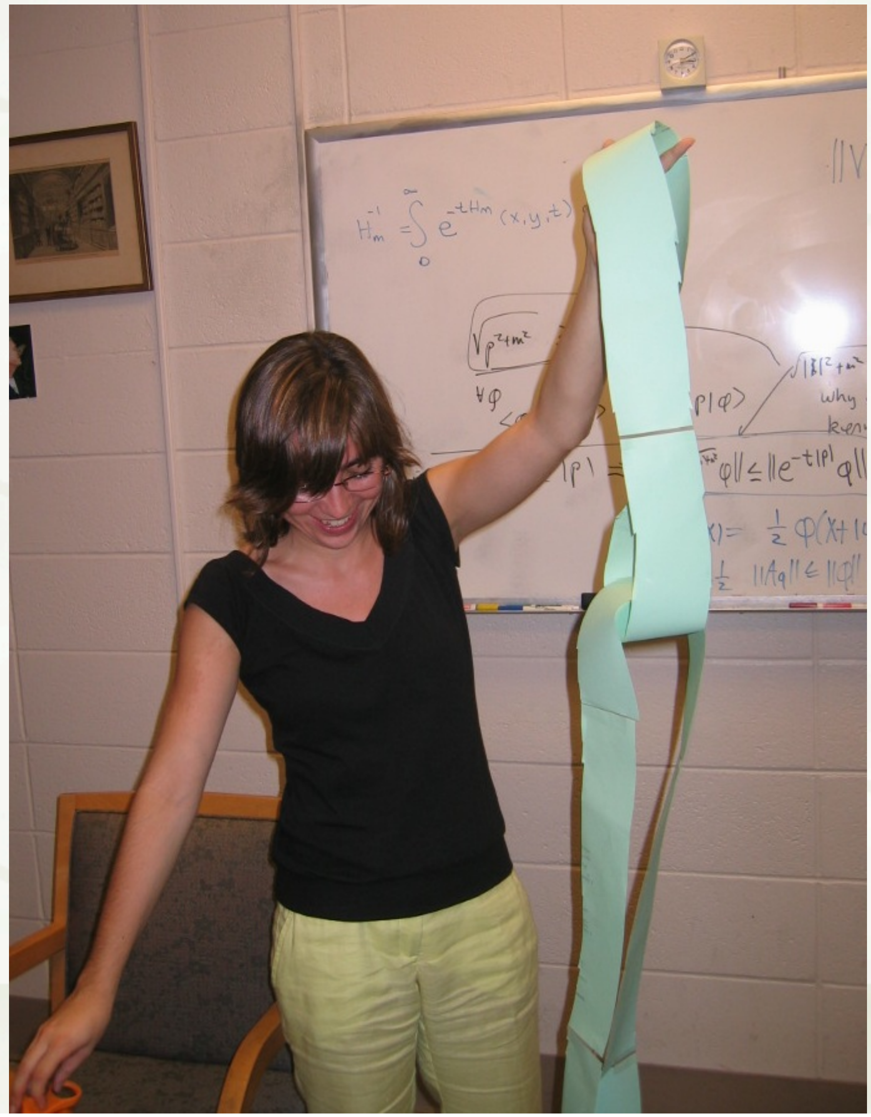
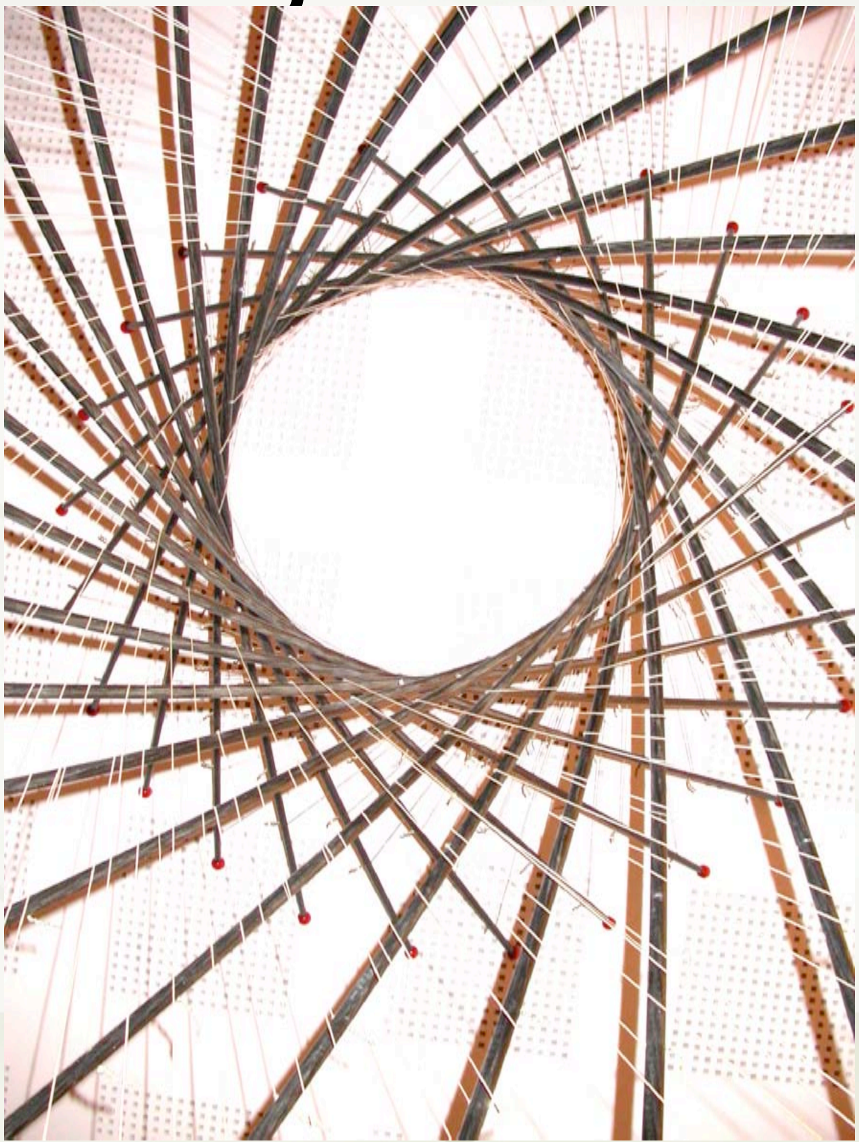


Recollections of last week

- I can really get in to some of these shapes!

Recollections of last week

- The view from inside the hyperboloid.



What is the derivative, really?

- In KG calculus, it was the slope of the tangent line at a point.
- For a vector function (same as curve), it still gave the direction of the tangent line at a point.
- *It tells us how to straighten out something that is curved, with the least error.*

Extrapolating with the ideal gas law

Ideal gas law. $PV = nRT$, so

$$V(P,T) = nRT/P \ .$$

If we fix P and change T to $T+ \Delta T$,

$$V(P,T+ \Delta T) \cong V(P,T) + (nR/P)\ \Delta T$$

$$\cong V(P,T)\ (1 + \Delta T/T)$$

Extrapolating with the ideal gas law

Ideal gas law. $PV = nRT$, so

$$V(P,T) = nRT/P\ .$$

If we fix T and change P to $P + \Delta P$,

$$V(P + \Delta P, T) \cong V(P,T) - (nRT/P^2)\,\Delta P$$

$$\cong V(P,T)\,(1 - \Delta P/P)\ .$$

Extrapolating with the ideal gas law

Ideal gas law. $PV = nRT$, so

$$V(P,T) = nRT/P\ .$$

What if both T and P change? Estimate the volume. We might guess

$$V(P + \Delta P, T + \Delta T) \cong V(P,T)\,(1 + \Delta T/T - \Delta P/P).$$

Is this right?

The partial derivative

Just pretend y is a constant and differentiate with respect to x. Call this $\partial F/\partial x$.

If you pretend x is a constant and differentiate with respect to y, that's $\partial F/\partial y$.

Anatomy of the partial derivative

We write $F_x(x, y)$ or $\frac{\partial F}{\partial x}$

"F sub x" "Dee F by dee x"

Why do we calculate partials?

1 Sometimes only interested in one variable. Example: If we only care how the concentration c(x,y) varies when we move in the x-direction, we want $\partial c/\partial x$. This function still depends on y as well as x.

Why do we calculate partials?

1. Sometimes only interested in one variable.
2. Nature *loves* partial derivatives:
   a. Heat equation

$$u(t,x), \qquad \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

   b. Wave equation

$$u(t,x), \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

   c. Potential equation

$$u(x,y,z), \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{4\pi\rho(x,y,z)}{\epsilon_0}$$

Why do we calculate partials?

1. Sometimes only interested in one variable.
2. Nature *loves* partial derivatives:
3. All the Calculus I stuff (max-min, slopes, tangent *planes* rather than lines) will use partial derivatives when there is more than one variable.

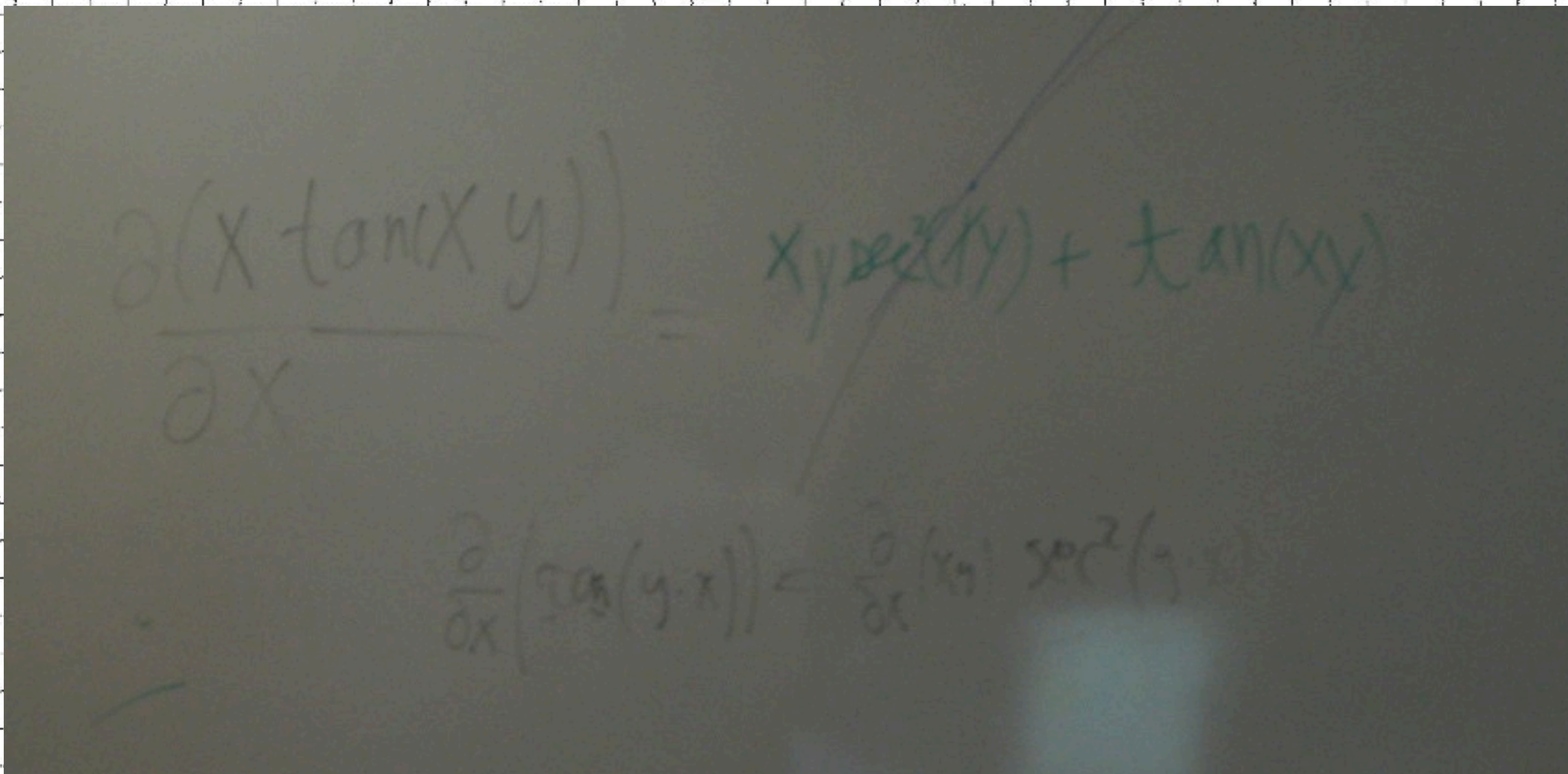
The partial derivative

Important! $\partial F/\partial x$ and $\partial F/\partial y$ are still functions of 2 variables. Let's do an example or two.

Examples of partial derivatives

(Let's play Stump the !)

Examples of partial derivatives



Sets

(something mathematicians are obsessed with)

- Neighborhood of a point
- Interior, boundary
- Open
- Closed
- Neither open nor closed

Sets

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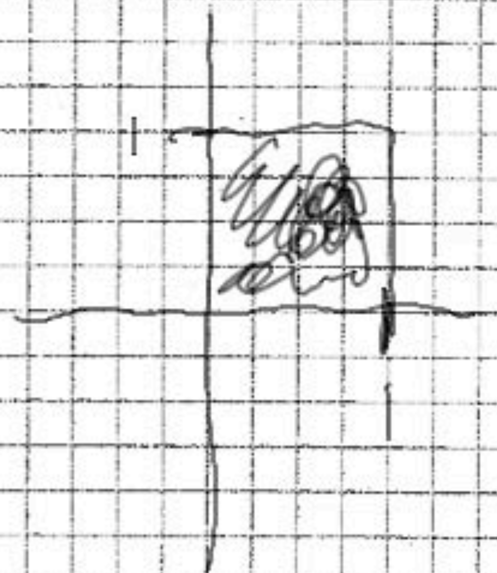
$$\{\vec{x} : |\vec{x} - \vec{x}_0| < \delta\}$$

How about some examples?



$\{(x,y): 0 \le x \le 1, 0 \le y \le 1\}$

"closed" U-Square

$(.72, .99999990)$

How about some examples?

S

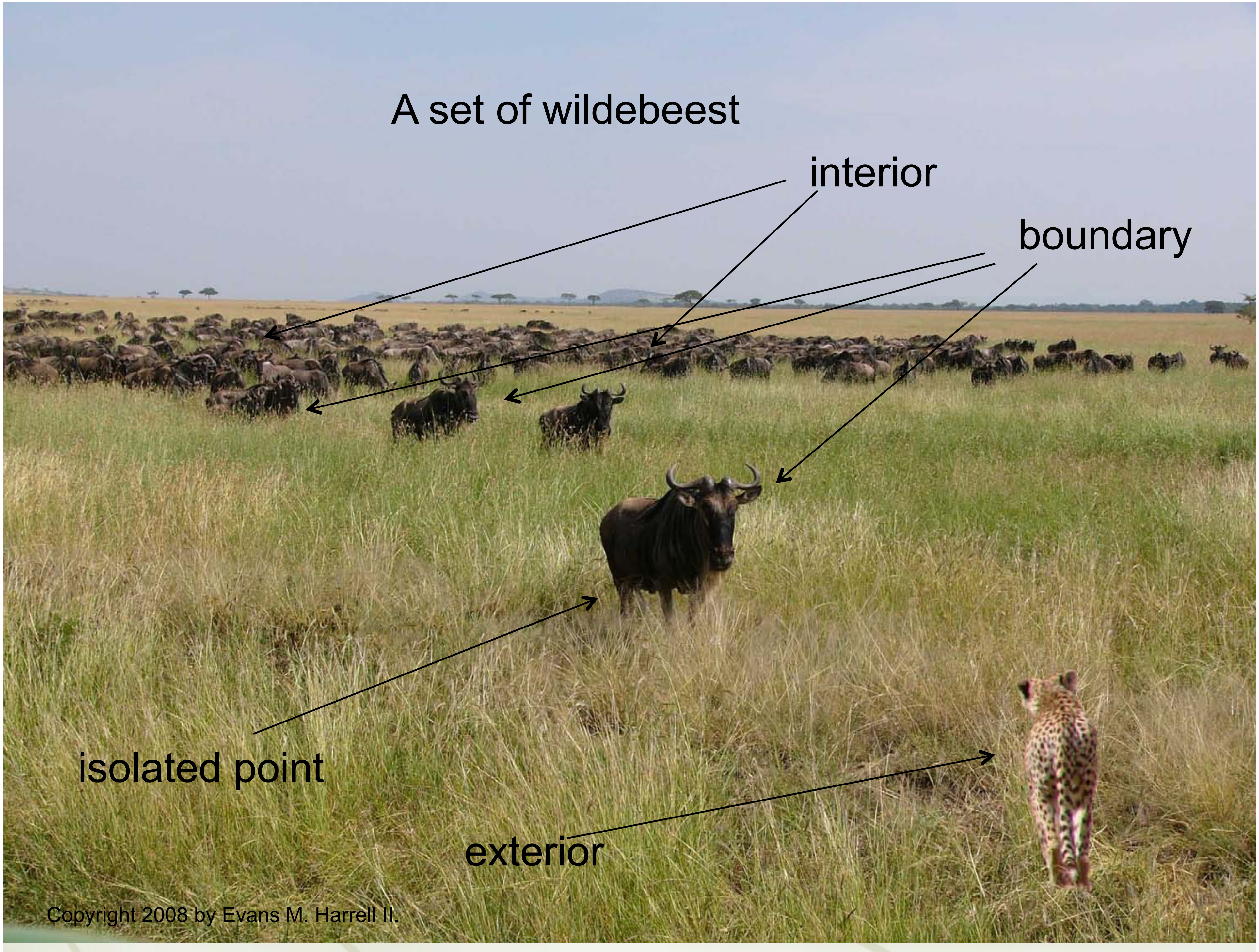
$\{(x,y): 0 \le x \le 1, 0 \le y \le 1\}$

"closed" u-square

$(.72, .99999990)$

∂S "bdry of S"

$= \{(x,y) = (0,y); 0 \le y \le 1\} \cup \{(x,0), 0 \le x \le 1\}$ $\cup$ 2 more


A set of wildebeest
interior
boundary
isolated point
exterior

Limits for scalar fields

$$\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r}) = L$$

$$\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{L}$$

Compare and contrast.

- Curves
  - scalar (time) in, vector (position) out
- Scalar fields
  - vector (position) in, scalar out

Limits for scalar fields

$$\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r}) = L$$

What does it mean?

Limits for scalar fields

nbd

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $|\vec{r} - \vec{r}_0| < \delta$, then $|f(r) - L| < \varepsilon$

$$\lim_{\mathbf{r} \to \mathbf{r}_0} f(\mathbf{r}) = L$$

$$\lim_{t \to t_0} \mathbf{F}(t) = \mathbf{L}$$

Compare and contrast.

This kind of limit can depend on *how* you get where you are going.

Example: What is

$$\lim_{\mathbf{r}\to\mathbf{0}} \frac{x^2 - y^2}{x^2 + y^2} \; ?$$

Try this. First set y = 0 and let x → 0. You get the limit 1.

But if you first set x = 0 and let y → 0. You get the limit -1.

Weird. We don't consider this function continuous.

Second partial derivatives

1 Since $f_x(x,y)$ and $f_y(x,y)$ still depend on both variables, it makes sense to calculate

$f_{xx}(x,y)$, $f_{yy}(x,y)$, $f_{xy}(x,y)$, $f_{yx}(x,y)$

Anatomy of the *second* partial derivative

$$\text{We write } F_{xy}(x,y) \text{ or } \frac{\partial^2 F}{\partial y \partial x}$$

Isn't there something funny about the x-y order?

$$(F_x)_y(x,y) \text{ or } \frac{\partial}{\partial y}\frac{\partial F}{\partial x}$$

Examples

$$F(x,y) = e^x \cos(\pi y) \Rightarrow F_x = e^x \cos(\pi y) \Rightarrow F_{xx} = e^x \cos(\pi y)$$

$$F_{xy} = -\pi e^x \sin(\pi y)$$

$$F_y = -\pi e^x \sin \pi y \Rightarrow F_{yx} = -\pi e^x \sin(\pi y)$$

$$F_{yy} = -\pi^2 e^x \cos(\pi y)$$

Same!

Examples

$$G(x,y) = \ln(x^2+y^2) \Rightarrow G_x = \frac{2x}{x^2+y^2} \Rightarrow G_{xx} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

$$G_{xy} = \frac{-4xy}{(x^2+y^2)^2}$$

$$G_y = \frac{2y}{x^2+y^2} \Rightarrow G_{yx} = \frac{-4xy}{(x^2+y^2)^2}$$

$$G_{yy} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

A strange example

```
In[1]:= F[x_, y_] := x y (x^2 - y^2) / (x^2 + y^2)

In[14]:= D[F[x, y], x]
```

Out[14]= $$-\frac{2 x^2 y (x^2 - y^2)}{(x^2 + y^2)^2} + \frac{2 x^2 y}{x^2 + y^2} + \frac{y (x^2 - y^2)}{x^2 + y^2}$$

```
In[15]:= Limit[%, x → 0]
```

Out[15]= $-y$

```
In[16]:= D[%, y]
```

Out[16]= -1

Therefore $F_{xy}[0,0] = -1$. Meanwhile, Out[14]= $-\frac{2 x^2 y (x^2 - y^2)}{(x^2 + y^2)^2} + \frac{2 x^2 y}{x^2 + y^2} + \frac{y (x^2 - y^2)}{x^2 + y^2}$

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In[15]:= Limit[%, x → 0]
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Out[15]= -y

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Out[16]= -1

Therefore $F_{xy}[0,0] = -1$. Meanwhile,

```
In[17]:= D[F[x, y], y]
```

Out[17]= $-\frac{2 x y^2 (x^2 - y^2)}{(x^2 + y^2)^2} - \frac{2 x y^2}{x^2 + y^2} + \frac{x (x^2 - y^2)}{x^2 + y^2}$

```
In[18]:= Limit[%, y → 0]
```

Out[18]= x

```
In[19]:= D[%, x]
```

Out[19]= 1

Therefore $F_{yx}[0,0] = +1$.

If we stay away from $x = y = 0$, then

In[20]:= `D[F[x, y], x, y]`

Out[20]= $$\frac{8x^2y^2(x^2-y^2)}{(x^2+y^2)^3} - \frac{2x^2(x^2-y^2)}{(x^2+y^2)^2} - \frac{2y^2(x^2-y^2)}{(x^2+y^2)^2} + \frac{2x^2}{x^2+y^2} - \frac{2y^2}{x^2+y^2} + \frac{x^2-y^2}{x^2+y^2}$$

In[21]:= `D[F[x, y], y, x]`

Out[21]= $$\frac{8x^2y^2(x^2-y^2)}{(x^2+y^2)^3} - \frac{2x^2(x^2-y^2)}{(x^2+y^2)^2} - \frac{2y^2(x^2-y^2)}{(x^2+y^2)^2} + \frac{2x^2}{x^2+y^2} - \frac{2y^2}{x^2+y^2} + \frac{x^2-y^2}{x^2+y^2}$$

which are the same, as usual. What went wrong at the origin?