MATH 2401 - Harrell

Which way is up?

Lecture 9

A set does not necessarily have boundaries

If it does have boundaries…

Are the boundaries part of the set or not?

Sets
(something mathematicians are obsessed with)

- Neighborhood of a point
- Interior, boundary
- Open – *every point is interior*
- Closed - *contains all of its boundary*
- Neither open nor closed

Open, closed, or neither?

- Closed rectangle:
  - {(x,y) : 0 ≤ x ≤ 1, 0 ≤ y ≤ 2}
- Open rectangle:
  - {(x,y) : 0 < x < 1, 0 < y < 2}
- Neither one:
  - {(x,y) : 0 ≤ x < 1, 0 ≤ y ≤ 2}

Does Connecticut have a highest point?

- At http://geology.com/state-high-points.shtml it states that Connecticut's highest point is "Mt. Frissell--S slope, CT "
- But the top of Mt. Frissell is in Massachusetts!

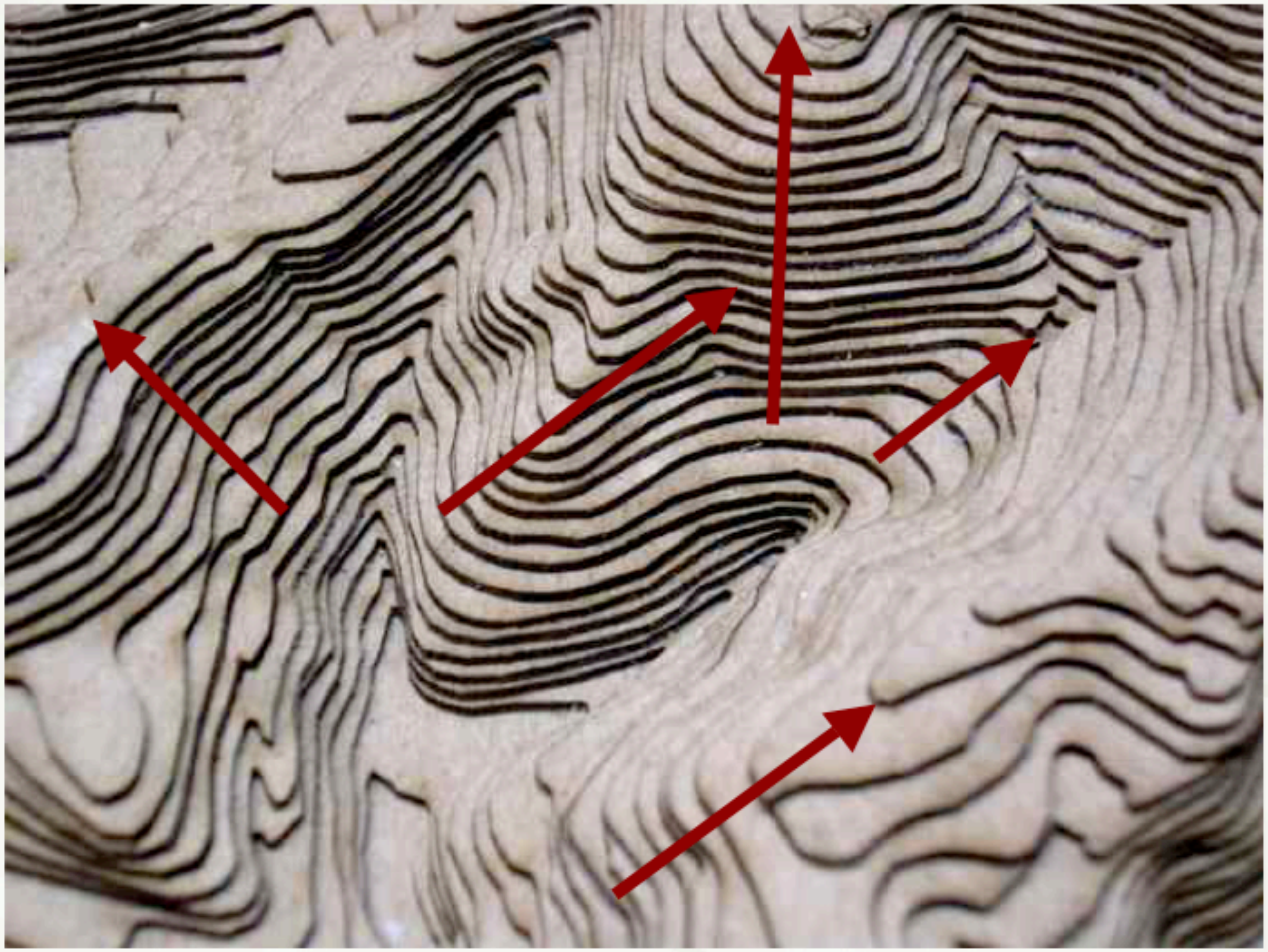
Who owns the boundary – CT or MA?

Does Connecticut have a highest point?

- If MA owns the boundary, from any point in Connecticut you can get higher without leaving the state.

There is no highest point.

Point uphill



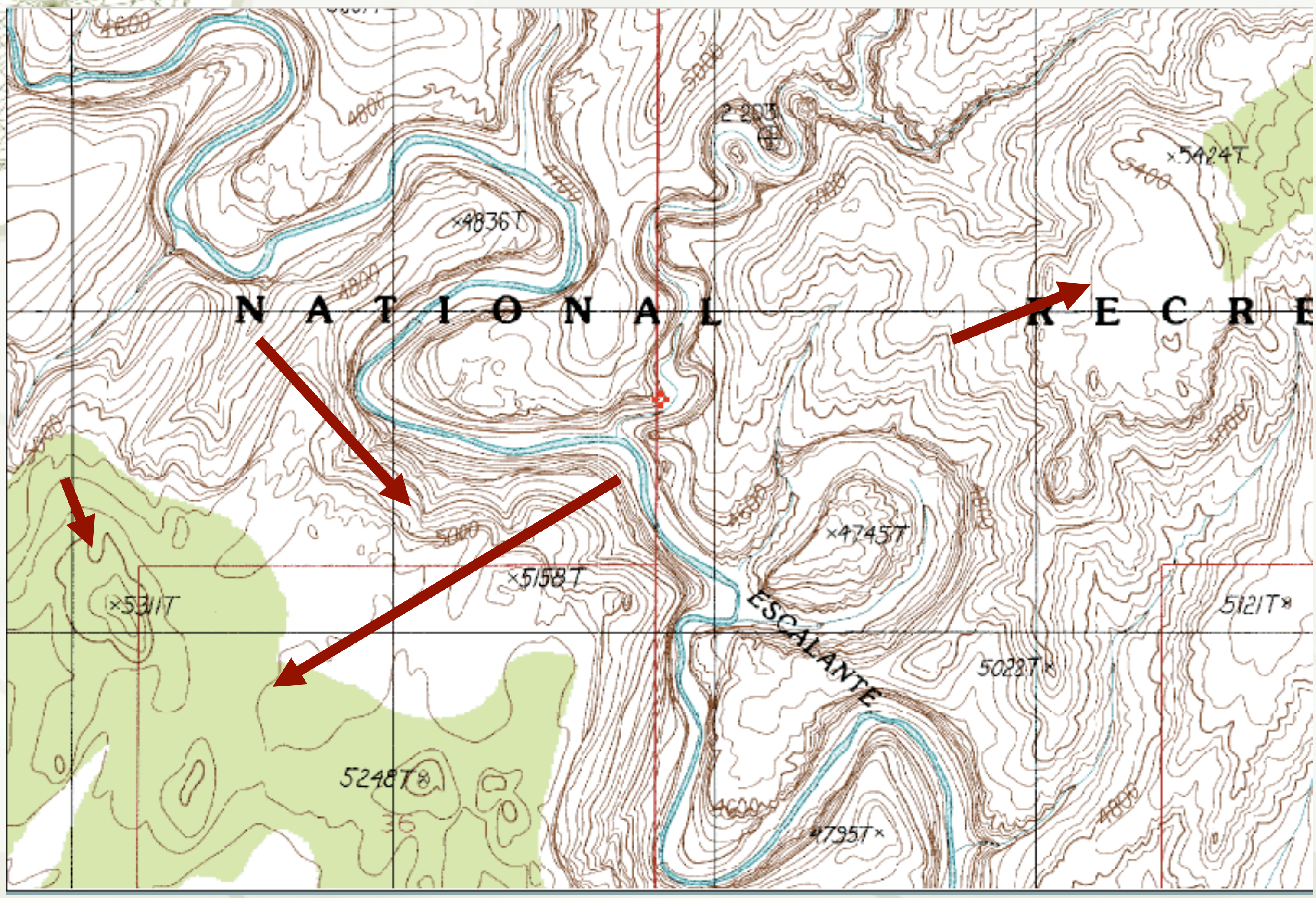
Our convention is that the slope vector applies to the point at its base, not its tip.

An important theorem

- A continuous function on a closed and bounded set *always* has a maximum value and a minimum value.

- Might or might not be true if the set is unbounded or not closed.

Point uphill



How to determine the gradient from a topographic map

- At a point on the Escalante map, we can move uphill by ___ contours, separated by 40 feet, over a distance _______ mile,
  - rise = ___*40 = ___
  - run = _________
  - slope = _______

How to determine the gradient from a topographic map

- At a point on the Escalante map, we can move uphill by 4½ contours, separated by 40 feet, over a distance 1/10 ~~mile~~, 500'
  - rise = 4½*40 = 180'
  - run = 500
  - slope = .36

"Uphill" as a vector

- Direction
- Magnitude = the slope in that direction, e.g., 5% grade.
- If it happens that "uphill" is along the positive x-axis, then the slope is

$$\partial f/\partial x.$$

Directional derivative

- Choose a direction, by taking a unit vector **u**.
- Measure rise/run over a small distance in the direction **u**.

Directional derivative

- What is the d.d. if the vector **u** is *tangent* to a level curve?
- What is the d.d. if the vector **u** is *normal (perpendicular)* to a level curve?
- Of all directions **u**, which one gives the greatest d.d.?

Partial derivatives as a vector

- In 2 dimensions, we have 2 partials,

  $\partial F/\partial x$ and $\partial F/\partial y$

- In 3 dimensions we have 3

  $\partial F/\partial x$, $\partial F/\partial y$, and $\partial F/\partial z$

- Same number as components of a vector, *hmmmmm*.

True vectorial differentiability

You can't divide by a vector, so you can't make sense of

$$\frac{f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})}{\mathbf{h}}$$

True vectorial differentiability

But you *could* hope that the function can be “linearized” in a neighborhood of **x**:

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \mathbf{y}\cdot\mathbf{h} + o(\mathbf{h})$$

Differentiation as a *three -dimensional* concept

- Since you can't divide by a vector,
- We say f is differentiable iff

  $$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \mathbf{y}\bullet\mathbf{h} + o(\mathbf{h})$$

  for some vector **y** (which will depend on **x**).
- This is like the 1-D formula of a tangent line,

  $$f(x+h) - f(x) = f'(x)\, h + o(h)$$

Differentiation as a *three -dimensional* concept

- Since you can't divide by a vector,
- We say f is differentiable iff

  $f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \mathbf{y} \cdot \mathbf{h} + o(\mathbf{h})$,
- **y** is called the *gradient* of f at x, and denoted $\nabla f(\mathbf{x})$.
- *The gradient* $\nabla f(\mathbf{x})$ *of a scalar function is a vector-valued function of a vector variable!*

- ∇
- “grad”
- “del”
- “nabla”

Some good news:

- It is easy to calculate the gradient.
- With the gradient you can easily calculate directional derivatives.
- “Uphill” is nothing other than the gradient!
- Tangent planes are also not hard to work out.

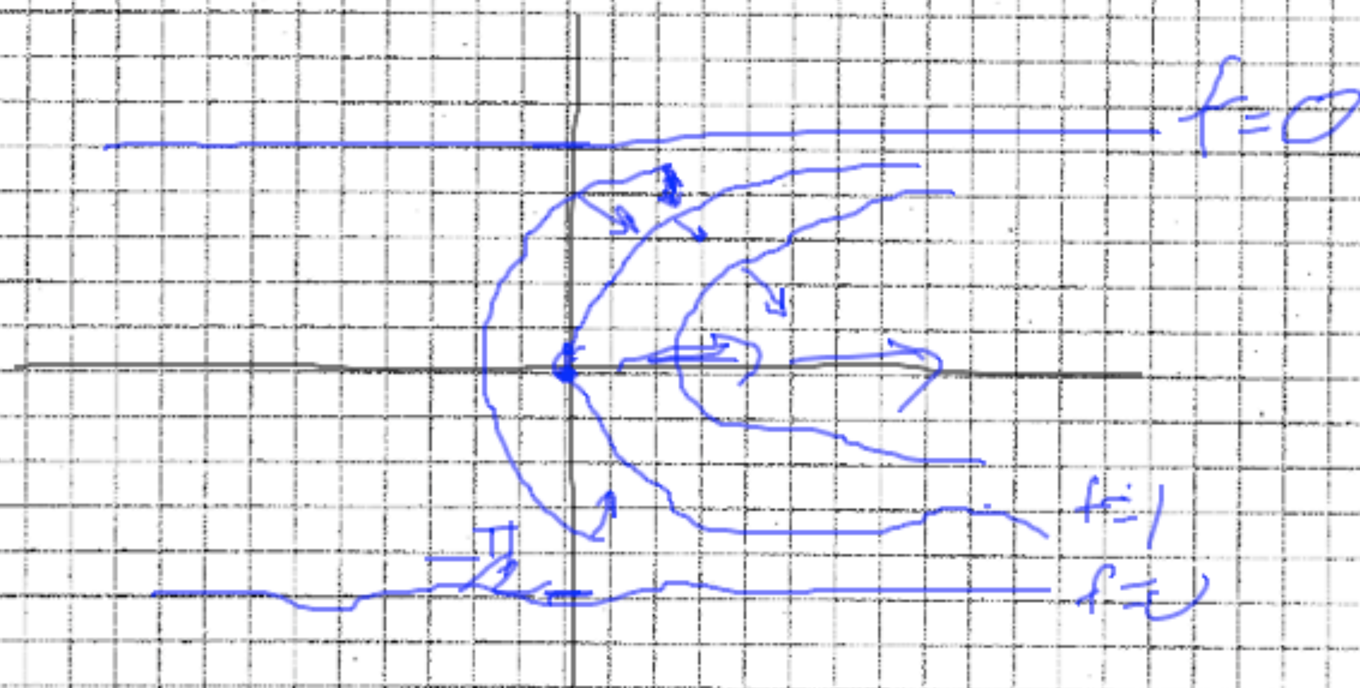
Now let's do some examples

- Let's consider $f(x,y) = e^x \cos y$. (You might see this function in a thermodynamics class, since it is a possible equilibrium temperature distribution in a homogeneous solid.)

Example: $f(x,y) = e^x \cos y$

Show gradient on contour plot (first with no calculations)

$e^x \cos(y)$



Example: $f(x,y) = e^x \cos y$

$\nabla f = e^x \cos y\, \mathbf{i} - e^x \sin y\, \mathbf{j}$

Magnitude is e^x.

Example: $f(x,y) = e^x \cos y$

Find the directional derivative in the "northeast" direction **i** + **j**, at $P = (1,\pi)$.

1. Calculate the gradient
2. Identify a *unit* vector in the direction you want: $2^{-1/2}\,\mathbf{i} + 2^{-1/2}\,\mathbf{j}$.
3. Take dot product

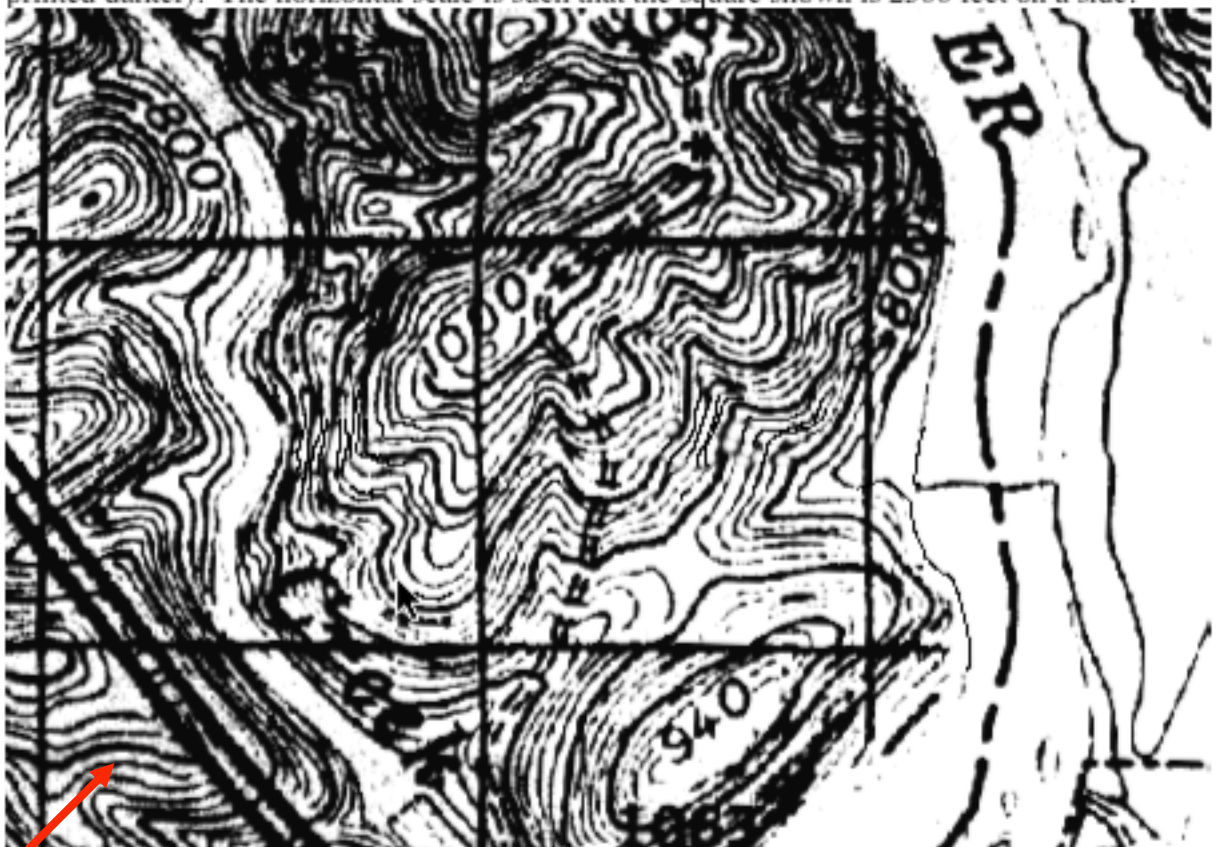
ANSWER: $2^{-1/2} e^{1}(\cos \pi - \sin \pi) = -e/2^{1/2}$.

How to determine the gradient from a topographic map

- Direction - perpendicular to the "contour" (= level curve) passing through the position of interest.
- Calculate slope by measuring off a short distance and counting the contours crossed in that distance.
- Given a direction and a magnitude, you have the vector.

A problem from an old final exam:

A topographic map such as the one shown here, of the Chattahoochee National Recreation Area in Sandy Springs, is a contour plot for a function f(x,y). The contours are level sets for values of z=f(x,y). Contours are separated by heights of 10 feet (every fifth contour is printed darker). The horizontal scale is such that the square shown is 2500 feet on a side.



The point P

Annotate the contour map as follows:

a) Find the top of a hill and label it with the letter **T**.

b) Find a saddle point and label it with the letter **S**.

c) There are several cliffs in the park. Find a cliff on this map and label it with the letter **C**.

d) In the lower left part of the map you will find a small arrow. Estimate the gradient at the point of the arrow. By the way, the arrow drawn is not meant to indicate the gradient at its base.) Draw a vector on the map with the same direction as the gradient, and estimate the magnitude of the gradient here:_______________