4.1 #8. Since

$$\begin{bmatrix} r-s+3t \\ 2r-t \\ -r+3s+2t \\ -2r+s+t \end{bmatrix} = r\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix},$$

we see that

$$S = \left\{ \begin{bmatrix} r-s+3t \\ 2r-t \\ -r+3s+2t \\ -2r+s+t \end{bmatrix} : r,s,t \in \mathbf{R} \right\} = \left\{ r\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} : r,s,t \in \mathbf{R} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} \right\},$$

so those three vectors listed above form a spanning set.

Remark: Looking ahead to Section 4.2, IF those three vectors listed above are ALSO independent, then we'll call them a *basis* for S. Are they?

4.1 #16. S is not closed under scalar multiplication. For example, $u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in s, but $-2u = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$ isn't in S.

4.1 #46. Find the standard matrix A for T. The columns of A are a spanning set for the range of T, and if you find the vector parametric form of the general solution to $Ax = 0$, you'll find a spanning set for the nullspace of T.

4.2 #6. Since

$$\left\{ \begin{bmatrix} 2s \\ -s+4t \\ s-3t \end{bmatrix} : s,t \in \mathbf{R} \right\} = \left\{ s\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix} : s,t \in \mathbf{R} \right\},$$

we see that

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix} \right\}$$

SPANS the given space. HOWEVER, you can't stop here—in order to be a basis, you must ALSO know that these vectors are independent. Sometimes, as in the procedure for computing a spanning set for a nullspace, we know in advance that that procedure always gives independent vectors. But that's not the case here, so we have to check whether those two vectors given above are independent. And indeed they are, since we have only two of them and they're not multiples of each other—but we have to verify that. Because those vectors span AND are independent, we can say that $\mathcal{B}$ is a basis for the space.

Remark: If we somehow knew in advance that the given space was 2-dimensional, then we could stop once we had two vectors that span the space. But in this problem, there's no way to know the dimension in advance.

4.2 #15. Find the reduced row echelon form R for the matrix A. Then find the vector parametric form of the general solution to $Ax = 0$, that leads you to a basis for the nullspace. You find that

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis for the nullspace (because they span the nullspace, and the procedure for finding these vectors always gives independent vectors). Next, pick out the pivot columns of A (NOT of R!!) to get a basis for the column space. You find that

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

is a basis for the column space.

4.2 #29. In #15 you found a basis for the nullspace of that matrix. Since that basis had two vectors in it, we know that Null(A) is 2-dimensional, hence EVERY basis for Null(A) must have two vectors in it. Moreover, any two independent vectors in Null(A) will automatically span the nullspace, and hence will give another basis for it. So, simply verify that the two vectors given in this problem are in the nullspace (by checking that $Ax = 0$), and then verify that they are independent (because they're not multiples of each other). Hence we've found another basis for the nullspace.

4.3 #1a. False. For example, any two lines in $\mathbf{R}^2$ through the origin are 1-dimensional subspaces of $\mathbf{R}^2$.

d. False. The dimension of the nullspace equals the number of free variables, which is the nullity of A, which is $n - \text{rank}(A)$.

e. False. The dimension of the column space equals the rank, not the nullity.

4.3 #22. The range of T is the column space of its standard matrix, and the nullspace of T is the nullspace of its standard matrix. So, find the standard matrix A and then find a basis for its nullspace and for its column space.

4.4 #14. a. Let v_1, v_2, v_3 be the three vectors in $\mathcal{B}$, create a matrix A that has v_1, v_2, v_3 as columns, and verify that the echelon form for A has three pivots. Since there's a pivot in every row and column, this shows that v_1, v_2, v_3 are independent and span $\mathbf{R}^3$, hence are a basis for $\mathbf{R}^3$.

b. Since $v = 0v_1 + 1v_2 - 4v_3$, the coordinates of v in the basis $\mathcal{B}$ are $[v]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}$.

4.4 #20. Let v_1, v_2, v_3 be the three vectors in the basis $\mathcal{B}$, and let A be the matrix that has those vectors as columns, i.e., $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Let $b = \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix}$ be the given vector. We need to solve $b = c_1v_1 + c_2v_2 + c_3v_3$, which means that we need to solve $Ax = b$. We row-reduce $[A \,|\, b]$ and find that $c_1 = 11$, $c_2 = -3$, and $c_3 = -5$, so $[b]_{\mathcal{B}} = \begin{bmatrix} 11 \\ -3 \\ -5 \end{bmatrix}$.