MATH 1522 EXAM #2 NAME: ____________________________

March 14, 2005

Answer the following questions clearly and completely. Unless otherwise specified, you must provide work justifying your solution.

There are 5 questions plus one extra credit problem, on 8 pages. The exam is worth 40 points total, plus up to 3 points extra credit.

1. Let $A = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 3 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 3 \end{bmatrix}$.

(6 points) a. Compute $\det(A)$ by using cofactors to reduce the 4×4 determinant to a sum of 3×3 determinants. You can then use any valid method you like to find those 3×3 determinants (if the scalar multiplying a particular 3×3 determinant is zero, you don't need to compute it).

(Problem 1 Continued)

(2 points) b. WITHOUT doing row reduction, give the reduced row echelon form for A. Hint: This follows immediately from part a. you do NOT need to do any more work to find it.

(2 points) c. What is Col(A)? Hint: This follows immediately from part b.

(4 points) 2. Define $T\colon \mathbf{R}^2 \to \mathbf{R}^3$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_1 x_2 \\ x_1 + x_2 \end{bmatrix}$.
Determine (with proof) whether T is linear.

(4 points) 3. Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. Which one of the following statements is true, and why?

A. If x is in Null(A), then x is in Null(AB).
B. If x is in Null(B), then x is in Null(AB).
C. If x is in Null(AB), then x is in Null(A).
D. If x is in Null(AB), then x is in Null(B).

4. Let $T\colon \mathbf{R}^3 \to \mathbf{R}^4$ be the linear transformation defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 3x_3 \\ x_2 + x_3 \\ 0 \\ x_1 + x_2 + 3x_3 \end{bmatrix}$.

(2 points) a. Find the standard matrix for T (no explanation required).

(5 points) b. Find a basis for range(T).

(Problem 4 Continued)

(1 point) c. Find the dimension of range(T).

(5 points) d. Find a basis for Null(T).

(Problem 4 Continued)

(2 points) e. Is T 1-1? Why or why not?

(2 points) f. Is $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ a basis for Null(T)? Why or why not?

(5 points) 5. For each part, circle T for True or F for False (no explanation required). Note: An answer of "sometimes true and sometimes false" or "cannot be determined" would count as False.

T F a. The column space of an $m \times n$ matrix equals the set $\{Ax : x \in \mathbf{R}^n\}$.

T F b. If A is an $m \times n$ matrix then $\text{rank}(A^{\text{T}}) = \dim(\text{Col}(A))$.

T F c. The columns of an $m \times n$ matrix form a basis for its column space.

T F d. If W is a subspace of $\mathbf{R}^n$ and $u_1, \ldots, u_k$ is a spanning set for W, then there are vectors $u_{k+1}, \ldots, u_\ell$ in $\mathbf{R}^n$ such that $u_1, \ldots, u_k, u_{k+1}, \ldots, u_\ell$ is a basis for W.

T F e. There is only one subspace of $\mathbf{R}^5$ that has dimension 5.

(3 points EXTRA CREDIT) Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix. Prove that $\text{rank}(AB) \leq \text{rank}(B)$. Hint: Use Problem 3.