May 5, 2005

Answer the following questions clearly and completely. Unless otherwise specified, you must provide work justifying your solution.

There are 6 questions plus one extra credit problem, on 12 pages. The exam is worth 80 points total, plus up to 5 points extra credit.

(8 points) 1. Let $A = \begin{bmatrix} 2 & 1 & 6 & -4 \\ 0 & 2 & 0 & 0 \\ 3 & 1 & 1 & 5 \\ 1 & 1 & 3 & 0 \end{bmatrix}$. Compute $\det(A)$ by using row reduction. Do not use cofactors to compute the determinant.

2. Let $u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 5 \\ 1 \\ 3 \end{bmatrix}$ and let $W = \text{span}\{u_1, u_2, u_3\}$.

(8 points) a. Apply the Gram–Schmidt process to find orthogonal vectors v_1, v_2, v_3 which have the same span as u_1, u_2, u_3.

(2 points) b. Show directly that your vectors v_1, v_2, v_3 are orthogonal.

(4 points) c. Let $u = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -2 \end{bmatrix}$. Find unique vectors $w \in W$ and $z \in W^\perp$ such that $u = w + z$.

(2 points) d. Find the distance from u to W.

(2 points) e. Find a vector v_4 so that v_1, v_2, v_3, v_4 is an orthogonal basis for $\mathbf{R}^4$.

3. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

(4 points) a. Find the characteristic polynomial of A.

(4 points) b. Find the eigenvalues of A and state the multiplicity of each.

(8 points) c. Find a basis for each eigenspace of A.

(8 points) d. If A is diagonalizable, write down a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. Use the Gauss-Jordan algorithm to find P^{-1}.

4. Let W be the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbf{R}^3$ which satisfy $2x_1 + x_2 - 3x_3 = 0$.

(5 points) a. Determine, with proof, whether W is a subspace of $\mathbf{R}^3$ or not.

(5 points) b. Find a basis for $W^\perp$, and find the dimension of $W^\perp$.

5. Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ 4 \\ -11 \end{bmatrix}$. Let $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation whose rule is

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1v_1 + x_2v_2 + x_3v_3.$$

(4 points) a. Determine whether v_1, v_2, v_3 are dependent or independent.

(4 points) b. Find the standard matrix for T.

(4 points) c. Determine whether T is 1-1 and whether T is onto.

(8 points) 6. For each part, circle T for True or F for False (no explanation required). Note: An answer of "sometimes true and sometimes false" or "cannot be determined" would count as False.

T [F] a. If A is an $n \times n$ matrix and $A - 3I$ is invertible, then 3 is an eigenvalue of A.

T [F] b. If λ is an eigenvalue of an $n \times n$ matrix A, then the multiplicity of λ is the dimension of $\mathrm{Null}(A - \lambda I)$.

T [F] c. Three vectors v_1, v_2, v_3 in $\mathbf{R}^4$ are linearly independent if no one of them is a multiple of the others.

[T] F d. The eigenvalues of a lower triangular matrix are equal to the diagonal entries of the matrix.

[T] F e. If A is a 4×4 matrix with rank 3 then 0 is an eigenvalue of A.

[T] F f. The row space of an $m \times n$ matrix A equals the row space of the row-reduced echelon form R of A.

[T] F g. If u_1, u_2, u_3, u_4 are vectors in $\mathbf{R}^4$ and $\mathrm{span}\{u_1, u_2, u_3, u_4\} = \mathbf{R}^4$, then u_1, u_2, u_3, u_4 are linearly independent.

T [F] h. If an $n \times n$ matrix A is diagonalizable, then the eigenvalues of A are distinct, i.e., each eigenvalue has multiplicity 1.

(5 points EXTRA CREDIT) Let $u_1, \ldots, u_k$ be k vectors in $\mathbf{R}^n$. Let $S = \{u_1, \ldots, u_k\}$. Use the definition of subspace to prove that $S^\perp$ is a subspace of $\mathbf{R}^n$. (Use the back of the page if you need more space.)