Below is a quick list of some of the terminology and other highlights from the sections of the text that we have covered. You should be understand each item and be able to use or define it as appropriate. THIS LIST MAY NOT BE COMPLETE! Add additional items to it as you study for the final exam. If you are not sure how to use or define something, ask!

As before, you will be allowed to write notes on the back of the exam instruction sheet and bring it to the final exam.

Chapter 1: Matrices, Vectors, and Systems of Linear Equations

1.1 Matrices and Vectors

(1) Scalar
(2) Vector in $\mathbf{R}^n$
(3) $\mathbf{R}^n$
(4) Vector addition and scalar multiplication
(5) Matrix notation, size of a matrix

1.2 Linear Combinations

(1) Linear combination of vectors
(2) Matrix-vector product Ax is a linear combination of the columns of A
(3) $n \times n$ identity matrix I_n
(4) Standard vectors $e_1, \ldots, e_n$
(5) Zero vector
(6) Zero matrix
(7) Transpose A^{T} of a matrix A
(8) Sum $A + B$ of matrices A and B
(9) Product cA of scalar c with a matrix A

1.3 Systems of Linear Equations

(1) System of linear equations
(2) Consistent and inconsistent systems of linear equations
(3) Elementary row operations
(4) Row echelon form, reduced row echelon form
(5) Basic variables, free variables
(6) Augmented matrix

1.4 Gaussian Elimination

(1) Pivots
(2) "Sweep down, sweep up" procedure
(3) How to find the general solution to $Ax = b$, how to tell when $Ax = b$ is consistent
(4) How to write the general solution in vector parametric form
(5) Rank, nullity

1.6 Span

(1) Span of vectors $v_1, \ldots, v_k \in \mathbf{R}^n$
(2) Fewer than n vectors can't span $\mathbf{R}^n$, n or more vectors might span $\mathbf{R}^n$
(3) Conditions for when the span is all of $\mathbf{R}^n$, especially pivot in every row of r.r.e.f.

1.7 Linear Independence

    (1) Linear independence, linear dependence
    (2) n or fewer vectors might be independent, more than n vectors in $\mathbf{R}^n$ must be dependent
    (3) Conditions for when vectors are independent, especially pivot in every column of r.r.e.f.

Chapter 2: Matrices and Linear Transformations

2.1 Matrix Multiplication

    (1) Columnwise definition of AB, entrywise computation of AB
    (2) Diagonal matrix
    (3) Symmetric matrix
    (4) Upper and lower triangular matrices
    (5) $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$

2.3 Invertibility and Elementary Matrices

    (1) Invertible matrix, inverse of A
    (2) $(AB)^{-1} = B^{-1} A^{-1}$ (IF A and B are invertible)
    (3) $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$ (IF A is invertible)
    (4) $A(BC) = (AB)C$ (associative law)
    (5) $A(B + C) = AB + AC$ (distributive law)
    (6) $AB \neq BA$ in general!! (matrix multiplication is usually NOT commutative)
    (7) Rowwise definition of AB
    (8) Elementary matrix (3 types)
    (9) Inverse of an elementary matrix
    (10) What is EA when E is an elementary matrix (for each of the 3 types)
    (11) $R = PA$ where R is the reduced echelon form of A and P is a product of elementary matrices
    (12) Pivot columns of A are linearly independent

2.4 Inverse of a Matrix

    (1) Conditions for when a square matrix is invertible
    (2) Algorithm for finding A^{-1}

2.6 Linear Transformations

    (1) Definition of linear transformation, domain, codomain, and range
    (2) How to test whether a transformation is linear
    (3) Linear transformation T_A where A is a matrix
    (4) Standard matrix A for a linear transformation
    (5) $\mathrm{range}(T) = \mathrm{span}\{\text{columns of } A\} = \mathrm{Col}(A) = \{Ax : x \in \mathbf{R}^n\}$

2.7 Composition and Invertibility of Linear Transformations

    (1) Definition of onto and 1-1
    (2) Conditions for when T is onto
    (3) Conditions for when T is 1-1
    (4) Nullspace of T

Chapter 3: Determinants

3.1 Cofactor Expansion

    (1) How to do 2×2 and 3×3 determinants without cofactors
    (2) How to compute a general determinant using cofactors
    (3) Determinant of a triangular matrix

3.2 Properties of Determinants

    (1) If one row is multiplied by a scalar c, then the determinant is multiplied by c
    (2) The determinant changes sign when two rows are interchanged
    (3) The determinant is unchanged when a multiple of one row is added to another row
    (4) $\det(AB) = \det(A)\det(B)$
    (5) A is invertible if and only if $\det(A) \neq 0$
    (6) $\det(A^{-1}) = 1/\det(A)$
    (7) $\det(A^{\mathrm{T}}) = \det(A)$
    (8) How to use row-reduction to find a determinant

Chapter 4: Subspaces

4.1 Subspaces

    (1) Definition of subspace
    (2) How to show that a set is a subspace
    (3) $\mathrm{span}\{v_1, \ldots, v_k\}$ is a subspace
    (4) Nullspace of a matrix or transformation is a subspace
    (5) Column space of a matrix or range of a transformation is a subspace

4.2 Basis and Dimension

    (1) Basis for a subspace
    (2) How to find a basis for the column space of a matrix
    (3) How to find a basis for the nullspace of a matrix
    (4) Dimension of a subspace
    (5) "Two out of three" test

4.3 Dimensions of Subspaces Associated with a Matrix

    (1) Dimension of column space = rank
    (2) Dimension of nullspace = nullity
    (3) Row space (= column space of A^{T})
    (4) Dimension of row space = rank = dimension of column space
    (5) $\mathrm{rank}(A) = \mathrm{rank}(A^{\mathrm{T}})$

Chapter 5: Eigenvalues and Eigenvectors

5.1 Eigenvalues and eigenvectors

    (1) Eigenvalue of an $n \times n$ matrix

    (2) Eigenvector of an $n \times n$ matrix

    (3) Eigenspace

    (4) The λ-eigenvectors are the nonzero vectors in $\text{Null}(A - \lambda I_n)$

    (5) λ is an eigenvalue if and only if $A - \lambda I_n$ is not invertible

    (6) $\lambda = 0$ is an eigenvalue if and only if A is not invertible

    (7) λ is an eigenvalue if and only if $\det(A - \lambda I_n) = 0$

5.2 Characteristic Polynomial

    (1) Characteristic polynomial is $\det(A - tI_n)$

    (2) Eigenvalues are the roots of the characteristic polynomial

    (3) Multiplicity of an eigenvalue

    (4) λ is a "repeated" eigenvalue if the multiplicity is > 1

    (5) Dimension of the λ-eigenspace is no more than the multiplicity of λ

    (6) Eigenvalues can be complex (but we'll only do real ones)

    (7) Multiplicities add to n (but might need complex eigenvalues to do this)

5.3 Diagonalization

    (1) A is diagonalizable if $A = PDP^{-1}$ where P is invertible and D is diagonal

    (2) What must P and D be in this case?

    (3) What has to be true about the dimensions of the eigenspaces and the multiplicities of the eigenvalues in order for A to be diagonalizable?

    (4) If the eigenvalues of A are all distinct (n different eigenvalues all having multiplicity 1), then A is sure to be diagonalizable

    (5) If the eigenvalues aren't distinct, A might still be diagonalizable! How can you tell?

    (6) What is A^k if A is diagonalizable? (Relate to $A = PDP^{-1}$, and identify the eigenvalues and eigenvectors of A^k)

Chapter 6: Orthogonality

6.1 The Geometry of Vectors

    (1) Length $\|v\|$ of a vector $v \in \mathbf{R}^n$

    (2) Distance $\|u - v\|$ between vectors $u, v \in \mathbf{R}^n$

    (3) Dot product $u \cdot v$ of vectors $u, v \in \mathbf{R}^n$, formula $u \cdot v = u^{\text{T}}v$

    (4) $u, v \in \mathbf{R}^n$ are orthogonal if $u \cdot v = 0$

    (5) Properties of dot products

    (6) Orthogonal projection of u onto the line through v is $p = \frac{u \cdot v}{\|v\|^2} v$ (this is a special case of orthogonal projection onto a subspace W, here the subspace is just 1-dimensional)

    (7) Cauchy-Schwarz inequality

    (8) Triangle inequality

6.2 Orthogonal Vectors

(1) Orthogonal set of vectors
(2) Orthonormal set of vectors
(3) Orthogonal basis for a subspace W
(4) Orthonormal basis for a subspace W
(5) The vector in a subspace W that is closest to a vector v is the orthogonal projection p of v onto W. The error vector $e = v - p$ is orthogonal to every vector in W
(6) If $w_1, \ldots, w_k$ is an ORTHONORMAL basis for a subspace W, then

$$v = (v \cdot w_1)w_1 + \cdots + (v \cdot w_k)w_k \qquad \text{for each } v \text{ that is IN } W$$

and for ALL v in $\mathbf{R}^n$, the orthogonal projection of v onto W is

$$p = (v \cdot w_1)w_1 + \cdots + (v \cdot w_k)w_k \qquad \text{for all } v \in \mathbf{R}^n$$

(note that when v is IN W then $p = v$)
(7) Modify the formulas above if $w_1, \ldots, w_k$ is an orthogonal but not orthonormal basis
(8) Gram-Schmidt process for 2 and 3 vectors (it's based on finding orthogonal projections p and error vectors e)
(9) Orthogonal complement $W^\perp$ of a subspace W
(10) Row(A) and Null(A) are orthogonal complements
(11) $\dim(W) + \dim(W^\perp) = n$
(12) If p is orthogonal projection of v onto W and $e = v - p$, then $p \in W$ and $e \in W^\perp$ and $v = p + e$. This is the ONLY way to write $v = w + z$ with $w \in W$ and $z \in W^\perp$

6.3 Least Squares

(1) Connection between best-fit straight line and orthogonal projection
(2) Orthogonal projection of a vector v onto a subspace W is $p = C(C^\mathrm{T}C)^{-1}C^\mathrm{T}v$ where the columns of C form a basis for W
(3) The orthogonal projection matrix is $P_W = C(C^\mathrm{T}C)^{-1}C^\mathrm{T}$, the orthogonal projection projection of v onto W is $p = P_W v$

6.4 Orthogonal Matrices

(1) Rotation operator
(2) Orthogonal matrix, orthogonal operator
(3) Properties of orthogonal matrices, especially $Q^\mathrm{T}Q = I_n$

6.5 Symmetric Matrices

(1) A symmetric matrix has real eigenvalues
(2) A symmetric matrix has an orthonormal basis of eigenvectors