1.3 Quantifiers

There are two logical quantifies:

For all symbol: $\forall$

There exists symbol: $\exists$

For all x, A_x

This statement is true if the statement A_x is true for every x.

There exists x such that A_x

This statement is true if there is at least one x such that A_x is true.

Symbolic abbreviations: $\forall x\ A_x$

$\exists x\ A_x$ or $\exists x$ s.t. A_x

s.t. = such that

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Usually it is clear from context what x is referring to. To be more precise, we would write something like

$\forall x \in X, \quad A_x$ where X is some set, $\in$ means "belongs to"

Negations

$$\text{Not}(\forall x, A_x) = \exists x \text{ s.t. } (\text{not } A_x)$$

if it's not true for every x, then there's some x it's not true for.

$$\text{Not}(\exists x, A_x) = \forall x, (\text{not } A_x)$$

if it's not true that there's an x it's true for, then it's false for every x.

Example

$$\text{Not}(A_x, \exists y \text{ s.t. } A_{xy})$$

$$= \exists x \; \forall y \; (\text{not } A_{xy})$$

Here's an example. We'll assume we're dealing with functions that map real numbers to real numbers, the kind of functions we study in Calc I.

Definition

A function f is <u>continuous</u> at a point x if

$$\forall \varepsilon > 0, \quad \exists \delta > 0 \text{ s.t. } \forall y,$$

$$\text{if } |x-y| < \delta \text{ then } |f(x) - f(y)| < \varepsilon. \quad \square$$

my "end of def." symbol

Theorem

$f(x) = x^2$ is continuous at the point $x = 0$.

Proof: (with extra discussion).

We have to show that something is true for every $\varepsilon > 0$. No matter what ε is, we have to show that "$\exists \delta > 0 \ldots$" is true. We usually start the proof by saying:

"Let $\varepsilon > 0$ be given"

1st real line of proof →

or "Choose any $\varepsilon > 0$."

This is just a way of saying that we're going to give

a proof that works for every value of ε, it isn't tied to some specific value. E.g., we couldn't just give a proof that works for the number $\varepsilon = \frac{1}{2}$ but not for any thing else.

So now we have to show that

"$\exists \delta > 0$ s.t. $\forall y$, if ... then ..."

is true for ε. Implicitly, there's an ordering here: if we want to be really explicit, we should have written the definition of continuity as

$$\forall \varepsilon > 0, \quad \exists \delta_\varepsilon > 0 \text{ s.t. } \forall y, \quad \text{if } |x-y| < \delta_\varepsilon \text{ then } \ldots$$

There's a δ for each ε,
not a single δ that works for <u>all</u> ε.

Implicitly, the ordering "$\forall \varepsilon > 0 \; \exists \delta > 0$" means that <u>$\delta$ depends on ε</u>.

Exercise: Think about what "$\exists \varepsilon > 0 \; \forall \delta > 0 \ldots$" would mean.

So, for "our" ε, we need to find a δ that goes with it. <u>How</u> you do that is what

scratchwork is for - you do not put it in the proof! All that you put in the proof is:

2nd line →

Let $\delta = \varepsilon^{1/2}$.

For all we know, this came to you in a dream. That's not the issue - the issue is to prove the theorem.

Next we must show that "$\forall y$, if ... then ..." So the next line is written:

3rd line →

Choose any y.

Now we get to the last part: we have to show that "if $|x-y| < \delta$ then $|f(x)-f(y)| < \varepsilon$." Recalling our if-then truth tables, the only thing to worry about is what happens when statement A is true. If A is false then "if A then B" is automatically true & there's nothing to prove. So our next line is

42 line →

Suppose $|x-y| < \delta$.

Now we go on, until we've shown that B is true. So here's the rest of the proof:

Since $x = 0$, our assumption is that $|y| < \delta$.
Also, $f(x) = f(0) = 0^2 = 0$, so

$$|f(x) - f(y)| = |0 - y^2|$$

$$= y^2$$

$$< \delta^2 \quad \text{(rules of arithmetic)}$$

$$= \varepsilon.$$

Thus we've shown that $|f(x) - f(y)| < \varepsilon$ is true, so the proof is complete. ∎

The proof is written in English, in complete sentences, proceeding logically to show that the statement "$\forall \varepsilon > 0 \; \exists \delta > 0 \ldots$" is true.

Exercise: What does it mean for f to not be continuous at x? That is, find

$$\text{NOT}\left(\forall \varepsilon > 0 \; \exists \delta > 0 \ldots \right)$$

and explain what this means in words.

Let's prove that if f is continuous at x, then $5f$ is continuous at x. $5f$ simply means the function that takes values that are 5 times what f takes, i.e., the rule for $5f$ is

$$(5f)(t) = 5 \cdot f(t), \quad \text{all } t \in \mathbb{R}.$$

Theorem
If f is continuous at a point x, then $5f$ is continuous at x.

Proof:
Suppose that f is continuous at x.

Choose any $\varepsilon > 0$. By definition, since f is continuous at x, there is a $\delta > 0$ such that

$$\forall y \in \mathbb{R}, \quad |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Therefore, if $y \in \mathbb{R}$ satisfies $|x-y| < \delta$, then

$$|(5f)(x) - (5f)(y)| = |5f(x) - 5f(y)|$$

$$= 5 \cdot |f(x) - f(y)|$$

$$< 5\varepsilon.$$

WELL, THAT DIDN'T WORK.
THIS BECOMES SCRATCHPAPER, THROW IT OUT

Choose any $\varepsilon > 0$. Let $\eta = \frac{\varepsilon}{5}$. Since η is a positive number and since f is continuous at the point x, there is a $\delta > 0$ such that

$$\forall y \in \mathbb{R}, \quad |x-y| < \delta \implies |f(x) - f(y)| < \eta.$$

Therefore, if $y \in \mathbb{R}$ and $|x-y| < \delta$, then

$$\begin{aligned} |(5f)(x) - (5f)(y)| &= |5f(x) - 5f(y)| \\ &= 5 \cdot |f(x) - f(y)| \\ &< 5 \cdot \eta \\ &= 5 \cdot \frac{\varepsilon}{5} \\ &< \varepsilon. \end{aligned}$$

Thus, we have shown that if $|x-y| < \delta$, then $|(5f)(x) - (5f)(y)| < \varepsilon$. Therefore $5f$ is continuous at x. ∎

Exercise

Let f and g be two functions that are continuous at a point x. Define a new function h by the rule

$$h(t) = f(t) + g(t), \qquad t \in \mathbb{R}.$$

Show that h is continuous at x.

Remarks

If we choose a number $\varepsilon > 0$, then since f is continuous at x, there is a number $\delta > 0$ such that ... Since g is also continuous at x, there is a $\delta > 0$ such that ... But the δ for f doesn't have to be the same number δ for g. To tell them apart, you need to give them different names. For example, you could write the following.

Choose any $\varepsilon > 0$. Since f is continuous at x, there is a number $\delta_1 > 0$ such that

$$\forall y \in \mathbb{R}, \quad |x-y| < \delta_1 \implies |f(x) - f(y)| < \varepsilon.$$

Also, since g is continuous at x, there is a number $\delta_2 > 0$ such that

$$\forall y \in \mathbb{R}, \quad |x-y| < \delta_2 \implies |g(x) - g(y)| < \varepsilon.$$

Hint: What happens if $|x-y| < \delta_1$ AND $|x-y| < \delta_2$?

Exercise: Suppose f & g are continuous at x. Define

$$k(t) = f(t) - g(t), \qquad t \in \mathbb{R}.$$

Show that k is continuous at x.

Exercise: Suppose f & g are continuous at x and a, b are any two real numbers. Define a function p by the rule

$$p(t) = a \cdot f(t) + b \cdot g(t), \qquad t \in \mathbb{R}.$$

Show that p is continuous at the point x.

Exercise

Suppose that f is continuous at a point x. Show that there exists a number $\delta > 0$ and a number $M \geq 0$ such that

$$\forall y \in \mathbb{R}, \quad |x - y| < \delta \implies |f(x)| \leq M.$$

Challenge

Suppose f & g are continuous at x. Define

$$h(t) = f(t) \cdot g(t), \qquad t \in \mathbb{R}.$$

Show that h is continuous at x.

Exercises on Limits

Assume that f, g are functions such that

$\lim_{y \to x} f(y)$ exists and $\lim_{y \to x} g(y)$ exists.

a. Define a function h by the rule $h(y) = f(y) + g(y)$.

Prove that $\lim_{y \to x} h(y)$ exists, and has the value

$$\lim_{y \to x} h(y) = \lim_{y \to x} f(y) + \lim_{y \to x} g(y).$$

b. Let a, b be any two real numbers, and define a function k by $k(y) = af(y) + bg(y)$. Prove that $\lim_{y \to x} k(y)$ exists, and has the value

$$\lim_{y \to x} k(y) = a \cdot \lim_{y \to x} f(y) + b \cdot \lim_{y \to x} g(y).$$