2.9 Direct Products

If S, T are two arbitrary sets, then their Cartesian product is the set of all ordered pairs of element of S with elements of T:

$$S \times T = \{(s,t) : s \in S,\ t \in T\}.$$

If S, T are finite, then $|S \times T| = |S||T|$.

When G, H are groups, their Cartesian product also has a natural group structure.

Exercise
Let G, H be groups. Define an operation on $G \times H$ by

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2,\ b_1 b_2).$$

Show that $G \times H$ is a group under this operation, with identity element

$$e_{G \times H} = (e_G, e_H)$$

and inverses

$$(a, b)^{-1} = (a^{-1}, b^{-1}).$$

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Example $\mathbb{Z}_3 \times \mathbb{Z}_2$ (with $\mathbb{Z}_3, \mathbb{Z}_2$ under addition)

As a set, $\mathbb{Z}_3 \times \mathbb{Z}_2$ has 6 elements:

$$\mathbb{Z}_3 \times \mathbb{Z}_2 = \{(0,0), (1,0), (2,0), (0,1), (1,1), (2,1)\}$$

Since the operation in both $\mathbb{Z}_3$ & $\mathbb{Z}_2$ is addition, "powers" of an element mean repeated additions. For example, consider the "powers" (multiples) of $(1,1)$:

$$0 \cdot (1,1) = (0,0)$$

$$1 \cdot (1,1) = (1,1)$$

$$2 \cdot (1,1) = (1,1) + (1,1) = (2,0)$$

$$3 \cdot (1,1) = (1,1) + (1,1) + (1,1) = (0,1)$$

$$4 \cdot (1,1) = \cdots = (1,0)$$

$$5 \cdot (1,1) = \cdots = (2,1)$$

$$6 \cdot (1,1) = \cdots = (0,0)$$

Hence $(1,1)$ has order 6, so $\mathbb{Z}_3 \times \mathbb{Z}_2$ is cyclic:

$$\mathbb{Z}_3 \times \mathbb{Z}_2 = \langle (1,1) \rangle.$$

Thus $\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$.

Exercise: Exhibit the isomorphism between $\mathbb{Z}_3 \times \mathbb{Z}_2$ & $\mathbb{Z}_6$ explicitly.

Exercise
Show that $\mathbb{Z}_2 \times \mathbb{Z}_2$ has order 4 but is NOT cyclic, so $\mathbb{Z}_2 \times \mathbb{Z}_2 \not\cong \mathbb{Z}_4$.

Exercise
Show that $\mathbb{Z}_4$ & $\mathbb{Z}_2 \times \mathbb{Z}_2$ are the "only" groups of order 4, in the sense that if $|G| = 4$, then either $G \cong \mathbb{Z}_4$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Exercise
Extend the definition of product to finitely many groups. That is, if $G_1, \ldots, G_n$ are groups, show that

$$G_1 \times \cdots \times G_n = \{(a_1, \ldots, a_n) : a_1 \in G_1, \ldots, a_n \in G_n\}$$

is a group under an appropriate operation. This group is called the (external) direct product of $G_1, \ldots, G_n$.

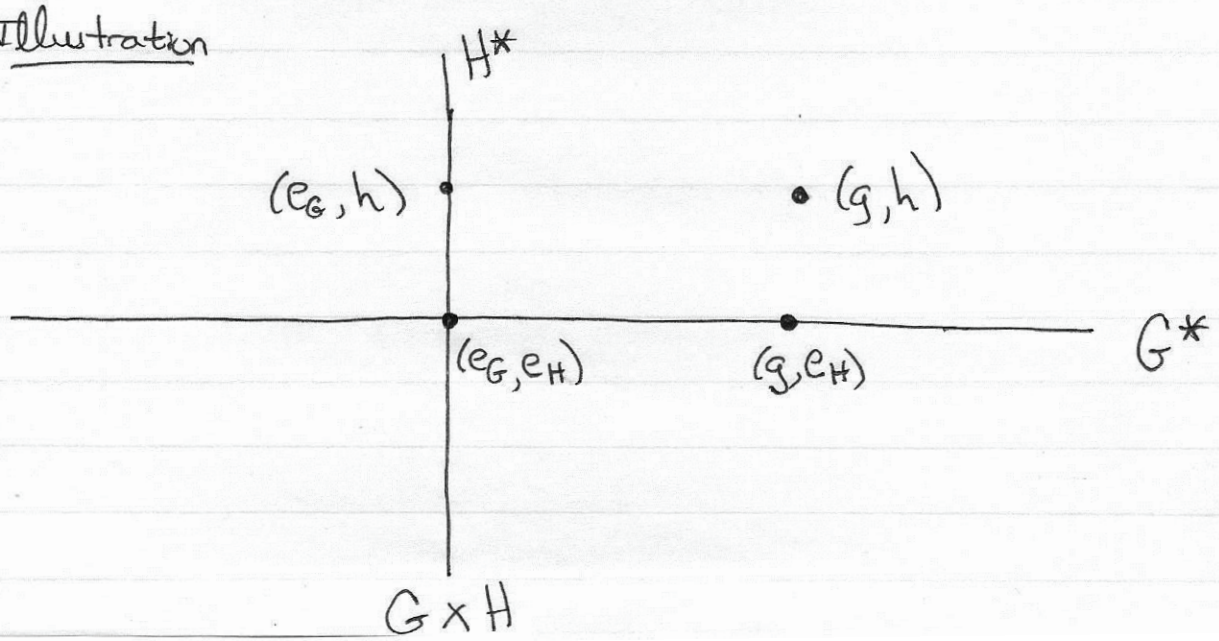
Exercise

Let G, H be groups. Show that $G \times H$ has a subgroup isomorphic to G, & a subgroup isomorphic to H, namely,

$$G^* = \{(g, e_H) : g \in G\}$$

$$H^* = \{(e_G, h) : h \in H\}$$

Illustration



Show further that:

a. $G^* \cap H^* = \{e_{G\times H}\} = \{(e_G, e_H)\}$

b. G^*, H^* are each normal in $G \times H$.

Let us show explicitly one more important fact.

c. $G \times H = G^* H^*$

Proof:

By definition,

$$\begin{aligned} G^* H^* &= \{xy : x \in G^*, y \in H^*\} \\ &= \{(g, e_H)(e_G, h) : g \in G, h \in H\} \\ &= \{(ge_G, e_H h) : g \in G, h \in H\} \\ &= \{(g, h) : g \in G, h \in H\} \\ &= G \times H. \end{aligned}$$

∎

This is an important fact: $G \times H$ has <u>normal</u> subgroups G^*, H^* which satisfy

a. $G^* \cap H^* = \{e_{G \times H}\}$

b. $G^* H^* = G \times H$.

Conversely, we'll show now that whenever a group has normal subgroups that satisfy analogous conditions, then that group is isomorphic to a direct product:

<u>Theorem</u>
If a group G has normal subgroups M, N s.t.

a. $M \cap N = \{e\}$

b. $MN = G$

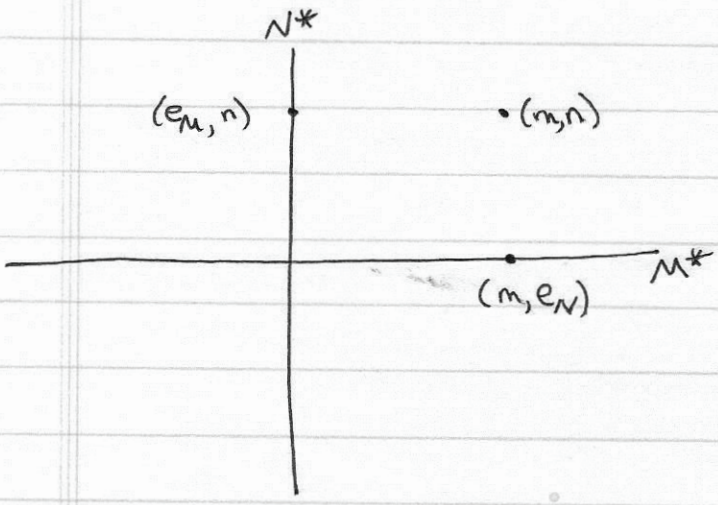
then $G \cong M \times N$.

Proof:

Define

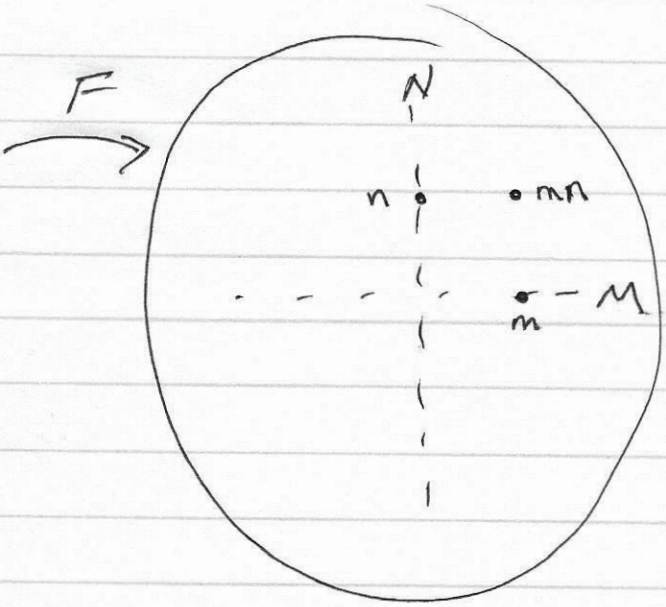
$$F : M \times N \longrightarrow G$$

$$F((m,n)) = mn$$



$M \times N$

Set of ordered pairs (m,n)



G

Set of elements

MN = subset of G consisting of elements mn with $n \in M$, $n \in N$

Homomorphism

Suppose (m,n) & (k,l) belong to $M \times N$.

Then:

$$F\big((m,n)(k,\ell)\big) = F\big((mk,n\ell)\big) = (mk)(n\ell)$$

while

$$F\big((m,n)\big)\,F\big((k,\ell)\big) = (mn)(k\ell).$$

So, we need to show that $mknl = mnk\ell$, which means that we must show that $kn = nk$.

Now $k \in M$ & $n \in N$, & both subgroups are normal.

Hence

$$nkn^{-1}k^{-1} = \underbrace{(nkn^{-1})}_{\in M}\underbrace{k^{-1}}_{\in M} \in M$$

and

$$nkn^{-1}k^{-1} = \underbrace{n}_{\in N}\underbrace{(kn^{-1}k^{-1})}_{\in N} \in N.$$

Thus $nkn^{-1}k^{-1} \in M \cap N = \{e\}$. Hence $nkn^{-1}k^{-1} = e$, so $nk = kn$, and therefore F is a homomorphism.

Injective

Suppose $F((m,n)) = e$. Then $mn = e$. Hence

$$m = n^{-1} \in N \quad \text{and} \quad m \in M$$

which implies $m \in M \cap N = \{e\}$. Hence $m = n = e$.

Thus $(m,n) = (e,e) = e_{M \times N}$, so F is injective.

Surjective

If $x \in G$ then since $G = MN$ we have $x = mn$

for some $m \in M$ & $n \in N$. Therefore

$F((m,n)) = mn = x$, so F is surjective.

Thus F is an isomorphism, so $G \cong M \times N$. ∎

Exercise

Let $G = \{e, r, r^2, r^3, a, b, c, d\}$ be the dihedral group of order 8. Let

$$M = \langle r \rangle = \{e, r, r^2, r^3\}, \quad N = \langle a \rangle = \{e, a\}.$$

Show that

$$G = MN \quad \text{and} \quad M \cap N = \{e\}.$$

Prove, however, that $G \not\cong M \times N$.

Why does this not contradict the preceding theorem?

Exercise

Show that if G & H are both abelian, then $G \times H$ is abelian.

Exercise:

Prove or find a counterexample: if G & H are cyclic groups then $G \times H$ is cyclic

Definition

If M, N are normal subgroups of a group G, then we say that G is the (internal) direct product of M & N if

- $G = MN$ and $M \cap N = \{e\}$.

Note that, by the preceding theorem, if G is the internal direct product of M & N then G is isomorphic to the external direct product $M \times N$.

Exercise

Show that if G is the internal direct product of normal subgroups M, N, then

$$x \in G \Rightarrow \exists \text{ unique } m \in M, n \in N \text{ s.t. } x = mn.$$

2.10 Finite Abelian Groups

We will not prove this result, but the finite abelian groups can be completely characterized.

Theorem
Every finite abelian group is a direct product of cyclic subgroups. Further, each cyclic subgroup can be taken to have prime-power order.

Corollary
If $|G| = p^n$, then

$$G \cong \mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_k}}$$

where $n_1 + \cdots + n_k = n$.

Example
An abelian group of order 27 is isomorphic to one of

$$\mathbb{Z}_{27}, \quad \mathbb{Z}_9 \times \mathbb{Z}_3, \quad \text{or} \quad \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$$

None of these three groups is isomorphic to the others.

Example

An abelian group of order 20 is isomorphic to one of

$\underbrace{\mathbb{Z}_4 \times \mathbb{Z}_5}_{\cong \mathbb{Z}_{20} \text{ (cyclic)}}$ or $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5}_{\text{not cyclic}}$