

4.4 Maximal Ideals

Definition

A ring R is simple if it has no nontrivial ideals, i.e., $\{0\}$ & R are the only ideals.

Definition

An ideal M in a ring R is a maximal ideal if

a. $M \subsetneq R$

b. There are no ideals I with $M \subsetneq I \subsetneq R$.

Note: Thus

R is a simple ring $\Leftrightarrow \{0\}$ is a maximal ideal.

Theorem

Let M be a proper ideal in a ring R . Then:

$$M \text{ is maximal} \iff R/M \text{ is simple.}$$

Proof:

\Leftarrow ~~Suppose~~ Suppose that R/M is simple. Suppose that

I ~~is~~ is an ideal in R & $M \subseteq I \subseteq R$.

Consider the canonical map

$$\begin{aligned} \pi: R &\longrightarrow R/M \\ \pi(a) &= a+M. \end{aligned}$$

Consider

$$J = \pi(I) = \{a+M : a \in I\}.$$

Exercise: Show J is an ideal in R/M .

But R/M is simple, so either $J = \{M\}$

(~~Since~~ since M is the zero element of R/M), or

$$J = R/M.$$

~~Exercise: Show J is an ideal in R/M .~~

Case 1: $J = \{M\}$.

We claim that this implies that $I = M$.

To see this, suppose that $a \in I$. Then, by definition of J , we have $a + M \in J$. But $J = \{M\}$, so this implies $a + M = M$. Hence $a \in M$, so we have shown that $I \subseteq M$.

Exercise: Show that $M \subseteq I$.

Thus, in this case we have $I = M$.

Case 2: $J = R/M$.

We claim that this implies that $I = R$.

Note that we have $I \subseteq R$, so we only have to show the reverse inclusion. Suppose that $a \in R$. Then $a + M \in R/M = J$, so $a + M = b + M$ for some $b \in I$. But then $a - b \in M$, and

$M \subseteq I$, so $a-b \in I$. Since $b \in I$, we conclude $a = (a-b) + b \in I$. Therefore $R = I$.


Thus there are only two possibilities for I : either $I = M$ or $I = R$. Therefore M is maximal.

\Rightarrow Exercise.

Hint: ~~Suppose J is an ideal in R/M .~~

~~Suppose J is an ideal in R/M .~~ Suppose that J is an ideal in R/M . Show that

$$I = \pi^{-1}(J) = \{a \in R : a + M \in J\}$$

is an ideal in R , and $M \subseteq I \subseteq R$. Since M is maximal, this leaves only two possibilities. 


Corollary

Let R be a commutative ring with identity, and let M be a proper ideal in R . Then:

M is maximal $\iff R/M$ is a field.

Proof:

\Leftarrow Suppose that R/M is a field. Then we know by earlier results that R/M is simple. The preceding theorem therefore implies that M is maximal.

\Rightarrow Suppose that M is maximal. ~~Assume that R/M is not a field.~~ Then the preceding theorem implies that R/M is simple. Thus R/M is a commutative ring with 1 that has no nontrivial ideals. We proved earlier that this implies that R/M is a field. 

Example/Exercise

Consider $R = \mathbb{Z}$. Suppose that $n > 0$ is composite, i.e., $n = kl$ with $k, l > 1$.

Exercise: Show $n\mathbb{Z} \subsetneq k\mathbb{Z} \subsetneq \mathbb{Z}$.

Thus $n\mathbb{Z}$ is not a maximal ideal. Therefore

$\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$ is not a field.

Suppose $p > 0$ is prime. Show that $p\mathbb{Z}$ is a maximal ideal, & conclude that $\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ is a field.