MATH 4107 EXAM #2 March 16, 2007

PLEASE READ THESE DIRECTIONS: Answer PROBLEM 1 (15 points) and choose TWO other problems to answer (10 points each). You may also answer (for up to 3 points extra credit) ONE additional problem. In this case, please specify which problem is the extra credit problem.

All statements require proof or justification. There are 35 points total, plus up to 3 points of extra credit.

1. Let G be an abelian group (note: G can be an infinite group!). Let T be the set of elements of G that have finite order:

$$T = \{a \in G : o(a) \text{ is finite}\}.$$

a. Prove that T is a subgroup of G.

b. Prove that G/T has no elements of finite order other than the identity element.

c. This part is not related to parts a or b. Let $\alpha = (1352)(2745)(6824) \in S_8$. Answer the following (no proof needed): write α as a product of disjoint cycles, compute the order of α, and determine if α is even or odd.

2. Suppose that τ is a k-cycle in S_n. Show that if σ is any permutation in S_n, then $\sigma\tau\sigma^{-1}$ is also a k-cycle in S_n.

3. Suppose that M is a normal subgroup of a group G, and N is a normal subgroup of a group H. Then the set $M \times N$ is a normal subgroup of $G \times H$ (you do not need to prove this).

a. Define $f\colon G \times H \to (G/M) \times (H/N)$ by

$$f\big((a,b)\big) = (Ma, Nb), \qquad (a,b) \in G \times H.$$

Show that f is a surjective homomorphism.

b. Use the First Homomorphism Theorem to show that

$$(G \times H)/(M \times N) \cong (G/M) \times (H/N).$$

4. Suppose that G is a finite abelian group with order $|G| > 1$. Suppose there exists a prime number p such that for each $a \in G$ there exists a positive integer k such that $a^{p^k} = e$ (the integer k depends on the element a). Show that $|G| = p^n$ for some integer n.