MATH 4107 FINAL EXAM May 4, 2007

PLEASE READ THESE DIRECTIONS: Answer PROBLEM 1 (16 points) and choose THREE other problems to answer (13 points each). You may also answer (for up to 5 points extra credit) ONE additional problem. In this case, please specify which problem is the extra credit problem. There are problems on BOTH SIDES of this page!

All statements require proof or justification. There are 55 points total, plus up to 5 points of extra credit.

1. The parts of this problem are not related, i.e., you do not have to do one part in order to solve the other part.

Let R be a ring. Then given a polynomial $p(x) = a_0 + a_1x + a_2x^2 \cdots + a_nx^n \in R[x]$, and given an element $a \in R$, we define $p(a)$ to be the element of R given by

$$p(a) = a_0 + a_1a + a_2a^2 \cdots + a_na^n.$$

a. Let $a \in R$ be fixed. Show that

$$I = \{p \in R[x] : p(a) = 0\}$$

is an ideal in $R[x]$, and use this to prove that $R[x]/I \cong R$.

b. Let F be a field, and let $p \in F[x]$ be a polynomial with $\deg(p) \geq 1$. Let a be a nonzero element of F. Prove that

$$p(a) = 0 \quad \Longleftrightarrow \quad \exists\, q \in F[x] \text{ such that } p(x) = q(x)(x - a).$$

Hint: The Division Algorithm.

c. Prove that there are infinitely many integers a such that the polynomial $p(x) = x^7 + 15x^2 - 30x + a$ is irreducible in $\mathbb{Q}[x]$.

d. What is purple and commutes? (All nonempty answers on this part will be counted as correct.)

2. Let p be a prime number. Show that if q is an irreducible polynomial in $\mathbb{Z}_p[x]$ of degree n, then $\mathbb{Z}_p[x]/(q)$ is a field that contains exactly p^n elements.

3. Show that if G is a group of order 42, then G has a normal subgroup H of order 7, and also a normal subgroup K of order 21.

4. Let R be a commutative ring with identity, and assume that M is a maximal ideal in R. Prove that if x, $y \in R$ are not in M, then $xy \notin M$.

5. Let $n > 2$ be a fixed integer, and let S_n denote the symmetric group of order n.

Note: You are not free to choose n in this problem: it is a fixed positive integer but you don't know what it is.

a. Show that if H is a subgroup of S_n such that every 2-cycle belongs to H, then $H = S_n$.

b. Either prove or find a counterexample: If H is a subgroup of S_n such that every 3-cycle belongs to H, then $H = S_n$.

6. Let F be a field, and let $F[x, y]$ be the ring of polynomials in two variables with coefficients in F (you may assume without proof that $F[x, y]$ is a ring). For example, xy and $x^2 + 2x + xy^3 + 2xy + y + 3$ would be elements of $\mathbb{R}[x, y]$.

a. Prove that

$$I = \{xp(x, y) + yq(x, y) : p, q, \in F[x, y]\}$$

is a nontrivial ideal in $F[x, y]$.

b. Prove that I is not a principal ideal in $F[x, y]$.

7. The two parts of this problem are not related, i.e., you do not have to prove one in order to do the other.

a. Prove that if $\varphi\colon \mathbb{Q}[x] \to \mathbb{Q}[x]$ is an isomorphism, then $\varphi(a) = a$ for each constant polynomial a.

b. Let F be field, and suppose that $\varphi\colon F[x] \to F[x]$ is an isomorphism that satisfies $\varphi(a) = a$ for every constant polynomial a. Show that there exist b, $c \in F$ with $b \neq 0$ such that $\varphi(p)(x) = p(bx + c)$ for all $p \in F[x]$.