MATH 4107 HOMEWORK #4 DUE: April 9, 2007

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Prove that $R = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a field.

2. Problem 4.2 #3.

3. Problem 4.2 #8.

4. Define c and c_0 to be the following sets of infinite sequences of real numbers:

$$\begin{aligned} c &= \Big\{x = (x_1, x_2, \dots) : x_k \in \mathbb{R} \text{ and } \lim_{k\to\infty} x_k \text{ exists}\Big\}, \\ c_0 &= \Big\{x = (x_1, x_2, \dots) : x_k \in \mathbb{R} \text{ and } \lim_{k\to\infty} x_k = 0\Big\}. \end{aligned}$$

You may assume without proof that these are both commutative rings under the following operations: if $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ then

$$x + y = (x_1 + y_1, x_2 + y_2, \dots) \qquad \text{and} \qquad xy = (x_1y_1, x_2y_2, \dots).$$

a. Determine (with proof) whether c or c_0 has a multiplicative identity.

b. Show that c_0 is an ideal in c.

c. Use the First Homomorphism Theorem to show that $c/c_0 \cong \mathbb{R}$.

d. Given $x \in c$, give an explicit description of the coset $x + c_0$.

5. Let I be a nontrivial ideal in the ring of integers $\mathbb{Z}$. Show that there exists an $n > 1$ such that $I = n\mathbb{Z}$.

Hint: I contains a smallest positive element.

Extra Credit (up to 3 points). Problem 4.1 #40.

Note that this is the same as Problem 4.2 #3, except that now the ring may be noncommutative.