MATH 4107 HOMEWORK #5 DUE: April ?, 2007

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Problem 4.4 #6. Let

$$M = 5\mathbb{Z} + 5\sqrt{2}\mathbb{Z} = \{5m + 5n\sqrt{2} : m, n \in \mathbb{Z}\}$$

Show that M is a maximal ideal in

$$R = \mathbb{Z} + \sqrt{2}\mathbb{Z} = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}.$$

Note: You may assume without proof that R is a subring of $\mathbb{R}$, but be sure to show that M is an ideal in R.

Hints for showing M is maximal: Suppose that I is an ideal such that $M \subseteq I \subseteq R$. You have to show that either $I = M$ or $I = R$. If it was the case that $I = M$ then you're done, so suppose that $I \neq M$. In this case, there exists some element $k + \ell\sqrt{2}$ that belongs to I but not to M. Consider $(k + \ell\sqrt{2})\,(k - \ell\sqrt{2})$; this belongs to what ideal? And then write $k = 5m + r$ and $\ell = 5n + s$ where r and s are the remainders after dividing by 5.

2. Problem 4.4 #7. With M and R as in Problem 4.4 #6, show that R/M is a field having 25 elements.

3. Problem 4.5 #14, part a only. Let $F = \mathbb{Z}_{11}$, the integers mod 11. Let $p(x) = x^2 + 1$. Show that p is irreducible in $\mathbb{Z}_{11}[x]$ and that $\mathbb{Z}_{11}[x]/(p)$ is a field containing 121 elements.

Extra Credit (up to 3 points). Problem 4.5 #18. Let F be a finite field. Show that $F[x]$ contains irreducible polynomials of arbitrarily high degree.

Hint: Try to imitate Euclid's proof that there are infinitely many prime numbers in $\mathbb{Z}$.