2.5 #15. Suppose that N is a normal subgroup of a group G, and that $f : GtoG'$ is a homomorphism of G onto G'. Prove that $f(N)$ is a normal subgroup of G'.

Solution

First we must prove that $f(N)$ is a subgroup, and then we must prove that it is normal. Remember that $f(N)$ is the direct image of N:

$$f(N) \;=\; \{f(n) : n \in N\}.$$

Suppose that x, y belong to $f(N)$. This means that $x = f(m)$ and $y = f(n)$ for some m, $n \in N$. Since N is a subgroup, we know that mn belongs to N. Hence $f(mn) \in f(N)$ by definition of $f(N)$. Since f is a homomorphism, we therefore have

$$xy \;=\; f(m)f(n) \;=\; f(mn) \;\in\; f(N).$$

Hence $f(N)$ is closed under compositions.

Now suppose that x belongs to $f(N)$. This means that $x = f(n)$ for some $n \in N$. Since N is a subgroup, we have $n^{-1} \in N$, and therefore $f(n^{-1}) \in f(N)$. Since f is a homomorphism, we therefore have

$$x^{-1} \;=\; f(n)^{-1} \;=\; f(n^{-1}) \;\in\; f(N).$$

Thus $f(N)$ is closed under inverses. So, we conclude that $f(N)$ is indeed a subgroup of G'.

Now we show that $f(N)$ is normal. Suppose that a is any element of G'. We must show that $af(N)a^{-1} \subseteq f(N)$. Recall that

$$af(N)a^{-1} \;=\; \{axa^{-1} : x \in f(N)\}.$$

So, we must show that if x is any element of $f(N)$, then axa^{-1} belongs to $f(N)$. Since f is onto, we know that $a = f(g)$ for some $g \in G$. And since x belongs to $f(N)$, we know that $x = f(n)$ for some $n \in N$. Since N is normal, gng^{-1} belongs to gNg^{-1}, which is equal to N. Hence $gng^{-1} \in N$, and therefore $f(gng^{-1}) \in f(N)$. Since f is a homomorphism, we therefore have

$$axa^{-1} \;=\; f(g)f(n)f(g)^{-1} \;=\; f(gng^{-1}) \;\in\; f(N).$$

Thus we have shown that every element of $af(N)a^{-1}$ belongs to $f(N)$, so $af(N)a^{-1} \subseteq f(N)$. Therefore $f(N)$ is normal by definition. $\square$