

Answer the following questions clearly and completely. Unless specifically stated otherwise, all statements require proof or justification. There are 30 points total.

(7 points) 1. Let G be a group. Let $G \times G$ be the direct product of G with itself. Define $\mu: G \times G \rightarrow G$ by $\mu(a, b) = ab$. Prove that if μ is a homomorphism, then G is abelian.

(7 points) 2. Prove that every integral domain R with a finite number of elements must be a field. Hint: Let R^* denote the set of nonzero elements of R . Show that for each $r \in R^*$, multiplication by r is a one-to-one mapping $R^* \rightarrow R^*$.

3. Let R be a commutative ring.

(6 points) a. Find a surjective ring homomorphism $\varphi: R[x] \rightarrow R$ such that $\ker(\varphi) = xR[x]$.

Hint: You want to find a way to define $\varphi(a_0 + a_1x + \cdots + a_nx^n)$. It must be an element of R , and you want any polynomial of the form $a_1x + \cdots + a_nx^n \in xR[x]$ to map to zero. Then show φ is indeed a homomorphism, that it is onto, and that its kernel is $xR[x]$.

(3 points) b. Prove that $R \cong R[x]/xR[x]$.

(7 points) 4. Let R be an integral domain. Prove that if $a \in R$ is prime, then it is irreducible.