MATH 4107 FINAL EXAM NAME: ______________________

December 10, 2002

Answer the following questions clearly and completely. All statements require proof or justification. There are 50 points total, plus one extra credit problem.

(6 points) 1. Let $G = D_4$ be the dihedral group of symmetries of the square.

a. Let $X = \{v_1, v_2, v_3, v_4\}$ be the set of four vertices (corners) of the square. Define $\varphi\colon D_4 \to \mathrm{Sym}(X)$ by letting $\varphi(\pi)$ be the induced permutation of the vertices. Compute the orbit and stabilizer of v_1 under this action.

b. Now let $X = \{e_1, e_2, e_3, e_4\}$ be the set of four edges of the square, and again let D_4 act on X in the natural way. Compute the orbit and stabilizer of e_1.

c. Finally, let $X = \{d_1, d_2\}$ be the set of two diagonals of the square, and let D_4 act on X in the natural way. Compute the orbit and stabilizer of d_1.

(8 points) 2. Let G be a group and let H and N be normal subgroups of G such that $H \subseteq N \subseteq G$.

a. Define $\varphi\colon G/H \to G/N$ by $\varphi(aH) = aN$. Show that φ is well-defined, is onto, is a homomorphism, and that $\ker(\varphi) = N/H = \{aH : a \in N\}$.

b. Prove that $(G/H)/(N/H) \cong G/N$. Hint: See part a.

3. Let R be a commutative ring. An element $a \in R$ is *nilpotent* if $a^k = 0$ for some positive integer k. Let N be the set of all nilpotent elements, i.e.,

$$N = \{a \in R : a^k = 0 \text{ for some } k > 0\}.$$

(6 points) a. Show that N is an ideal in R. Hint: Closure under addition is the most difficult part.

(6 points) b. Show that if S is an integral domain and $\varphi\colon R \to S$ is a ring homomorphism, then $N \subseteq \ker(\varphi)$.

(6 points) c. Show that R/N has no nilpotent elements other than the zero element of R/N (which is N itself).

CONTINUED ON REVERSE

(5 points) 4. Let $a \in \mathbb{Q}$ be a positive number such that $a^{1/2} \notin \mathbb{Q}$. Find the minimal polynomial for $a^{1/4}$ over $\mathbb{Q}$.

(5 points) 5. Prove that $\mathbb{Q}(\sqrt{7}+\sqrt{3}) = \mathbb{Q}(\sqrt{7}, \sqrt{3})$. Hint: What is $(7-3)/(\sqrt{7}+\sqrt{3})$?

(8 points) 6. Let $K \subseteq L$ be fields such that $\dim_K(L)$ is finite. Suppose that D is an integral domain such that $K \subseteq D \subseteq L$. Prove that D is a field. Hint: If $\alpha \in D$, how many powers of α can be independent?

(5 points EXTRA CREDIT) 7. Let $K \subseteq L \subseteq M$ be fields. Prove that if L is algebraic over K and M is algebraic over L, then M is algebraic over K.