MATH 4317 Real Analysis I

SOME RECOMMENDED PROBLEMS WITH SOLUTIONS

Here are a few practice problems with solutions. Try to work these WITHOUT looking at the solutions! After you write your own solution, you can compare to my solution. Your solution does not need to be identical—there are often many ways to solve a problem—but it does need to be CORRECT.

Problem 1 M. If B_1 and B_2 are subsets of B and if $B = B_1 \cup B_2$, then

$$A \times B \;=\; (A \times B_1) \cup (A \times B_2).$$

Solution

Suppose that $x \in A \times B$. Then $x = (a, b)$ for some $a \in A$ and some $b \in B$. Since $b \in B = B_1 \cup B_2$, we have that b is either in B_1 or it is in B_2 (or both). Therefore either $(a, b) \in A \times B_1$ or $(a, b) \in A \times B_2$, so $x = (a, b) \in (A \times B_1) \cup (A \times B_2)$. This shows that $A \times B \subseteq (A \times B_1) \cup (A \times B_2)$.

For the reverse inclusion, suppose that $x \in (A \times B_1) \cup (A \times B_2)$. Then either $x \in A \times B_1$ or $x \in A \times B_2$. If $x \in A \times B_1$ then $x = (a, b)$ with $a \in A$ and $b \in B_1$. But $B_1 \subseteq B$, so $b \in B$ and therefore $x = (a, b) \in A \times B$. On the other hand, if $x \in A \times B_2$ then $x = (a, b)$ with $a \in A$ and $b \in B_2$. But $B_2 \subseteq B$, so $b \in B$ and therefore $x = (a, b) \in A \times B$. Thus in any case we have $x \in A \times B$. This shows that $(A \times B_1) \cup (A \times B_2) \subseteq A \times B$ and completes the proof. $\square$

Problem 2 G. Let f and g be functions and suppose that $(g \circ f)(x) = x$ for all x in $D(f)$. Show that f is an injection and that $R(f) \subseteq D(g)$ and $R(g) \supseteq D(f)$.

Solution

First we show that f is an injection. Suppose that $x, y \in D(f)$ were such that $f(x) = f(y)$. Then, by definition of composition and by the hypotheses on f and g, we have

$$x \;=\; (g \circ f)(x) \;=\; g(f(x)) \;=\; g(f(y)) \;=\; (g \circ f)(y) \;=\; y.$$

Thus $f(x) = f(y)$ implies $x = y$, and therefore f is an injection.

Now we show that $R(f) \subseteq D(g)$. Here we are trying to show that one set is contained in another set, so we must show that every element of the first set is also an element of the second set. Therefore, suppose that $y \in R(f)$. Then, by definition, $y = f(x)$ for some $x \in D(f)$.

Hence $g(y) = g(f(x)) = (g \circ f)(x) = x$. Since $g(y)$ is defined, this implies that $y \in D(g)$. Thus every element of $R(f)$ is also an element of $D(g)$, so $R(f) \subseteq D(g)$.

Finally, we show that $R(g) \supseteq D(f)$. Suppose that $x \in D(f)$, and let $y = f(x)$. Then, by hypothesis, $x = (g \circ f)(x) = g(f(x)) = g(y)$. Thus x is the image of y, i.e., $x = g(y) \in R(g)$. Hence $D(g) \subseteq R(g)$. ☐

Problem 3 C. Exhibit a one-to-one correspondence between $\mathbf{N}$ and a proper subset of $\mathbf{N}$.

Solution

There are of course many possible proper subsets and many possible one-to-one correspondences to choose from. Here are two.

Set $S = \mathbf{N} \setminus \{1\} = \{2, 3, 4, \dots\}$. Define $f \colon \mathbf{N} \to S$ by $f(n) = n + 1$ for $n \in \mathbf{N}$. Then f is injective since if $f(n) = f(m)$ then $n + 1 = m + 1$ and hence $m = n$. Further, f is surjective since if $n \in S$ then n is an integer bigger than 1, and therefore $n - 1 \in \mathbf{N} = D(f)$, and therefore there is an $m \in \mathbf{N}$ such that $f(m) = n$, namely $m = n - 1$. Thus f is a bijection of $\mathbf{N}$ onto the proper subset S.

For a second example, set $E = \{2n : n \in \mathbf{N}\} = \{2, 4, 6, \dots\}$. Define $g \colon \mathbf{N} \to E$ by $f(n) = 2n$. Then it is again easy to see that g is a bijection of $\mathbf{N}$ onto the proper subset E. ☐

Problem 4 F. Use the argument in Theorem 4.7 to show that there does not exist a rational number s such that $s^2 = 6$.

Solution

Suppose on the contrary that $s = p/q$ is such that $(p/q)^2 = 6$, where p and q are integers with $q \neq 0$. By dividing out common factors, we may assume that p and q have no common integer factors. We are given that $p^2 = 6q^2$. Hence p^2 is even. Now, p is either even or odd. some integer k, or $p = 2k+1$ for some integer k. If p is odd then $p = 2k+1$ for some integer k, so $p^2 = 4k^2 + 4k + 1$, which is odd. This is impossible, so p must be even, i.e., $p = 2k$ for some k. Therefore $6q^2 = p^2 = 4k^2$, so $3q^2 = 2k^2$ and thus $3q^2$ is even. By reasoning similar to that used above (fill in the details), it follows that q must be even. Hence both p and q are divisible by 2, which contradicts the fact that p and q have no divisors in common. Therefore there can be no such integers p and q. ☐

Problem 4 H. If $\xi \in \mathbf{R}$ is irrational and $r \in \mathbf{R}$, $r \neq 0$, is rational, show that $r + \xi$ and $r\xi$ are irrational.

Solution

a. Since r is rational, we know that $r = m/n$ for some integers m and n with $n \neq 0$. Suppose that $r + \xi = p/q$ for some integers p and q with $q \neq 0$. Then $\xi = p/q - r = p/q - m/n = (np - mq)/(nq)$, and therefore ξ is rational. We have thus proved that $r + \xi$ rational implies ξ rational, which is the contrapositive of the statement ξ irrational implies $r + \xi$ rational. Note that we really didn't need the assumption $r \neq 0$ in this part, it's still true even if $r = 0$.

b. Again, since r is a NONZERO rational, we know that $r = m/n$ for some integers m and n with $m \neq 0$ AND $n \neq 0$. Suppose that $r\xi = p/q$ for some integers p and q with $q \neq 0$. Then $\xi = p/(qr) = (np)/(qm)$, so ξ is rational since np and qm are integers with $qm \neq 0$. Thus we have proved the contrapositive statement $r\xi$ rational implies ξ rational. Note that we did use the assumption $r \neq 0$ in this part. In fact, the problem would be false without this assumption, since $0\xi = 0$ is rational for every ξ. $\square$

Problem 5 C. If $a > -1$, $a \in \mathbf{R}$, show that $(1+a)^n \geq 1 + na$ for all $n \in \mathbf{N}$. This inequality is called *Bernoulli's Inequality*. (Hint: use mathematical induction.)

Solution

The hint suggests using mathematical induction, so let use it. Mathematical induction says that we just have to prove the following two things: (a) Base step: Show the statement is true for the specific case $n = 1$, and (b) Inductive step: Show that IF the statement is true for some n THEN it is also true for $n + 1$.

Base step $n = 1$. We have to show that $(1+a)^1 \geq 1 + 1 \cdot a$. This is trivial because $(1+a)^1 = 1 + a$ and $1 + 1 \cdot a = 1 + a$. Hence, not only is it true that $(1+a)^1 \geq 1 + 1 \cdot a$, but it is in fact true that $(1+a)^1 = 1 + 1 \cdot a$ (although for general n we cannot put in an equality, only the inequality).

Inductive step. ASSUME that the statement is true for some $n \geq 1$. We then have to somehow show that the statement is also true for $n + 1$. Here is the reasoning:

$$\begin{aligned}
(1+a)^{n+1} &= (1+a)\,(1+a)^n && \text{by definition} \\
&\geq (1+a)\,(1+na) && \text{by our assumption that the statement is true for } n \\
&= 1 + (n+1)a + na^2 && \text{algebra} \\
&\geq 1 + (n+1)a && \text{since } na^2 \geq 0.
\end{aligned}$$

Therefore the statement is true for $n + 1$ as well. $\square$

Problem 6 C. Give an example of a set of rational numbers which is bounded but does not have a rational supremum.

Solution

There are of course many ways to construct such a set, here is just one way. For each $n \in \mathbf{N}$ define $x_n = \pi - 1/n$. Then x_n is not rational since π is not rational, but since $x_n < \pi$ we know from Theorem 6.10(a) that there is a rational number r_n such that $x_n < r_n < \pi$. Let R be the set of these rational numbers, i.e., $R = \{r_n : n \in \mathbf{N}\}$. This set is bounded both above and below since

$$\pi - 1 \;\le\; \pi - 1/n \;=\; x_n \;<\; r_n \;<\; \pi. \tag{1}$$

That is, $\pi - 1$ is a lower bound, and π is an upper bound. I claim that π is the least upper bound, i.e., $\pi = \sup R$, and therefore that R does not have a rational supremum.

To prove this claim, I'll use Lemma 6.3. We just have to show that (i) there are no elements $r \in R$ such that $\pi < r$, and (ii) if $x < \pi$ then there is an element $r \in R$ such that $x < r$. Part (i) is true simply because π is an upper bound for R, or by considering equation (1). To show that (ii) is true, suppose that $x < \pi$. Then there is an n such that $x + 1/n < \pi$. But then $x < \pi - 1/n = x_n < r_n$. Since $r_n \in R$, we have shown that (ii) holds. It therefore follows from Lemma 6.3 that π is the supremum of R.

Of course, in this argument you could have used any irrational number ξ in place of π, and everything would still work the same. □

Problem 7 E. Let $I_n = (n, +\infty)$ for $n \in \mathbf{N}$. Show that this sequence of intervals is nested, but that there is no common point.

Solution

First, let's show that the intervals I_n are nested. That means that we have to show that $I_n \supseteq I_{n+1}$ for every n. But $I_n = \{x \in \mathbf{R} : x > n\}$ and $I_{n+1} = \{x \in \mathbf{R} : x > n+1\}$, so if $x \in I_{n+1}$ then $x > n + 1 > n$, and hence $x \in I_n$.

Next, let's show that these intervals share no common point, i.e., that $\bigcap\limits_{n=1}^{\infty} I_n = \emptyset$. Suppose that $x \in I_n$ for every n. Then, by definition of I_n, we must have $x > n$ for every n. But we know that if $x \in \mathbf{R}$ then there is a natural number m such that $x < m$ (Archimedean Property 6.6), so this is impossible. Hence there cannot be any x that lies in every I_n. □

Problem 8 D. If w_1 and w_2 are strictly positive, show that the definition

$$(x_1, x_2) \cdot (y_1, y_2) \;=\; x_1 y_1 w_1 + x_2 y_2 w_2$$

yields an inner product on $\mathbf{R}^2$. Generalize this to $\mathbf{R}^p$.

NOTE: This is an example of a *weighted inner product.* It allows some directions in the coordinate system to be weighted more heavily than others.

Solution

We have to show that properties (i)–(v) of the definition of an inner product in Definition 8.3 are satisfied.

(i) Given $x = (x_1, x_2) \in \mathbf{R}^2$, we have

$$x \cdot x \;=\; (x_1, x_2) \cdot (x_1, x_2) \;=\; x_1 x_1 w_1 + x_2 x_2 w_2 \;=\; x_1^2 w_1 + x_2^2 w_2. \tag{2}$$

Since $x_1^2,\ x_2^2 \ge 0$ and $w_1,\ w_2 > 0$, it follows that $x \cdot x \ge 0$.

(ii) Suppose that $x \cdot x = 0$. Then, as in equation (2), we have $x_1^2 w_1 + x_2^2 w_2 = 0$. But since every term in this sum is positive, we must then have $x_1^2 w_1 = x_2^2 w_2 = 0$. Since $w_1,\ w_2 \ne 0$, it therefore follows that $x_1^2 = x_2^2 = 0$, which implies $x = (x_1, x_2) = (0, 0) = 0$.

Conversely, suppose that $x = 0$. Then $x \cdot x = 0^2 \cdot w_1 + 0^2 \cdot w_2 = 0$.

(iii) If $x = (x_1, x_1) \in \mathbf{R}^2$ and $y = (y_1, y_2) \in \mathbf{R}^2$ then

$$x \cdot y \;=\; (x_1, x_2) \cdot (y_1, y_2) \;=\; x_1 y_1 w_1 + x_2 y_2 w_2 \;=\; x_2 y_2 w_2 + x_1 y_1 w_1 \;=\; (y_1, y_2) \cdot (x_1, x_2) \;=\; y \cdot x.$$

(iv) If $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2) \in \mathbf{R}^2$ then

$$\begin{aligned} x \cdot (y + z) \;=\; (x_1, x_2) \cdot (y_1 + z_1, y_2 + z_2) \;&=\; x_1(y_1 + z_1) w_1 + x_2(y_2 + z_2) w_2 \\ &=\; x_1 y_1 w_1 + x_2 y_2 w_2 \;+\; x_1 z_1 w_1 + x_2 z_2 w_2 \\ &=\; (x_1, x_2) \cdot (y_1, y_2) \;+\; (x_1, x_2) \cdot (z_1, z_2) \\ &=\; x \cdot y + x \cdot z. \end{aligned}$$

This fact, combined with the commutative property (iii), can easily be shown to imply that we also have $(x + y) \cdot z = x \cdot z + y \cdot z$.

(v) If $a \in \mathbf{R}$ and $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbf{R}$, then

$$(ax) \cdot y \;=\; (ax_1, ax_2) \cdot (y_1, y_2) \;=\; (ax_1) y_1 w_1 + (ax_2) y_2 w_2 \;=\; a(x_1 y_1 w_1 + x_2 y_2 w_2) \;=\; a(x \cdot y).$$

Combining this with the commutative property (iii), we see that we also have $x \cdot (ay) = a\,(x \cdot y)$.

The fact that this definition of $x \cdot y$ satisfies properties (i)–(v) means that it is an inner product for $\mathbf{R}^2$. In particular, each choice of w_1 and w_2 leads to a distinct inner product for

$\mathbf{R}^2$, with the particular choice $w_1 = w_2 = 1$ corresponding to the "usual" dot product on $\mathbf{R}^2$. Thus the same vector space can have many different inner products. Which one is "best" usually depends on the problem that we are trying to solve.

It's easy to see how to extend this definition to higher dimensions: just define the inner product of $x = (x_1, \dots, x_p)$ with $y = (y_1, \dots, y_p)$ by

$$(x_1, \dots, x_p) \cdot (y_1, \dots, y_p) \;=\; x_1 y_1 w_1 + \cdots + x_p y_p w_p,$$

where $w_1, \dots, w_p$ are fixed real numbers. $\square$

Problem 9 C. Prove that the intersection of any *finite* collection of open sets is open in $\mathbf{R}^p$. (Hint: use 9.3(b) and induction.)

Solution

Theorem 9.3(b) says that the intersection of any *two* open sets is open. We will use this to prove that if $U_1, \dots, U_n$ are n open sets then $U_1 \cap \cdots U_n$ is open. The proof is by induction.

Note that the base step corresponds to $n = 2$ and is true by Theorem 9.3(b). Therefore we can move directly to the inductive step. Assume that the intersection of n open sets is open. Let $U_1, \dots, U_n, U_{n+1}$ be any $n+1$ open sets. Then, by our inductive assumption, $U = U_1 \cap \cdots \cap U_n$ is open. Further, we know that the intersection of any two open sets is open. Therefore $U \cap U_{n+1}$ is open. But

$$U \cap U_{n+1} \;=\; (U_1 \cap \cdots \cap U_n) \cap U_{n+1} \;=\; U_1 \cap \cdots \cap U_n \cap U_{n+1}.$$

Hence $U_1 \cap \cdots \cap U_{n+1}$ is open, and therefore the intersection of any $n+1$ open sets is open. $\square$

Problem 10 C. A point x is a cluster point of a set $A \subseteq \mathbf{R}^p$ if and only if every neighborhood of x contains infinitely many points of A.

Solution

$\Rightarrow$. Suppose that x is a cluster point of A, and let N be any neighborhood of x. Then N contains an open set G that contains x, and by definition of open set there exists an $r > 0$ such that $B_r(x) \subseteq G \subseteq N$. Now consider the smaller balls $B_{r/n}(x)$ for $n \in \mathbf{N}$. They are all contained in $B_r(x)$, and hence in N. Furthermore, each $B_{r/n}(x)$ is a neighborhood of x, so it follows from the fact that x is cluster point of A that each $B_{r/n}(x)$ contains a point $y_n \in A \cap B_{r/n}(x)$ with $y_n \neq x$. Note that each y_n lies in N. Furthermore, although some of the y_n may coincide, there must be infinitely many distinct values represented, since $\|x - y_n\| < r/n$ (if there were only finitely many distinct values, then there would be a point closest to x, but then we could choose n so that r/n was closer than that point). Hence $\{y_n\}_{n\in\mathbf{N}}$ is an infinite set of points that lie in both A and N.

$\Leftarrow$. Suppose that every neighborhood of x contains infinitely many points of A. Let N be any neighborhood of x. Then we know that there exist infinitely many points $\{y_n\}_{n\in\mathbf{N}}$ that lie in both A and N. Hence at least one of these points, say $y = y_n$, must be different from x. But then $y \in A \cap N$ with $y \neq x$, so x satisfies the requirements of a cluster point. $\square$

Problem 11 B. Prove directly that the entire space $\mathbf{R}^2$ is not compact.

Solution

Of course, $\mathbf{R}^p$ is not bounded, so it can't be compact by the Heine–Borel Theorem. But the point is: can you show *directly from the definition* of compact set that $\mathbf{R}^p$ isn't compact?

Here is one method. Let $U_n = B_n(0)$, i.e., U_n is the open ball of radius n centered at the origin. Note that $U_1 \subseteq U_2 \subseteq \cdots$. Further, we certainly have $\mathbf{R}^p \subseteq \bigcup_{n\in\mathbf{N}} U_n$. However, if we select only *finitely* many sets out of this open cover, say $U_{k_1}, \dots, U_{k_n}$ with $k_1 < \cdots < k_n$, then the largest one, U_{k_n}, contains all the balls $U_{k_1}, \dots, U_{k_n}$, so $U_{k_1} \cup \cdots \cup U_{k_n} = U_{k_n} \neq \mathbf{R}^p$. Hence $\{U_n : n \in \mathbf{N}\}$ is one cover of $\mathbf{R}^p$ by open sets that contains no finite subcover of $\mathbf{R}^p$. Therefore $\mathbf{R}^p$ isn't compact. $\square$

Problem 12 A. If A and B are connected subsets of $\mathbf{R}^p$, give examples to show that $A \cup B$, $A \cap B$, $A \backslash B$ can be either connected or disconnected.

Solution

Let $A = [0, 2]$ and $B = [1, 3]$. Then $A \cup B = [0, 3]$, $A \cap B = [2, 3]$, and $A \backslash B = [0, 1)$ are all connected.

On the other hand, let $A = [0, 1]$ and $B = [2, 3]$. Then $A \cup B = [0, 1] \cup [2, 3]$ is disconnected (what is one disconnection?). If $A = [0, 3]$ and $B = [1, 2]$, then $A \backslash B = [0, 1) \cup (2, 3]$ is disconnected. However, in one dimension, the connected sets are just the intervals, and there is no example of connected intervals A, B such that $A \cap B$ not connected. But if we move to $\mathbf{R}^2$, then we can find such sets. For example, let A be the top half of a semicircle of radius 1 and let B be the bottom half, i.e.,

$$A = \{(x, y) : x^2 + y^2 = 1,\ y \geq 0\}, \qquad B = \{(x, y) : x^2 + y^2 = 1,\ y \leq 0\}.$$

Then $A \cap B = \{(-1, 0),\ (0, 1)\}$ contains only two points, hence is disconnected.

It is easy to create analogues of these examples in higher dimensions as well. $\square$

Problem 14 E. Let $X = (x_n)$ be a sequence in $\mathbf{R}^p$ and let $\lim(\|x_n\|) = 0$. Show that $\lim(x_n) = 0$. However, give an example in $\mathbf{R}$ to show that the convergence of $(|x_n|)$ may not imply the convergence of (x_n).

Solution

Assume that $\lim(\|x_n\|) = 0$, and fix any $\varepsilon > 0$. Then

$$\exists\, N > 0 \text{ such that } \quad n \geq N \implies \big|\|x_n\| - 0\big| < \varepsilon.$$

Therefore, if $n \geq N$ then

$$\|x_n - 0\| \;=\; \|x_n\| \;=\; \big|\|x_n\| - 0\big| < \varepsilon.$$

This exactly says that $x_n \to 0$. In summary, if the norms of the x_n converge to the number zero, then the x_n themselves converge to the zero vector.

The point of the second part of this problem is that this statement doesn't have to remain true if we change the word "zero" to something else. For example, consider the sequence of real numbers that alternates from 1 to -1, i.e., $x_n = (-1)^n$. The norms (= absolute values in one dimension) converge to 1 because $|x_n| = 1$ for every n. But the vectors x_n themselves do NOT converge. $\square$

Problem 14 O. Give an example of a convergent sequence (x_n) of strictly positive real numbers such that $\lim(x_n^{1/n}) = 1$. Give an example of a divergent sequence with this property.

Solution

Problem 14M showed that if $\lim(x_n^{1/n}) < 1$ then the x_n converge to zero, while Problem 14N showed that if $\lim(x_n^{1/n}) > 1$ then the x_n diverge. The point of this problem is that in the remaining case, $\lim(x_n^{1/n}) = 1$, either scenario can happen. In other words, we can have numbers that converge and whose nth roots converge, or we can have numbers that diverge and yet whose nth roots converge. Maybe this isn't such a surprise: the numbers can be growing, but this growth could be countermanded when we take nth roots.

The first example is easy: what number has the easiest nth root? Simply take $x_n = 1$ for every n. Then certainly (x_n) converges, to the number 1 of course, and the nth roots are exactly the same thing, so they converge to 1 as well.

On the other hand, suppose we let $x_n = n$. Then certainly (x_n) diverges. But it is also true that $x_n^{1/n} \to 1$. To see this you could note that by L'Hopital's rule for limits that have the form ∞/∞,

$$\ln x_n^{1/n} \;=\; \ln n^{1/n} \;=\; \frac{\ln n}{n} \;=\; \frac{1/n}{1} \;\to\; 0.$$

Therefore, $x_n^{1/n} = e^{\ln x_n^{1/n}} \to e^0 = 1$.

Another example of a divergent sequence (x_n) whose nth roots $(x_n^{1/n})$ do converge to 1 is $(x_n) = (1, 2, 1, 2, 1, 2, \ldots)$. This sequence doesn't converge, but $(1, 2^{1/2}, 1, 2^{1/4}, 1, 2^{1/6}, \ldots)$ does converge to 1. $\square$

Problem 15 D. If X and Y are sequences in $\mathbf{R}^p$ and if $X + Y$ converges, do X and Y converge and have $\lim(X + Y) = \lim X + \lim Y$?

Solution

A theorem in the books states that if X and Y do converge, then $X + Y$ converges as well. The point of this problem is that the *converse* of this statement doesn't need to be true: just because $X + Y$ converges, it need not be true that X and Y themselves converge. For example, consider the sequences of numbers given by $x_n = (-1)^n$ and $y_n = -(-1)^n$. Neither $X = (x_n)$ nor $Y = (y_n)$ converge, but $X + Y = (x_n + y_n) = (0, 0, 0, \ldots)$ does converge. $\square$