MATH 4317 Real Analysis I

SOME RECOMMENDED PROBLEMS WITH SOLUTIONS

Here are a few practice problems with solutions. Try to work these WITHOUT looking at the solutions! After you write your own solution, you can compare to my solution. Your solution does not need to be identical—there are often many ways to solve a problem—but it does need to be CORRECT.

Problem 16 H. Let $X = (x_n)$ and $Y = (y_n)$ be sequences in $\mathbf{R}^p$ and let $Z = (z_n)$ be the "shuffled" sequence defined by

$$z_1 = x_1, \quad z_2 = y_1, \quad \ldots, z_{2n} = x_n, \quad z_{2n+1} = y_n, \ldots.$$

Is it true that Z is convergent if and only if X and Y are convergent and $\lim X = \lim Y$?

Solution

Yes, it's true.

$\Rightarrow$. Suppose that Z is convergent. Since X and Y are both subsequences of Z, they must then be convergent and must converge to the same limit as Z, i.e., $\lim X = \lim Z = \lim Y$.

$\Leftarrow$. Suppose X and Y both converge and that $\lim X = \lim Y = x$. We claim that Z also converges to x. To see this, choose $\varepsilon > 0$. Then:

$$\exists\, N_1 > 0 \text{ such that} \quad n \geq N_1 \implies \|x - x_n\| < \varepsilon,$$

$$\exists\, N_2 > 0 \text{ such that} \quad n \geq N_2 \implies \|x - y_n\| < \varepsilon.$$

Let $N = \max 2N_1, 2N_2 - 1$. Suppose $n \geq N$. If n is even, say $n = 2k$, then $k \geq N_1$, so $\|z_n - x\| = \|x_k - x\| < \varepsilon$. On the other hand, if n is odd, say $n = 2j - 1$, then $j \geq N_2$, so $\|z_n - x\| = \|y_j - x\| < \varepsilon$. In any case, $\|z_n - x\| < \varepsilon$ for all $n \geq N$, so $z_n \to x$. $\square$

Problem 16 I. Show directly that the following are Cauchy sequences.

(a) $(1/n)$.

Solution

The sequence is $x_n = 1/n$. So, if $m > n$ then

$$|x_m - x_n| = \left|\frac{1}{m} - \frac{1}{n}\right| = \frac{1}{n} - \frac{1}{m} \leq \frac{1}{n}.$$

Let $\varepsilon > 0$ is given, and let $N > 1/\varepsilon$. If $m, n \geq N$, then

$$|x_m - x_n| \;\leq\; 1/n \;\leq\; 1/N \;<\; \varepsilon.$$

Therefore (x_n) is Cauchy. $\square$

(b) $\left(\dfrac{n+1}{n}\right)$.

<u>Solution</u>

With $x_n = (n+1)/n$ and $m > n$, we have

$$|x_m - x_n| \;=\; \left|\frac{m+1}{m} - \frac{n+1}{n}\right| \;=\; \frac{n-m}{mn} \;\leq\; \frac{n}{mn} \;\leq\; \frac{1}{m}.$$

Again, take $N > 1/\varepsilon$. Then, if $m, n \geq N$, then $|x_m - x_n| \leq 1/m \leq 1/N < \varepsilon$. $\square$

(c) $\left(1 + \dfrac{1}{1!} + \cdots + \dfrac{1}{n!}\right)$.

<u>Solution</u>

Note that

$$k! \;=\; 1 \cdot 2 \cdot 3 \cdots k \;\geq\; 2^{k-1}.$$

Therefore, with $x_n = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!}$ and $m > n$, we have

$$|x_m - x_n| \;=\; \sum_{k=n+1}^{m} \frac{1}{k!} \;\leq\; \sum_{k=n+1}^{\infty} \frac{1}{k!} \;\leq\; \sum_{k=n+1}^{\infty} \frac{1}{2^{k-1}} \;=\; \sum_{k=n}^{\infty} \frac{1}{2^k} \;=\; \frac{1}{2^{n-1}}.$$

Therefore, if we take $N > 1 + \log_2(1/\varepsilon)$, then for $n \geq N$ we have

$$|x_m - x_n| \;\leq\; 2^{-n+1} \;\leq\; 2^{-N+1} \;<\; 2^{-\log_2(1/\varepsilon)} \;=\; \varepsilon. \quad \square$$

Problem 17 B. For each $n \in \mathbf{N}$, let g_n be defined for $x \geq 0$ by the formula

$$g_n(x) \;=\; \begin{cases} nx, & 0 \leq x \leq 1/n, \\ \dfrac{1}{nx}, & 1/n < x. \end{cases}$$

Show that $\lim(g_n(x)) = 0$ for all $x > 0$.

Solution

Before you do anything—draw a picture!!

Let $x > 0$ be fixed. Hence, then there exists $N > 0$ such that $1/N < x$. Therefore, if $n \geq N$ then $1/n \leq 1/N < x$, so $g_n(x) = 1/(nx)$. But $r = 1/x$ is just a fixed, positive number, so for $n \geq N$ we have $g_n(x) = r/n \to 0$.

This proof is rigorous enough, but if we like, we could also write it out with explicit ε's. To do this, $\varepsilon > 0$. Then let $N > 0$ be large enough so that both $1/N < x$ and $1/N < \varepsilon x$. Then, $n \geq N$ implies $1/n \leq 1/N < x$, so $g_n(x) = 1/(nx) \leq 1/(Nx) < \varepsilon$. This shows that $g_n(x) \to 0$ for each individual x.

NOTE: Since $g_n(1/n) = 1$ for every n, it is NOT true that $g_n \to 0$ uniformly on $[0, \infty)$. For uniform convergence, we would have to have $\|g_n\|_\infty \to 0$, but we don't have this because $\|g_n\|_\infty = 1$ for every n. On the other hand, you will show in problem 17L that the convergence *is* uniform if we restrict the domain to $[c, \infty)$ with $c > 0$. □

Problem 17 I. Suppose that (x_n) is a convergent sequence of points which lies, together with its limit x, in a set $D \subseteq \mathbf{R}^p$. Suppose that (f_n) converges on D to the function f. Is it true that $f(x) = \lim f_n(x_n)$?

Solution

No. For example, think about the very simplest case of convergent functions, i.e., when every f_n is just f. Then $\lim f_n(x_n) = \lim f(x_n)$. Does this have to equal $f(x)$? The answer is no if f has a discontinuity at x. For example, define

$$f_n(x) \;=\; f(x) \;=\; \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Then let $x_n = 1/n$ and $x = 0$. Then $f_n(x_n) = f(x_n) = f(1/n) = 1$ for every n, but $f(x) = f(0) = 0$. □

Problem 17 L. Show that the convergence in Exercise 17B is not uniform on the domain $x \geq 0$, but that it is uniform on a set $x \geq c$, where $c > 0$.

Solution

I already pointed out that the convergence is not uniform on the domain $[0, \infty)$, because $g_n(1/n) = 1$ for every n. But suppose that we consider instead the domain $D = [c, \infty)$, where

c is a fixed real number greater than 0. We have to show that $\|g_n\|_\infty \to 0$, when we compute the L^∞ norm on this domain $[c, \infty)$.

To do this, let N be such that $1/N < c$. If $n \geq N$ then $1/n \leq 1/N < c$. Hence when we consider $x \in [c, \infty)$ we automatically have $x \geq c > 1/n$ for all $n \geq N$. Hence for $n \geq N$ we have $g_n(x) = 1/(nx)$ for all $x \in [c, \infty)$. On the domain $[c, \infty)$, the function $g_n(x) = 1/(nx)$ is decreasing, and therefore is at its largest when $x = c$. Thus, on the domain $[c, \infty)$,

$$\|g_n\|_\infty \;=\; \sup_{x \geq c} |g_n(x)| \;=\; \sup_{x \geq c} g_n(x) \;=\; \sup_{x \geq c} \frac{1}{nx} \;=\; \frac{1}{nc} \;\to\; 0.$$

Therefore $g_n \to 0$ uniformly on this domain. $\square$