MATH 4317 Real Analysis I

SOME RECOMMENDED PROBLEMS WITH SOLUTIONS

Here are a few practice problems with solutions. Try to work these WITHOUT looking at the solutions! After you write your own solution, you can compare to my solution. Your solution does not need to be identical—there are often many ways to solve a problem—but it does need to be CORRECT.

Problem 20 J. Let h be defined for $x \neq 0$, $x \in \mathbf{R}$, by

$$h(x) = \sin(\tfrac{1}{x}), \qquad x \neq 0.$$

Show that no matter how h is defined at $x = 0$, it will be discontinuous at $x = 0$.

Solution

Suppose that we were able to define $h(0)$ so that h was continuous at $x = 0$. Consider the sequences $x_n = 1/(2\pi n)$ and $y_n = 1/(\frac{\pi}{2} + 2\pi n)$. We have $x_n \to 0$ and $y_n \to 0$, but $h(x_n) = 0$ for every n while $h(y_n) = 1$ for every n, which contradicts that fact that we must have both $0 = h(x_n) \to h(0)$ and $1 = h(y_n) \to h(0)$. Thus there is no way to define $h(0)$ so that h becomes continuous at $x = 0$. $\square$

Problem 21 E. Let g be any linear function from $\mathbf{R}^2$ to $\mathbf{R}^3$. Show that not every element of $\mathbf{R}^3$ is the image under g of a vector in $\mathbf{R}^2$.

Solution

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ be the two standard basis vectors in $\mathbf{R}^2$. Then $g(e_1)$ and $g(e_2)$ are two particular vectors in $\mathbf{R}^3$. Every vector $g(x)$ in the range of g is a linear combination of these two vectors, because if $x = (a, b) = ae_1 + be_2$ then $g(x) = ag(e_1) + bg(e_2)$. All of these vectors lie on the plane determined by $g(e_1)$ and $g(e_2)$ (or possibly on a single line if $g(e_1)$ and $g(e_2)$ lie on one line through the origin). So, we believe that the range of g can be at most a plane within $\mathbf{R}^3$, and cannot be the entire space.

To prove this precisely, let z be any nonzero vector in $\mathbf{R}^3$ that is perpendicular to both $g(e_1)$ and $g(e_2)$ (there are many ways to show that such a vector exists, for example by using cross products). If z was in the range of g then we would have $z = g(x) = ag(e_1) + bg(e_2)$ for some vector $x = (a, b) \in \mathbf{R}^2$. But z is perpendicular to both $g(e_1)$ and $g(e_2)$, so it is perpendicular to any linear combination of them, and hence is perpendicular to itself:

$z \cdot z = z \cdot (ag(e_1) + bg(e_2)) = a(z \cdot g(e_1)) + b(z \cdot g(e_2)) = 0$. This is a contradiction because z is nonzero, so $z \cdot z = \|z\|^2 \neq 0$. Hence this vector z cannot be in the range of g. $\square$

Problem 22 B. Let $h\colon \mathbf{R} \to \mathbf{R}$ be defined by

$$h(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Exhibit an open set G such that $h^{-1}(G)$ is not open in $\mathbf{R}$, and a closed set F such that $h^{-1}(F)$ is not closed in $\mathbf{R}$.

<u>Solution</u>

If $G = (0, 2)$ then $h^{-1}(G) = [0, 1]$, while if $F = \{0\}$ then $h^{-1}(F) = (-\infty, 0) \cup (1, \infty)$. $\square$

Problem 22 E. If $f\colon \mathbf{R}^p \to \mathbf{R}^q$ is continuous on $\mathbf{R}^p$ and $\alpha < \beta$, show that the set

$$\{x \in \mathbf{R}^p : \alpha \leq f(x) \leq \beta\}$$

is closed in $\mathbf{R}^p$.

<u>Solution</u>

Just note that $\{x \in \mathbf{R}^p : \alpha \leq f(x) \leq \beta\} = f^{-1}([\alpha, \beta])$ and appeal to the Global Continuity Theorem. Or, to do it directly, suppose that x_n is a sequence of points in this set that converges to x, and show that x is a also in the set. This is easy because you're assuming $\alpha \leq f(x_n) \leq \beta$ for every n, and f is continuous, so $f(x) = \lim_{n\to\infty} f(x_n)$. $\square$

Problem 22 F. A subset $D \subseteq \mathbf{R}^p$ is disconnected if and only if there exists a continuous function $f\colon D \to \mathbf{R}$ such that $f(D) = \{0, 1\}$. Hint: Theorem 22.1.

<u>Solution</u>

"$\Rightarrow$" Suppose that $D \subseteq \mathbf{R}^p$ is disconnected, and let (A, B) be a disconnection of D. Let $D_1 = D \cap A$ and $D_2 = D \cap B$. Then, by definition of disconnection, $D_1 \cap D_2 = \emptyset$. Therefore, we can define a function $f\colon D \to \mathbf{R}$ by setting

$$f(x) = \begin{cases} 0, & x \in D_1, \\ 1, & x \in D_2. \end{cases}$$

Since D_1, D_2 are both nonempty, we have $f(D) = \{0, 1\}$. Hence the only thing that we have to show is that f is continuous. There are many ways to do this; I will do it by using the Global Continuity Theorem (Theorem 22.1).

Let G be any open set in $\mathbf{R}$. We must show that there exists an open set $G_1 \in \mathbf{R}^p$ such that $f^{-1}(G) = G_1 \cap D$. There are several cases to consider.

If G contains neither 0 nor 1 then $f^{-1}(G) = \emptyset = \emptyset \cap D$. The set $G_1 = \emptyset$ is open, so this case is done.

If G contains both 0 and 1 then $f^{-1}(G) = D = \mathbf{R}^p \cap D$, and $G_1 = \mathbf{R}^p$ is open.

If G contains 0 but not 1 then $f^{-1}(G) = D_1 = A \cap D$, and $G_1 = A$ is open by definition of disconnection.

Finally, if G contains 1 but not 0 then $f^{-1}(G) = D_2 = B \cap D$, and $G_1 = B$ is open.

"$\Leftarrow$" Suppose that there exists a continuous function $f\colon D \to \mathbf{R}$ such that $f(D) = \{0, 1\}$. The interval $I_1 = (-1/2, 1/2)$ is open in $\mathbf{R}$, so by the Global Continuity Theorem, there exists an open set $A \subseteq \mathbf{R}^p$ such that $f^{-1}(I_1) = A \cap D$. Similarly, $I_2 = (1/2, 3/2)$ is open, so there exists an open set $B \subseteq \mathbf{R}^p$ such that $f^{-1}(I_2) = B \cap D$. I will show that (A, B) is a disconnection of D.

First, $R(f) = f(D) = \{0, 1\}$ and $0 \in I_1$, so $A \cap D = f^{-1}(I_1) \neq \emptyset$. Similarly, $1 \in I_2$, so $B \cap D = f^{-1}(I_2) \neq \emptyset$.

Second,

$$(A \cap D) \cap (B \cap D) \;=\; f^{-1}(I_1) \cap f^{-1}(I_2) \;=\; f^{-1}(I_1 \cap I_2) \;=\; f^{-1}(\emptyset) \;=\; \emptyset.$$

Finally, since $R(f) \subseteq I_1 \cup I_2$, we have

$$(A \cap D) \cup (B \cap D) \;=\; f^{-1}(I_1) \cup f^{-1}(I_2) \;=\; f^{-1}(I_1 \cup I_2) \;=\; D.$$

Since A, B are both open, we conclude that (A, B) is indeed a disconnection and therefore D is disconnected. □

Problem 23 L. If $f\colon [0, 1] \to [0, 1]$ is continuous, show that f has a fixed point in $[0, 1]$. Hint: Consider $g(x) = f(x) - x$.

<u>Solution</u>

We must show that there is an $x \in [0, 1]$ such that $g(x) = 0$.

If $g(0) = 0$ or $g(1) = 0$ then we are done. So, suppose that $g(0) \neq 0$ and $g(1) \neq 0$. Then $g(0) = f(0) - 0 = f(0) > 0$ and $g(1) = f(1) - 1 < 0$ since $f(0)$ and $f(1)$ are both between 0 and 1. Hence g is strictly positive at $x = 0$ and strictly negative at $x = 1$. Since g is continuous, it therefore follows from the Intermediate Value Theorem that there must be some x such that $g(x) = 0$. □