# 15. Subsequences & Combinations

Given $X = (x_n)_{n=1}^{\infty} = (x_1, x_2, x_3, \ldots)$

Given indices $n_1 < n_2 < n_3 < \cdots$

Let $y_k = x_{n_k}, \quad k = 1, 2, \ldots$

Then $(y_k)_{k=1}^{\infty} = (y_1, y_2, y_3, \ldots)$

$$= (x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$$

$$= (x_{n_k})_{k=1}^{\infty}$$

is a subsequence of $(x_n)_{n=1}^{\infty}$.

Ex: $(x_2, x_3, x_5, x_7, x_{11}, \ldots)$ is a subsequence

$y_1 \quad y_2 \quad y_3$

$(x_7, x_{11}, x_2, x_3, \ldots)$ is not a subsequence.

## Lemma

$x_n \longrightarrow x \iff x_{n_k} \longrightarrow x \ \forall$ subsequence $(x_{n_k})$

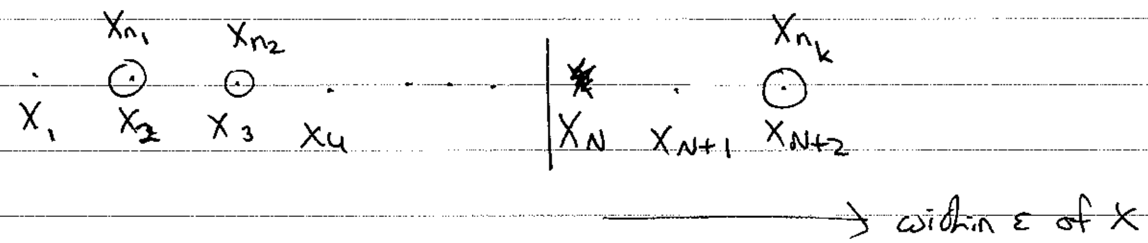
**Proof:**

$\Rightarrow$ Assume $x_n \to x$. Choose $\varepsilon > 0$. Must show:

Goal. $\boxed{\exists K > 0 \text{ s.t. } k > K \Rightarrow \|x - x_{n_k}\| < \varepsilon.}$

We know

$$\exists N > 0 \text{ st. } n > N \Rightarrow \|x - x_n\| < \varepsilon.$$




Since $n_1 < n_2 < \cdots$ is an increasing sequence of integers, $\exists K$ st. $n_K > N$.

Hence if $k > K$ then $n_k > n_K > N$, so

$\|x - x_{n_k}\| < \varepsilon$.

$\Leftarrow$ Trivial $n_k = k$: the full sequence is one of the subsequences.

$\blacksquare$

Exercise:

~~Suppose~~ Let $(x_n)$ be a sequence in $\mathbb{R}^p$ and $x \in \mathbb{R}^p$.
Suppose that

$\forall$ subsequence $(y_k)$ of $(x_n)$,

$\exists$ subsequence $(z_j)$ of $(y_k)$ s.t. $z_j \longrightarrow x$.

Prove that $x_n \longrightarrow x$.

Hint: What does $x_n \not\rightarrow x$ mean? Proof by contradiction