23. Uniform Continuity

In an earlier theorem, we showed that f is continuous at all points in its domain if:

$$\forall a \in D(f),\ \forall \varepsilon > 0,\quad \exists \delta > 0 \text{ such that}$$

$$x \in D(f)\ \&\ \|x - a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon.$$

The value of δ can depend on both a & ε.

If δ doesn't depend on a, we say that f is <u>uniformly continuous</u>. Precisely:

<u>Definition</u>

f is <u>uniformly continuous on $A \subseteq D(f)$</u> if:

$$\forall \varepsilon > 0,\quad \exists \delta > 0 \text{ such that}$$

$$\forall a, x \in A,\qquad \|x - a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon.$$

Main point: Given ε, there is a <u>single</u> δ such that

$\|x - a\| < \delta \Rightarrow \|f(x) - f(a)\|$ for <u>all</u> x, a !

δ is <u>independent</u> of a !

Exercise: $f(x) = \frac{1}{x}$ is NOT uniformly continuous on $(0, \infty)$.

It is uniformly continuous on $[c, \infty)$ if $c > 0$.

Theorem

If f is continuous & $K \subseteq D(f)$ is compact, then f is uniformly continuous on K.

Proof:

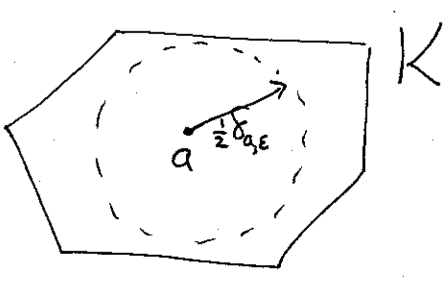
Since f is continuous on K, for each $a \in K$ & each $\varepsilon > 0$ there exists a number $\delta_{a,\varepsilon}$ such that

$$x \in K \ \& \ \|x - a\| < \delta_{a,\varepsilon} \implies \|f(x) - f(a)\| < \frac{\varepsilon}{2}$$

Then for a given ε the collection of open balls

$$\left\{ B_{\frac{1}{2}\delta_{a,\varepsilon}}(a) : a \in K \right\}$$

covers all of K (why?)



Since K is compact, it must be covered by finitely many of these balls, say the ones centered at the points $a_1, \ldots, a_N$. That is,

$$K \subseteq B_{\delta_{a_1,\varepsilon}}(a_1) \cup \cdots \cup B_{\delta_{a_N,\varepsilon}}(a_N). \quad (*)$$

Define

$$\delta_\varepsilon = \frac{1}{2}\min\{\delta_{a_1,\varepsilon}, \ldots, \delta_{a_N,\varepsilon}\}.$$

We will show that

$$\forall a, x \in K, \quad \|x-a\| < \delta_\varepsilon \implies \|f(x)-f(a)\| < \varepsilon.$$

This is exactly what we have to show to prove that f is uniformly continuous on K.

So, to do this, suppose $a, x \in K$ are such that $\|x-a\| < \delta_\varepsilon$.

From $(*)$, we must have

$$a \in B_{\frac{1}{2}\delta_{a_i,\varepsilon}}(a_i) \quad \text{for some } i = 1, \ldots, N.$$

This means that

$$\|a - a_i\| < \tfrac{1}{2}\delta_{a_i,\varepsilon} < \delta_{a_i,\varepsilon}.$$

But then

$$\begin{aligned}
\|x - a_i\| &= \|x - a + a - a_i\| \\
&\leq \|x - a\| + \|a - a_i\| \\
&< \delta_\varepsilon + \tfrac{1}{2}\delta_{a_i,\varepsilon} \\
&\leq \tfrac{1}{2}\delta_{a_i,\varepsilon} + \tfrac{1}{2}\delta_{a_i,\varepsilon} \\
&= \delta_{a_i,\varepsilon}.
\end{aligned}$$

Therefore

$$\|f(a) - f(a_i)\| < \frac{\varepsilon}{2} \quad \& \quad \|f(x) - f(a_i)\| < \frac{\varepsilon}{2},$$

so

$$\begin{aligned}
\|f(a) - f(x)\| &= \|f(a) - f(a_i) + f(a_i) - f(x)\| \\
&\leq \|f(a) - f(a_i)\| + \|f(a_i) - f(x)\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon. \quad \blacksquare
\end{aligned}$$

Contractions

A function f is Lipschitz if $\exists K > 0$ s.t.

$$\forall x, y \in D(f), \qquad \|f(x) - f(y)\| \leq K \|x - y\|.$$

Note K is not unique: if one value of K works, then so will any $K' \geq K$.

If we can take $K < 1$ then f is called a contraction.

Fixed Point Theorem

Let $f : \mathbb{R}^p \longrightarrow \mathbb{R}^p$ be a contraction. (Note $D(f) = \mathbb{R}^p$, $R(f) \subseteq \mathbb{R}^p$).

Then f has a unique fixed point.

That is, $\exists$ unique $u \in \mathbb{R}^p$ s.t. $f(u) = u$.

Proof:

By def., $\exists K < 1$ s.t.

$$\forall x, y \in \mathbb{R}^p, \qquad \|f(x) - f(y)\| \leq K \|x - y\|.$$

Let x_1 be any particular point in $\mathbb{R}^p$.

Define

$$x_2 = f(x_1)$$
$$x_3 = f(x_2)$$
$$x_4 = f(x_3)$$
$$\vdots$$

We'll show that (x_n) converges in $\mathbb{R}^p$, & that the limit is the unique fixed point of f.

Note:

$$\|x_3 - x_2\| = \|f(x_2) - f(x_1)\| \leq K\|x_2 - x_1\|.$$

$$\|x_4 - x_3\| = \|f(x_3) - f(x_2)\|$$
$$\leq K\|x_3 - x_2\| \leq K^2\|x_2 - x_1\|$$

etc. By induction,

$$\|x_{n+1} - x_n\| \leq K^{n-1}\|x_2 - x_1\|.$$

Hence if $m > n$ then

$$\|x_m - x_n\| \leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+1} - x_n\|$$
$$\leq \left(K^{m-2} + K^{m-3} + \cdots + K^{n-1}\right)\|x_2 - x_1\|$$
$$\leq \frac{K^{n-1}}{1-K}\|x_2 - x_1\|.$$

Exercise: It follows from this that (x_n) is a Cauchy sequence, hence converges. Let $u = \lim_{n\to\infty} x_n$.

Then since f is continuous,

$$f(u) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = u.$$

Thus u is a fixed point of f.

Suppose v was also a fixed point, i.e., $f(v)=v$.

Then

$$\|u-v\| = \|f(u)-f(v)\| \leq K\|u-v\|.$$

If $u \neq v$ then we could divide by $\|u-v\|$ to get $1 \leq K$, which is a contradiction. Therefore we must have $u=v$, so the fixed point is unique. $\boxed{\xi}$