$\mathbb{R}$ is the simplest example of a vector space. Optimization of functions: What is the closest function?

4. Algebraic Properties of $\mathbb{R}$

Addition & Multiplication of real numbers are binary operations on $\mathbb{R}$

$+$ is a function from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$

$$+((a,b)) = a+b$$

Just use this notation for simplicity

Ex: $+((2,3)) = 5 = 2+3$

Usual Properties hold

See p28 of Bartle.

$$a+b = b+a$$
$$(a+b)+c = a+(b+c)$$
$$\exists\, 0 \in \mathbb{R} \text{ s.t. } a+0 = a$$
$$\forall a \in \mathbb{R}\ \exists\, (-a) \in \mathbb{R} \text{ s.t. } a+(-a) = 0$$

Define $a-b = a+(-b)$

$$a\cdot b = b\cdot a$$
$$(a\cdot b)\cdot c = a\cdot(b\cdot c)$$
$$\exists\, 1 \in \mathbb{R} \text{ s.t. } a\cdot 1 = a$$
$$\forall a \in \mathbb{R}\setminus\{0\}\ \exists\, \left(\tfrac{1}{a}\right) \in \mathbb{R} \text{ s.t. } a\cdot\left(\tfrac{1}{a}\right) = 1$$

$$a\cdot(b+c) = a\cdot b + a\cdot c$$
$$(a+b)\cdot c = a\cdot c + b\cdot c$$

Familiar consequences of these properties. $a+x=b$ is solvable

Lemma k^2 even $\Leftrightarrow$ k even

Rational numbers 2

$a/b = a \cdot (\frac{1}{b})$ $\quad\quad \mathbb{Q} = \{a/b : a, b \in \mathbb{Z},\ b \neq 0\}$

Theorem

$\nexists\, r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Proof

Suppose $r \in \mathbb{Q}$, $r = p/q$, has the property $(p/q)^2 = 2$.

Assume p, q have no common factors. (p/q is in lowest terms)

$p^2 = 2q^2$ so $2 \mid p^2$. p^2 is even. p is even.

$p = 2k. \Rightarrow 4k^2 = 2q^2 \Rightarrow q^2 = 2k^2$

$\Rightarrow q^2$ even $\Rightarrow q$ even

Contradiction. $\blacksquare$

Still leaves the question

Q. Does $\exists\, r \in \mathbb{R}$ s.t. $r^2 = 2$?

How would you prove this?