MATH 6337 Real Analysis Fall 2007

Lecture notes by C. Heil
following the text "Real Analysis", 2nd Ed. by G. Folland

Chapter 0: Prologue

This is background material, not all of which will be needed immediately. Review especially:

0.1 The Language of Set Theory

Notation $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

0.3 Cardinality

Countable versus uncountable sets.

0.5 The Extended Real Number System

Shorthand: $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$
Extended real line

We often let functions take extended real values, e.g., we may take $f: \mathbb{R} \to \overline{\mathbb{R}}$, which allows the possibility that $f(x) = \infty$ or $f(x) = -\infty$.

NOTE: Always remember that "∞" is <u>not a real number</u>. It is a convenient shorthand, but is not a number.

Example: If $S \subseteq \mathbb{R}$ is bounded above, i.e., $\exists\, M \in \mathbb{R}$ s.t. $s \leq M \;\forall s \in S$, then the completeness property of $\mathbb{R}$ says that S has a least upper bound. We call this the <u>supremum</u> of S:

$$\sup(S) = \text{least upper bound for } S.$$

On the other hand, if S is not bounded above, then we <u>define</u>

$$\sup(S) = \infty.$$

There is no upper bound for S, but still, in the sense of the extended reals, S has a supremum.

Notational conventions:

$$x \pm \infty = \pm \infty \quad \text{for } x \in \mathbb{R}$$

$$x \cdot (\pm \infty) = \pm \infty \quad \text{for } x > 0$$

$$\infty + \infty = \infty$$

$$-\infty - \infty = -\infty$$

$$\infty - \infty \text{ is UNDEFINED}$$

Somewhat counterintuitively, it is usually "correct" to take

$$0 \cdot (\pm \infty) = 0$$

Be familiar with

$$\liminf_{n \to \infty} x_n = \sup_{k \geq 1} \inf_{n \geq k} x_n \quad \text{(smallest accumulation point of } (x_n)\text{)}$$

$$\limsup_{n \to \infty} x_n = \inf_{k \geq 1} \sup_{n \geq k} x_n \quad \text{(largest ...)}$$

$\lim_{n \to \infty} x_n$ exists & is finite $\iff$ $\liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n$ and are finite.

0.6 Metric Spaces

Each metric on a set X induces a topology, i.e., a definition of which sets are open or closed.

If we have a topology, then we can define what it means for a function on X to be continuous.

For later: 0.2 Orderings.

At some point, we may need to use the Axiom of Choice in the form of Zorn's Lemma.