MATH 6337 (C. Heil) HOMEWORK #1 Due date: September 4, 2007

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

NOTES. Your grade on this and every homework is based on my ability to understand and evaluate what you have written. Therefore it is ESSENTIAL that you COMMUNICATE CLEARLY in your writing. ALL problems require either a proof or an explanation of what you have done, using complete, correct English sentences (mathematical symbols are simply abbreviations for words or phrases and may be used as parts of sentences). I will read EXACTLY what you write and will take what you write literally. I will not fill in missing steps or guess at what you "really mean." Any symbols that you introduce that are not standard must be explained. YOUR EXPLANATIONS NEED NOT BE LENGTHY TO BE CLEAR, but you must carefully and logically demonstrate the validity of your solution to each problem. The words "show," "demonstrate," etc. are all synonyms for "prove." All statements that you make require proof, e.g., if I ask "is statement A true?" then it is not sufficient to answer "yes" or "no", you must explain WHY the answer is yes or no.

1. Problem 1.2 #4. An algebra $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is closed under countable increasing unions.

2. Problem 1.3 #8. If $(X, \mathcal{M}, \mu)$ is a measure space and $E_j \in \mathcal{M}$ for $j \in \mathbb{N}$, then $\mu(\liminf E_j) \le \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \ge \limsup \mu(E_j)$ provided that we have $\mu(\cup E_j) < \infty$.

3. Problem 1.3 #11. A finitely additive measure is a measure if and only if it is continuous from below. If $\mu(X) < \infty$, then a finitely additive measure is a measure if and only if it is continuous from above.

4. Problem 1.3 #14. If μ is a semifinite measure and $\mu(E) = \infty$, for $C > 0$ there exists $F \subseteq E$ with $C < \mu(F) < \infty$.

Hint: Define

$$C = \sup\{\mu(F) : F \subseteq E, \mu(F) < \infty\}.$$

If $C < \infty$, then there exist sets $F_k \subseteq E$ with finite measure such that $\mu(F_k) \to C$. Consider the sets $F = \cup F_k$ and $A = E \backslash F$.