MATH 6337 HOMEWORK #4 Due date: October 4, 2007

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Suppose that (X, Mc, μ) is a complete measure space, and let $f,\, g\colon X \to \bar{\mathbb{R}}$ Show that if f is measurable and $f = g$ μ-a.e., then g is measurable.

2. Let $f_k,\, f\colon X \to \mathbb{R}$ be measurable functions that are finite a.e. We say that f_k *converges in measure to* f, and write $f_k \overset{\mathrm{m}}{\to} f$, if for each $\varepsilon > 0$, the measure of the set where f_k differs from f by more than ε converges to zero, i.e.,

$$\forall\, \varepsilon > 0, \quad \mu\{|f - f_k| > \varepsilon\} \to 0.$$

Show that if $f_k \overset{\mathrm{m}}{\to} f$ and $g_k \overset{\mathrm{m}}{\to} g$, then $f_k + g_k \overset{\mathrm{m}}{\to} f + g$.

3. Let $(X, \mathcal{M}, \mu)$ be a complete measure space, and suppose that $f_n\colon X \to \mathbb{R}$ is measurable and that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for almost every x. Prove that f is measurable, no matter how we define it at the remaining points.

4. The measure space throughout this problem is $[a, b]$ under Lebesgue measure.

a. Give an example of a discontinuous function $f\colon [a, b] \to \mathbb{R}$ such that there exists a continuous function $g\colon [a, b] \to \mathbb{R}$ with $f = g$ a.e. Observe that Problem#1 above implies that f is measurable.

b. Give an example of a function $f\colon [a, b] \to \mathbb{R}$ that is continuous at almost every point, i.e.,

$$\lim_{y\to x} f(y) = f(x) \quad \text{for almost every } x \in [a, b],$$

but such that there is no continuous g with $f = g$ a.e. (Proof not required, just give the example.)

c. Prove that any function $f\colon [a, b] \to \mathbb{R}$ that is continuous at almost every point is measurable.

Hint: One way is to construct a sequence of simple functions that converge to f almost everywhere.