MATH 6337 (C. Heil) HOMEWORK #5 Due date: October 18, 2007

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Problem 2.2 #14. If $f\colon X \to [0,\infty]$ is measurable, and let $\lambda(E) = \int_E f\,d\mu$ for $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$, and for any measurable $g\colon X \to [0,\infty]$, we have $\int g\,d\lambda = \int fg\,d\mu$.

2. Problem 2.2 #17. Assume Fatou's Lemma and deduce the Monotone Convergence Theorem from it.

3. Problem 2.3 #19. Suppose $f_n \in L^1(X)$ and $f_n \to f$ uniformly.

a. If $\mu(X) < \infty$, then $f \in L^1(X)$ and $\int f_n \to \int f$ (this is the *Uniform Convergence Theorem*).

b. Give an example where $\mu(X) = \infty$ and the conclusion of part a fails.
Hint: Consider Lebesgue measure on $\mathbb{R}$.

4. Prove the following "absolute continuity" property of the integral: If $f \in L^1(X)$ and $\varepsilon > 0$, then there exists a $\delta > 0$ such that

$$\mu(A) < \delta \implies \int_A |f| < \varepsilon.$$

5. Problem 2.3 #20. (The Generalized Dominated Convergence Theorem). Suppose f_n, g_n, f, $g \in L^1(X)$, suppose $f_n \to f$ pointwise a.e. and $g_n \to g$ pointwise a.e., suppose $|f_n| \le g_n$ a.e., and suppose that $\int g_n \to \int g$. Then $\int f_n \to \int f$.
Hint: Rework the proof of the DCT.