MATH 6337 HOMEWORK #8 Due date: December 4, 2007

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

NOTE: Some parts of some of the problems are propositions in the text, and proofs are sketched there. However, I want complete, detailed proofs.

1. Let ν be a σ-finite signed measure on $(X, \mathcal{M})$ and μ a σ-finite positive measure on $(X, \mathcal{M})$. Assume that $\nu \ll \mu$, and let $g = d\nu/d\mu$, i.e., $d\nu = g\,d\mu$.

(a) Show that if $g \geq 0$, then for any measurable $f \geq 0$ we have

$$\int f\,d\nu = \int fg\,d\mu. \tag{1}$$

(b) Now let g be an arbitrary extended μ-integrable function. Show that if $f \in L^1(\nu)$, then $fg \in L^1(\mu)$, and equation (1) is still valid.

2. Problem 3.2 #16. Suppose that μ, ν are positive, σ-finite measures on $(X, \mathcal{M})$ such that $\nu \ll \mu$. Let $\lambda = \mu + \nu$.

(a) Show that $\nu \ll \lambda$ and $\mu \ll \lambda$.

(b) Let $f = d\nu/d\lambda$. Show that $0 \leq f < 1$ μ-a.e. and that $d\nu/d\mu = f/(1-f)$ μ-a.e.

3. Let ν be a complex measure on $(X, \mathcal{M})$. Show that the following statements hold.

(a) $|\nu(E)| \leq |\nu|(E)$ for all $E \in \mathcal{M}$.

(b) $\nu \ll |\nu|$, and there exists g with $|g| = 1$ $|\nu|$-a.e. such that $d\nu = g\,d|\nu|$.

(c) If $f \in L^1(\nu)$, then $\left|\int f\,d\nu\right| \leq \int |f|\,d|\nu|$.

4. Problem 3.3 #21. Let ν be a complex Borel measure on $(X, \mathcal{M})$. Prove the following equivalent characterizations of $|\nu|$ (there are hints in the lecture notes).

(a) $|\nu|(E) = \sup\left\{ \sum_{k=1}^{n} |\nu(E_k)| : n \in \mathbb{N}, E_k \in \mathcal{M}, E = \bigcup_{k=1}^{n} E_k \text{ disjointly} \right\}$.

(b) $|\nu|(E) = \sup\left\{ \sum_{k=1}^{\infty} |\nu(E_k)| : E_k \in \mathcal{M}, E = \bigcup_{k=1}^{\infty} E_k \text{ disjointly} \right\}$.

(c) $|\nu|(E) = \sup\left\{ \left| \int_E f\,d\nu \right| : |f| \leq 1 \ |\nu|\text{-a.e.} \right\}$.