MATH 7337 HOMEWORK #1 DUE: September 11, 2006

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

Definition.

$$C_b^m(\mathbb{R}) = \{f\colon \mathbb{R} \to \mathbb{C} \ :\ f, f', \dots, f^{(m)} \text{ exist and are continuous and bounded}\},$$

$$C_c^m(\mathbb{R}) = \{f \in C_b^m(\mathbb{R}) \ :\ f \text{ is compactly supported}\}.$$

1. Prove Young's Inequality: If $1 \le p, q \le \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, then

$$\forall\, f \in L^p(\mathbb{R}), \quad \forall\, g \in L^q(\mathbb{R}), \quad \|f * g\|_r \ \le\ \|f\|_p\, \|g\|_q.$$

2. a. Prove that if $f \in L^1(\mathbb{R})$ and $g \in C_c^m(\mathbb{R})$ then $f * g \in C_b^m(\mathbb{R})$, and furthermore

$$D^j(f * g) \ =\ f * D^j g, \quad j = 1, \dots, m.$$

b. Prove that if we assume in addition that f has compact support, then $f * g \in C_c^m(\mathbb{R})$.

3. Let M be a closed subspace of $L^1(\mathbb{R})$. Prove that the following two statements are equivalent.

a. M is translation-invariant, i.e., if $f \in M$ and $a \in \mathbb{R}$ then $T_a f \in M$.

b. M is an ideal in $L^1(\mathbb{R})$, i.e., if $f \in M$ and $g \in L^1(\mathbb{R})$ then $f * g \in M$.

Hint for a $\Rightarrow$ b: Hahn–Banach and $(L^1)^* = L^\infty$.

4. Let $\{k_\lambda\}_{\lambda>0}$ be an approximate identity.

a. Prove that if $f \in C_0(\mathbb{R})$ then $f * k_\lambda$ converges to f uniformly as $\lambda \to \infty$.

b. Fix $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Prove that if $f \in L^p(\mathbb{R})$ and $g \in L^{p'}(\mathbb{R})$ then $f * g \in C_0(\mathbb{R})$.

Hint: Let f_n, $g_n \in C_c(\mathbb{R})$ converge to f, g.

5. Define $f_k = \chi_{[-1,1]} * \chi_{[-k,k]}$.

a. Find an explicit formula for f_k and show that $\|f_k\|_\infty = 2$.

b. Show that $\lim_{k\to\infty} \|\widehat{f_k}\|_1 = \infty$.
Hints: Take the Fourier transform of f_k and make the change of variables $\eta = 2\pi k\xi$.

c. Show that $A(\mathbb{R}) = \text{range}(\mathcal{F}) \subsetneq C_0(\mathbb{R})$.
Hints: You can assume that $\mathcal{F}$ is injective. Consider the Inverse Mapping Theorem and the fact that f_k, $\widehat{f_k} \in L^1(\mathbb{R})$, so the Inversion Theorem applies.

d. Show that $C_c^2(\mathbb{R}) \subset A(\mathbb{R})$.

e. Show that $A(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.

6. Suppose that $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and that

$$f(0+) \;=\; \lim_{t\to 0^+} f(t), \qquad f(0-) \;=\; \lim_{t\to 0^-} f(t)$$

both exist, but $f(0-) \neq f(0+)$.
Let $G(x) = e^{-\pi x^2}$ be the Gauss kernel, and set $G_\lambda(x) = \lambda G(\lambda x)$.

a. Prove that

$$\lim_{\lambda\to\infty} (f * G_\lambda)(0) \;=\; \lim_{\lambda\to\infty} \langle f, G_\lambda\rangle \;=\; \frac{f(0+) + f(0-)}{2}.$$

Note: Although f need not be in $L^2(\mathbb{R})$, the "inner product" $\langle f, G_\lambda\rangle = \int f(x)\,\overline{G_\lambda(x)}\,dx$ is well-defined since G_λ is bounded.

b. Prove that $(\hat{f}\,\widehat{G_\lambda})^\vee = f * G_\lambda$.

c. Prove that $\langle \hat{f}, \widehat{G_\lambda}\rangle = \langle f, G_\lambda\rangle$.

d. Suppose $\hat{f}(\xi)$ is real for all ξ, and that $\hat{f}(\xi) > 0$ for all $|\xi| > R$. Prove that $\int_{|\xi|>R} \hat{f}(\xi)\,d\xi < \infty$.
Hint: Consider $\liminf_{\lambda\to\infty} \int_{|\xi|>R} \hat{f}(\xi)\,\widehat{G_\lambda}(\xi)\,d\xi$.

e. Prove that there is no f satisfying the assumptions on $\hat{f}$ in part d. Can you show that if $\hat{f}$ is real then it must change sign infinitely many times?