MATH 7337 HOMEWORK #2 DUE: September 29, 2006

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Prove that our definition of the Fourier transform on $L^2(\mathbb{R})$ is well-defined, i.e., it is independent of the choice of sequence $\{f_n\}_{n\in\mathbb{N}}$.

2. Suppose that $f \in L^1(\mathbb{R})$ is such that $\hat{f} \in L^2(\mathbb{R})$. Prove that $f \in L^2(\mathbb{R})$.

3. Let H be a Hilbert space. A continuous linear operator $A\colon H \to H$ is *Hilbert–Schmidt* if there exists an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ for H such that

$$\|A\|_{\mathrm{HS}}^2 \;=\; \sum_{n=1}^{\infty} \|Ae_n\|^2 \;<\; \infty.$$

a. Prove that $\|A\|_{\mathrm{HS}}$ does not depend on the choice of orthonormal basis, i.e., if $\{f_n\}_{n\in\mathbb{N}}$ is another orthonormal basis for H, then $\sum \|Ae_n\|^2 = \sum \|Af_n\|^2$.

b. Prove that the Hilbert–Schmidt norm dominates the operator norm, i.e., $\|A\| \le \|A\|_{\mathrm{HS}}$.

c. Suppose that A is a compact, self-adjoint operator. Then by the Spectral Theorem, there exists an orthonormal basis $\{e_n\}$ for $\overline{\mathrm{range}(A)}$ consisting of eigenvectors of A, and corresponding nonzero real eigenvalues λ_n, such that $Af = \sum \lambda_n \langle f, e_n\rangle\, e_n$. Prove that A is Hilbert–Schmidt if and only if $\sum |\lambda_n|^2 < \infty$, and in this case $\|A\|_{\mathrm{HS}}^2 = \sum |\lambda_n|^2$.

d. Suppose that A is an integral operator on $L^2(\mathbb{R})$ with kernel k, i.e.,

$$Af(x) \;=\; \int k(x,y)\, f(y)\, dy.$$

Prove that if $k \in L^2(\mathbb{R}^2)$, then A is Hilbert–Schmidt and $\|A\|_{\mathrm{HS}} = \|k\|_2$.

4. a. Let A, B be self-adjoint but possibly unbounded operators on a Hilbert space H. Prove that if $f \in \mathrm{domain}(AB) \cap \mathrm{domain}(BA)$, then

$$\|Af\|\,\|Bf\| \;\ge\; \frac{1}{2}\left|\bigl\langle [A,B]f,\, f\bigr\rangle\right|,$$

where $[A,B] = AB - BA$ is the *commutator* of A and B.

b. Show that equality holds in part a if and only if $Af = icBf$ for some $c \in \mathbb{R}$.

c. Apply part a to the position and momentum operators $Pf(x) = xf(x)$ and $Mf(x) = \frac{1}{2\pi i} f'(x)$ to derive the Classical Uncertainty Principle (for the case $x_0 = \xi_0 = 0$).