MATH 7337 HOMEWORK #4 DUE: November 13, 2006

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

Definition 1. Given f_k, $f \in C_c(\mathbb{R})$, we say that $f_k \to f$ in $C_c(\mathbb{R})$ if there exists a compact set $K \subset \mathbb{R}$ such that $\operatorname{supp}(f_k) \subset K$ for every k, and $\|f - f_k\|_\infty \to 0$ as $k \to \infty$.

This is the convergence criterion for the inductive limit topology $C_c(\mathbb{R})$ corresponding to writing $C_c(\mathbb{R}) = \bigcup\{C(K) : K \text{ compact}\}$. Here $C(K) = \{f \in C(\mathbb{R}) : \operatorname{supp}(f) \subset K\}$ is a Banach space with respect to the L^∞-norm topology.

1. Let δ_n denote the point mass at n measure, and let $\mu = \sum_{n\in\mathbb{Z}} \delta_n$ be the delta train. We proved earlier that $\mu \in \mathcal{S}'(\mathbb{R})$.

a. Show that μ is an unbounded positive Borel measure on $\mathbb{R}$.

b. Show directly that $f \mapsto \langle f, \mu\rangle$ is an unbounded linear functional on $C_c(\mathbb{R})$ with respect to the L^∞-norm topology on $C_c(\mathbb{R})$.

c. Show directly that $f \mapsto \langle f, \mu\rangle$ is a continuous linear functional on $C_c(\mathbb{R})$ with respect to the topology given in Definition 1.

2. Prove that δ' (distributional derivative of δ) does not belong to $M_b(\mathbb{R})$.

3. Let δ_n denote the point mass at n measure, and set

$$\mu_n = \frac{1}{n}(\delta_1 + \cdots + \delta_n).$$

a. Show that $\mu_n \xrightarrow{\text{w}^*} 0$.

b. Show that $\varphi(\xi) = \lim_{n\to\infty} \widehat{\mu_n}(\xi)$ exists for every ξ, but φ is not continuous. In particular, $\widehat{\mu_n} \not\to \hat{0}$, i.e., weak* convergence does not imply convergence of the corresponding Fourier transforms to the expected limit even if the Fourier transforms do converge.

c. Suppose that ν_n, $\nu \in M_b(\mathbb{R})$ and $\nu_n \xrightarrow{\text{w}^*} \nu$. Show that if $\varphi \in C(\mathbb{R})$ and $\widehat{\nu_n}(\xi) \to \varphi(\xi)$ pointwise, then $\varphi = \hat{\nu}$. Thus, weak* convergence combined with convergence of the corresponding Fourier transforms to a *continuous* function implies that the limit is the expected function.

Hints: Weak* convergent sequences are bounded (this is a consequence of the Uniform Boundedness Principle). Show that $\langle \hat{f}, \varphi\rangle = \langle f, \nu\rangle$ for all $f \in \mathcal{F}C_c^\infty(\mathbb{R})$.

4. Let $\varphi \in C(\mathbb{R})$ be given. Let w be the Fejer kernel, and set $w_\lambda(x) = \lambda w(\lambda x)$. Define $f_\lambda = (\varphi\,\widehat{w_\lambda})^\vee$, i.e.,

$$f_\lambda(x) \;=\; \int_{-\lambda}^{\lambda} \varphi(\xi)\left(1 - \frac{|\xi|}{\lambda}\right) e^{2\pi i\xi x}\, d\xi.$$

Prove that the following statements are equivalent.

a. $\varphi = \hat{\nu}$ for some $\nu \in M_b(\mathbb{R})$.

b. $f_\lambda \in L^1(\mathbb{R})$ for every λ, and $\sup_\lambda \|f_\lambda\|_1 < \infty$.

Hints for b $\Rightarrow$ a. Define $\nu_n = f_n\, dx \in M_b(\mathbb{R})$. Apply Alaoglu's Theorem to find a weak* convergent subsequence.