Hints and Solution Sketches for Exercises and Additional Problems

Exercises from Chapter 1

1.8 Hint: The Lebesgue Dominated Convergence Theorem.

1.14 Hint: (c) First prove the result assuming that $f \in C_c(\mathbb{R})$, so f is uniformly continuous. Then use the fact that $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for p finite to approximate an arbitrary $L^p(\mathbb{R})$ function by a function in $C_c(\mathbb{R})$.

1.16 Hint: Show that $|e^{2\pi i\eta x}-1| \le \max\bigl\{2, 2\pi|\eta x|\bigr\}$ (see the "proof by picture" in Figure 1.8). Hence, for any x we have $|e^{2\pi i\eta x}-1| \to 0$ as $\eta \to 0$. For $p<\infty$, apply the Lebesgue Dominated Convergence Theorem.

1.23 Hints: (a) Here are three approaches, all variations on the same theme.
First, show that

$$|(f*g)(x)| \;\le\; \int \Bigl(\bigl|f(y)\bigr|\,\bigl|g(x-y)\bigr|^{1/p}\Bigr)\,\bigl|g(x-y)\bigr|^{1/p'}\,dy,$$

and apply Hölder's Inequality with exponents p and p' to the two factors.
Second, recall that

$$\|f*g\|_p \;=\; \sup\bigl\{|\langle f*g, h\rangle| : \|h\|_{p'} = 1\bigr\}.$$

Show that

$$|\langle f*g, h\rangle| \;\le\; \int |f(y)|\,\bigl\langle |T_y g|,\, |h|\bigr\rangle\,dy,$$

and apply Hölder's Inequality to $\bigl\langle |T_y g|,\, |h|\bigr\rangle$.
Third, write out $\|f*g\|_p$ as an iterated integral, and apply Minkowski's Integral Inequality (Problem B.8).

(b) Show that

$$|(f*g)(x)| \;\le\; \int \Bigl(\bigl|f(y)\bigr|^{p/r}\,\bigl|g(x-y)\bigr|^{q/r}\Bigr)\,\bigl|f(y)\bigr|^{p(\frac{1}{p}-\frac{1}{r})}\,\bigl|g(x-y)\bigr|^{q(\frac{1}{q}-\frac{1}{r})}\,dy,$$

and apply Hölder's Inequality for a product of three functions (see Problem B.12) using exponents r, p_1, p_2, where

$$\frac{1}{p_1} = \frac{1}{p} - \frac{1}{r}, \qquad \frac{1}{p_2} = \frac{1}{q} - \frac{1}{r}.$$

1.32 Hints: (a) To follow the method of Theorem 1.29, show uniform continuity directly by using Hölder's Inequality and the fact that translation is strongly continuous on $L^{p'}(\mathbb{R})$.

To follow the method of Exercise 1.31, use the fact that $1 < p < \infty$ implies $1 < p' < \infty$, and hence $C_c(\mathbb{R})$ is dense in both $L^p(\mathbb{R})$ and $L^{p'}(\mathbb{R})$.

(b) For a counterexample, consider g identically 1.

1.33 Solution sketch for $m = 1$. We have that

$$\frac{(f*g)(x+h) - (f*g)(x)}{h} = \int f(y)\, \frac{g(x+h-y) - g(x-y)}{h}\, dy.$$

The integrand converges pointwise a.e. to $f(y)\, g'(x-y)$ as $h \to 0$. Use the Mean Value Theorem to show that, as a function of y,

$$\left| f(y)\, \frac{g(x+h-y) - g(x-y)}{h} \right| \le |f(y)|\, \|g'\|_\infty \in L^1(\mathbb{R}).$$

Then apply the Lebesgue Dominated Convergence Theorem.

1.37 Hint: Show that if $\widehat{g}(\xi) \neq 0$ for a.e. ξ then $g \notin g * L^1(\mathbb{R})$ (examples are the Dirichlet function d or the Gaussian function $g(x) = e^{-x^2}$).

1.46 Hint: Apply an argument similar to the one used in Theorem 1.45, using the fact that Theorem B.90 implies that the Fundamental Theorem of Calculus holds for g on every interval $[a, b]$.

1.52 Hints: Apply Hölder's Inequality to

$$\int |f(x) - f(x-t)|\, |k_\lambda(t)|^{1/p} \cdot |k_\lambda(t)|^{1/p'}\, dt,$$

or apply Minkowski's Integral Inequality.

1.53 Hint: Show that

$$\|f - f * k_\lambda\|_\infty \le \int \|f - T_t f\|_\infty\, |k_\lambda(t)|\, dt,$$

and then split the integral into $|t| < \delta$ and $|t| \ge \delta$.

1.54 Hints: For the last part of the problem, by Exercise 1.32 we know that $f * k_\lambda$ will be uniformly continuous on $\mathbb{R}$. Show that if $f * k_\lambda \to f$ uniformly on $\mathbb{R}$, then f is uniformly continuous. However, not every element of $C_b(\mathbb{R})$ is uniformly continuous, e.g., consider $f(x) = \sin x^2$.

1.61 Hints: (a) ⇒ (b). Choose $f \in J$ and $g \in L^1(\mathbb{R})$. If $f * g \notin J$ then by the Hahn–Banach Theorem there exists $\varphi \in L^1(\mathbb{R})^* = L^\infty(\mathbb{R})$ such that $\langle h, \varphi \rangle = 0$ for all $h \in J$ while $\langle f * g, \varphi \rangle \neq 0$.

(b) ⇒ (a). Let $\{k_\lambda\}_{\lambda>0}$ be an approximate identity, and consider $T_a f * k_\lambda$.

1.62 Hints: Show that J is translation-invariant and hence is an ideal.

For the opposite inclusion, consider $T_a g * k_\lambda$ where $\{k_\lambda\}_{\lambda>0}$ is an approximate identity.

1.64 Hints: By making a change of variables, using the half-angle formula $\sin^2 x = (1 - \cos 2x)/2$, and integration by parts, show that

$$\int w \;=\; \int \left(\frac{\sin \pi x}{\pi x}\right)^2 dx \;=\; \lim_{R\to\infty} \frac{1}{\pi} \int_{-R}^{R} \frac{\sin x}{x}\, dx.$$

Note that while $\frac{\sin x}{x}$ is not an integrable function, the improper Riemann integral $\int_0^\infty \frac{\sin x}{x}\, dx$ does exist, and equals $\frac{\pi}{2}$ (see Problem B.17).

1.77 Hints: (b) Take the Fourier transform of f_k and make the change of variables $\eta = 2\pi k \xi$.

(c) Since f_k, $\widehat{f_k} \in L^1(\mathbb{R})$, the Inversion Theorem applies.

1.78 Hints: (a) Even though $\frac{\sin x}{x}$ is not integrable, show that

$$K \;=\; \sup_{0<a<b<\infty} \left| \int_a^b \frac{\sin x}{x}\, dx \right| \;<\; \infty.$$

Then use the fact that f is odd to write $\widehat{f}(\xi) = 2i \int_0^\infty f(x)\, \sin(2\pi\xi x)\, dx$. Substitute this into $\left|\int_1^b \frac{\widehat{f}(\xi)}{\xi}\, d\xi\right|$, and use Fubini's Theorem to justify interchanging the integrals. Show that $K\, \|f\|_1$ is a bound for the desired supremum.

1.85 Hint: (c) Write $\Phi(0)^2 = \left(\int e^{-\pi x^2}\, dx\right) \left(\int e^{-\pi y^2}\, dy\right)$, and switch to polar coordinates.

1.88 Hint: Break into intervals $|x| \le 1$ and $|x| > 1$. For the latter, write $|f^{(n)}(x)| = |x^2 f^{(n)}(x)|/|x^2|$.

1.102 Hint: $\check{\chi}_N(x)$ is a geometric series in the variable $\omega = e^{2\pi i x}$.

1.104 Hints: Note that $\check{\chi}_N(x) = \sum_{n=-N}^{N} e^{2\pi i n x}$. Use Exercise 1.72 to write

$$\check{W}_N(x) \;=\; \sum_{n=-N}^{N} \Bigl(1 - \frac{|n|}{N+1}\Bigr) e^{2\pi i n x} \;=\; \frac{\check{\chi}_0(x) + \cdots + \check{\chi}_N(x)}{N+1}.$$

Now substitute

$$\check{\chi}_n(x) \;=\; \frac{\sin(2n+1)\pi x}{\sin \pi x} \;=\; \frac{e^{(2n+1)\pi i x} - e^{-(2n+1)\pi i x}}{e^{\pi i x} - e^{-\pi i x}},$$

and simplify the resulting geometric series.

1.107 Hint: To obtain the lower estimate, use the fact that $|\sin x| \le |x|$ and make a change of variables to write

$$\begin{aligned}\frac{1}{2}\,\|d_N\|_1 \;&\ge\; \int_0^{1/2} \frac{|\sin(2N+1)\pi x|}{\pi|x|}\,dx \\ &=\; \int_0^{N+\frac{1}{2}} \frac{|\sin \pi x|}{\pi|x|}\,dx \\ &\ge\; \sum_{k=0}^{N-1} \int_k^{k+1} \frac{|\sin \pi x|}{\pi|x|}\,dx.\end{aligned}$$

For the upper estimate, note that

$$f(x) \;=\; \frac{1}{\sin \pi x} - \frac{1}{\pi x}$$

is odd and increasing on $[-1/2, 1/2]$. Consequently,

$$\frac{1}{|\sin \pi x|} \;\le\; \frac{1}{\pi|x|} + \Bigl(1 - \frac{2}{\pi}\Bigr), \qquad |x| \le \frac{1}{2}.$$

Hence

$$\begin{aligned}\frac{1}{2}\,\|d_N\|_1 \;&\le\; \int_0^{1/2} \frac{|\sin(2N+1)\pi x|}{\pi|x|}\,dx \;+\; \Bigl(1-\frac{2}{\pi}\Bigr)\int_0^{1/2} |\sin(2N+1)\pi x|\,dx \\ &\le\; \int_0^{N+\frac{1}{2}} \frac{|\sin \pi x|}{\pi|x|}\,dx \;+\; \Bigl(1-\frac{2}{\pi}\Bigr)\frac{1}{2} \\ &\le\; \int_0^1 \frac{\sin \pi x}{\pi x}\,dx \;+\; \frac{1}{\pi}\sum_{k=1}^{N} \int_k^{k+1} \frac{|\sin \pi x|}{\pi|x|}\,dx \;+\; \Bigl(\frac{1}{2}-\frac{1}{\pi}\Bigr).\end{aligned}$$

1.108 Hint: To show $\int w_N = 1$, use the form

$$w_N(x) \;=\; \sum_{n=-N}^{N} \bigl(1 - \frac{|n|}{N+1}\bigr)\, e^{2\pi i n x}.$$

To show requirement (c) in the definition of approximate identity, use the form

$$w_N(x) \;=\; \frac{1}{N+1}\Bigl(\frac{\sin(N+1)\pi x}{\sin \pi x}\Bigr)^2.$$

1.109 Hint: Compare Theorem 1.51.

1.117 Hints: (b) ⇒ (c). $S_N^{\mathrm{a}} f = S_N f - \widehat{f}(N)\, e^{2\pi i N x}$,

(d) ⇒ (e). $i\, S_N^{\mathrm{t}} f(x) = S_N^{\mathrm{o}} f(x) - S_N^{\mathrm{o}} f^*(-x)$, where $f^*(x) = f(-x)$.

(e) ⇒ (d). $2\,S^{\mathrm{o}}_N f = S_{2N} f + i S^{\mathrm{t}}_N f + \widehat{f}(0) = -S^{\mathrm{t}}_N S^{\mathrm{t}}_N f + i S^{\mathrm{t}}_N f + 2\,\widehat{f}(0)$.

1.125 Hint: To show $\widehat{f}(n)$ and $\hat{\varphi}(n)$ are equal, use the periodicity of the exponentials, specifically, $e^{2\pi i n(x+m)} = e^{2\pi i n x}$ for $m \in \mathbb{Z}$.

Additional Problems from Chapter 1

1.2 Remark: To say that a function $f \in L^1(\mathbb{R})$ is even means that there is an even function g such that $f = g$ a.e.

1.4 Hint: Fix ξ and let $\alpha = e^{-2\pi i\theta}$ be the complex number of modulus 1 such that $|\widehat{f}(\xi)| = \alpha \widehat{f}(\xi)$. Then consider $\widehat{f}(0) - |\widehat{f}(\xi)|$.

1.5 Remark. For $z \notin \mathbb{Z}$, the Gamma function satisfies the functional equation

$$\Gamma(z)\,\Gamma(z-1)\,\sin \pi z \;=\; \pi.$$

However, $\sin \pi z \neq 0$ when $z \notin \mathbb{Z}$, so $\Gamma(z) \neq 0$ for $z \notin \mathbb{Z}$. Also, for $z = n \in \mathbb{N}$ we have $\Gamma(n) = (n-1)! \neq 0$. Hence $\Gamma(z) \neq 0$ for all z for which it is defined.

1.6 Hint: Lusin's Theorem.

1.8 Hints: $\int e^{-2y^2}\,dy = (\pi/2)^{1/2}$ and $\int y^2 e^{-2y^2}\,dy = (\pi/2)^{1/2}/4$.

1.11 Hint: Set $f(p) = \ln A_p^2$ and show that

$$f(p) \;=\; \frac{(p-1)\,\ln(p-1) - (p-2)\,\ln p}{p}, \qquad f'(p) \;=\; \frac{2 - 2\ln p + \ln(p-1)}{p^2}.$$

Conclude from this that f has critical points at

$$\frac{e^2 \pm e\sqrt{e^2-4}}{2} \;\approx\; 1.19243,\; 6.19662.$$

1.12 Hint: Consider

$$f(x) \;=\; \begin{cases} \dfrac{1}{|x|}, & |x| > 1, \\ 1, & |x| \le 1, \end{cases} \qquad g(x) \;=\; \begin{cases} \dfrac{1}{\ln|x|}, & |x| > e, \\ 1, & |x| \le e. \end{cases}$$

Then $f \in L^p(\mathbb{R})$ for $p > 1$ and $g \in C_0(\mathbb{R})$, but $(f * g)(x) = \infty$ for every x.

1.18 Hints: (a) Show that $\chi_E * \chi_{-E} \in C_0(\mathbb{R})$.

(b) E cannot have zero measure because $\cup_{r\in\mathbb{Q}}\,(E+r) = \mathbb{R}$.

(c) Suppose that $|A|_e > 0$, and let E be as in part (b). Let $A_r = A \cap (E+r)$ for $r \in \mathbb{Q}$. Then the A_r are disjoint sets whose union is A. Use part (b) to show that if A_r is measurable then it must have measure zero.

1.19 Hint: (a) Write $\widehat{f}(\xi) = 2\int_0^{1/2}\cos(6\pi x)\cos(2\pi\xi x)\,dx$, and apply a trigonometric identity to rewrite the integrand as a sum of two cosines. Alternatively, write $f = \frac{1}{2}M_{6\pi}\chi_{[-1/2,1/2]} - \frac{1}{2}M_{-6\pi}\chi_{[-1/2,1/2]}$, use the duality between modulation and translation together with Exercise 1.7 to compute $\widehat{f}$, and apply trig identities.

1.20 Let $g(x) = e^{-x}\chi_{[0,\infty)}(x)$, and observe that $f'(x) = g(x) - f(x)$.

1.22 Hint: To show P is unbounded, consider $f_n = \chi_{[n,n+1]}$. To show M is unbounded, consider $f_n(x) = n^{1/p}f(nx)$ for an appropriate f.

1.25 Hint: If $\int k = 0$, let $m \in L^1(\mathbb{R})$ be any function such that $\int m = 1$, and consider that both $\{(k+m)_\lambda\}_{\lambda>0}$ and $\{m_\lambda\}_{\lambda>0}$ are approximate identities.

1.26 Hint: $g * k_\lambda$ belongs to $g * L^1(\mathbb{R})$.

1.29 Hint: Note that the Inversion Formula is not applicable. Instead, consider the Fourier transform of $g(x) = \big(f(x)+f(-x)\big)/2$ and apply the Uniqueness Theorem.

1.30 Hint: Show that the Inversion Formula applies to $f(x) = e^{-2\pi|x|}$ (consider Problem 1.1), and that it also applies to $\chi_{[-1/2,1/2]} * \widehat{f}$. Use this to relate the integral in question to $(\chi_{[-1/2,1/2]} * \widehat{f}\,)(1/2)$.

1.32 Hint: Suppose that $\sum_{k=1}^{N} T_{a_k}g = 0$ a.e. Take the Fourier transform of both sides, and consider the fact that a nontrivial trigonometric polynomial can have only countably many zeros (see Section F.3).

1.33 Hint: (b) Use the Inversion Formula to write $f(x+h) - f(x)$ as an integral involving $\widehat{f}$, and then estimate the integral by breaking it into the regions where $|\xi| \le 1/|h|$ and $|\xi| > 1/|h|$.

1.36 Hints: (b) Either compute directly, or show that $(T_1B_n - B_n)^\wedge = \widehat{B_n'}$ and apply the Uniqueness Theorem. Note that $\chi = T_{1/2}\chi_{[-1/2,1/2]}$, so $\widehat{\chi}(\xi) = M_{-1/2}d_\pi(\xi) = e^{-\pi i\xi}\,\frac{\sin\pi\xi}{\pi\xi}$, and therefore you can write an explicit formula for $\widehat{B_n}(\xi)$.

(c) Use the Inversion Formula to show that the decay of $\widehat{B_n}$ in frequency implies that B_n must be smooth in the time variable. To show that $B_n^{(n-1)}$ is piecewise linear, use the relation proved in part (b).

(d) Note that $\chi(x) = \chi(2x) + \chi(2x-1)$. Let $c_0 = c_1 = 1$ and set $c_k = 0$ for all other k. Show that $(\chi * \chi)(x) = \frac{1}{2}\sum_{k\in\mathbb{Z}}(c*c)_k\,(\chi*\chi)(2x-k)$, where $c*c$ is the discrete convolution of the sequence $c = (c_k)_{k\in\mathbb{Z}}$ with itself (see Definition 1.38). Note that $c*c$ has only three nonzero terms.

1.39 Hint: Show that the technique of Exercise 1.85 carries over to complex parameters.

1.41 Hint: Justify the equality $(f * \phi_\lambda)(0) = (f * \phi_\lambda)^{\wedge\vee}(0)$, and consider $\liminf_{\lambda\to\infty} \int_{|\xi|>R} \widehat{f}(\xi)\, \widehat{\phi_\lambda}(\xi)\, d\xi$.

1.42 Hint: $\widehat{g}(\xi) = (4\pi\xi)/(1 + 4\pi^2\xi^2)$.

1.44 Hint: Apply the product rule $(fg)^{(n)} = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}\, g^{(n-j)}$.

1.46 Hint: Let $K \in C_c^\infty(\mathbb{R})$ be such that $K(0) = 1$, and construct an approximate identity from $k = (K)^\vee$.

1.48 Hints: (a) Suppose that ξ_1, ξ_2 are distinct characters on G. Then there exists an $x \in G$ such that $\xi_1(x) \neq \xi_2(x)$. Let $z = \xi_1(x)\, \xi_2(x)^{-1}$. Since $|z| = 1$, we can write $z = e^{2\pi i\theta}$. Show that if z is a root of unity, i.e., θ is rational, then there exists an m such that $|z^m - 1| \geq \sqrt{3}$. On the other hand, if θ is irrational, then *Weyl's Equidistribution Theorem* implies that $\{z^m\}_{m\in\mathbb{Z}}$ is dense in S^1.

(b) Since the topology on $\widehat{G}$ is induced from a norm, all open balls $B_r(\xi) = \bigl\{\eta \in \widehat{G} : \|\xi - \eta\|_\infty < r\bigr\}$ are open subsets of $\widehat{G}$.

1.49 Hint: (a) Haar measure is translation-invariant.

1.54 Hint: The Fourier coefficients are

$$\widehat{f}(n) \;=\; \begin{cases} 0, & n \text{ even}, \\ -(2i)/(\pi n), & n \text{ odd}. \end{cases}$$

1.55 Hint: (b) Apply the Plancherel formula to f' and use part (a).

1.56 Hints: (b) Suppose that $f \in A(\mathbb{T})$. For each $n \in \mathbb{Z}$, let g_n be any complex number such that $g_n^2 = \widehat{f}(n)$. Then $(g_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, so $g(x) = \sum_{n\in\mathbb{Z}} g_n\, e^{2\pi i n x}$ belongs to $L^2(\mathbb{T})$. Show that $g * g = f$.

(c) For each $N \in \mathbb{N}$, define

$$F_N \;=\; \biggl\{ f \in A(\mathbb{T}) : \sum_{n=-\infty}^{\infty} |\widehat{f}(n)| \leq N \biggr\},$$

so $A(\mathbb{T}) = \cup F_N$. Show that each F_N is a closed subset of $C(\mathbb{T})$. Since $A(\mathbb{T})$ is a dense subspace $C(\mathbb{T})$, it contains no open subsets of $C(\mathbb{T})$. Therefore, F_N contains no interior points, so is nowhere dense, and consequently $A(\mathbb{R})$ is meager.

1.57 Hint: (b) Let $f_n(x) = xe^{2\pi i n x}$. If $\{f_n\}_{n\in\mathbb{Z}}$ was a Schauder basis, then there would be a finite constant C (its *basis constant*) such that $1 \leq \|f_n\|_2\, \|g_n\|_2 \leq 2C$ for all n (see Exercise C.108).