

# DISSIPATIVE DYNAMICS IN SEMICONDUCTORS AT LOW TEMPERATURE

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ABSTRACT. A mathematical model is introduced which describes the dissipation of electrons in lightly doped semi-conductors. The dissipation operator is proved to be densely defined and positive and to generate a Markov semigroup of operators. The spectrum of the dissipation operator is studied and it is shown that zero is a simple eigenvalue, which makes the equilibrium state unique. Also it is shown that there is a gap between zero and the rest of its spectrum which makes the return to equilibrium exponentially fast in time.

*The sciences do not try to explain, they hardly even try to interpret, they mainly make models. By a model is meant a mathematical construct which, with the addition of certain verbal interpretations, describes observed phenomena. The justification of such a mathematical construct is solely and precisely that it is expected to work.* (J. von Neumann [51])

## 1. INTRODUCTION

This article is dedicated to the construction and the fundamental properties of a model of dissipative transport, describing the electron or hole transport in semiconductors at very low temperature. By “very low” it is meant that the temperature is low enough so as to confine the charge carriers to the impurity band. Without dissipation, the transport is *coherent* and is likely to be described by an Anderson model, namely a Schrödinger operator on a discrete lattice with a random potential. In *lightly doped* semiconductors this model has to be considered in the *strong localization regime* as will be explained in Section 2. The main source of dissipation in all solids, including semiconductors, is coming from the *electron-phonon* interaction. Namely, the Coulomb interaction between electrons and nuclei leads to the slow nuclei motion when an electron is passing by. The harmonic interaction between nuclei leads, in turn, to the production of acoustic waves, that are quantized, at least if produced in small quantities. These quanta of acoustic waves are called *phonons*. These waves propagate in the crystal at the speed of sound and can kick another electron eventually, leading to loss of information about the electron quantum state. There are other sources of dissipation for the electron motion like the direct Coulomb interaction between them, spin coupling or photon emission. However,

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it has been shown [29] that the first process is quantitatively more important than all others. Still, even the dominant mechanism for dissipation can be considered as weak in most cases, in particular for the problem considered in the present paper. Therefore, it is legitimate to approximate the system by using the so-called *Markov approximation*. As a consequence, the dissipative dynamic can be described through a Markov semigroup whose generator is a Lindblad operator [28], also called nowadays *Lindbladian*. Like in Atomic Physics [12], this Lindbladian could be computed from their second order perturbation theory, called the *Fermi golden rule*. It will not be the method used here. The purpose of the present work will be to construct a Lindblad operator that describes phenomenologically the dominant sources of dissipation, and to investigate its spectral properties. The precise description of the model is given in Subsection 2.5. The spectral properties are summarized in Section 5. The main result is the Theorem 5.5, showing that (i) such an operator and the dynamical semigroup it generates are well defined even if the system is out of equilibrium and that (ii) it forces the return to equilibrium if there is no gradient of chemical potential or of temperature in the system. However, the main new contribution of the present work lies in the mathematical framework as explained below. But since the explicit model is strongly dependent of the physical regime that it intends to describe, it will be necessary to describe the physics in detail in order to make sure that the model is realistic (see Section 2).

The problem investigated here is motivated by the mechanism called *variable range hopping conductivity* [39]. It has dominated the study of semiconductors for almost two decades since the work of Miller and Abrahams in 1960 [31] and the seminal contribution of Mott [32] predicting the behavior of the conductivity as a function of the temperature. It has been suggested [36] that this regime is also the dominant contribution in the Quantum Hall Effect [5] that explains in particular the amazing accuracy of the experiment. For indeed, the relative error with which the Hall conductivity can be measured in a QHE experiment is of the order of  $10^{-10}$  due to the smallness of the direct conductivity on the plateau of conductivity [34]. The theory of the integer QHE has been made rigorous through the use of the formalism of Non-commutative Geometry in [5]. In this latter work, the problem of dissipative transport was investigated within the so-called *relaxation time approximation* (RTA). The RTA reduces the dissipative mechanism to only one time scale and allows to consider the charge carriers as independent particles. As shown in [5], when applied to the QHE, this approximation leads to a relative error of  $10^{-4}$  with the best data, namely six orders of magnitude larger than what is actually observed !! The reason is that the charge carriers conductivity is suppressed by the variable range hopping, as was shown by Mott [32]. However, such a mechanism involves an infinite number of time scales. It was advocated in [6] that such a mechanism can be represented through a Markov semigroup, the generator of which is a Lindblad operator. Unfortunately previous attempts to implement this idea have provided mixed results. The main reason is that the charge carriers, electrons or holes, are Fermions and any approximation leading to consider these particles in a semi classical regime fails to include the statistical correlation induced by the electron-phonon interactions. In order to successfully represent this

mechanism, a mathematical model must take second quantization into account. The main new contribution of the present paper is precisely to work with a many-body formalism. Since this approach is technically very demanding, the model will be simplified. The main simplification consists in replacing the Anderson model, describing the coherent part of the motion, by a purely potential contribution, neglecting the kinetic part, which, in real semiconductors is indeed extremely small. This kinetic part will be reintroduced in the dissipative mechanism through a contribution of the tunneling effect between impurities.

Even with so many simplifications, the formalism is heavy and will occupy most of this paper. This is because the random character of the distribution of impurities breaks the translation invariance. Since the early eighties, thanks to using the ideas of Non-commutative Geometry [15], the formalism required to replace the translation group is known (for instance, see [6] and references therein): a groupoid replaces the group of translations. The inclusion of the many-body formalism in this framework was developed in the PhD Thesis of Dominique Spehner [41]. This leads to replace the observable algebra by a *continuous field* of  $C^*$ -algebras over the space describing the disorder. The notion of continuous field of Banach spaces was introduced by Tomyama [47, 48, 49] in the context of the spectral theory for  $C^*$ -algebras and later developed by Dixmier [17]. While the concept is easy to understand, it is technically demanding. Then, the coherent dynamics can be defined as a continuous field of dynamics, leading to a continuous field of KMS-states describing the equilibrium of the electron gas in the solid. In much the same way, the dissipative dynamics is defined by a continuous field of Markov semigroups. In the present work, various existence results for the dynamics are proposed. One, mainly the Theorem 4.4, is based on the estimates used by Bratteli and Robinson [10] to prove the existence of the dynamics in the many-body theory. The other one, mainly the Theorem 4.9, uses the continuous field of Hilbert spaces generated by the field of KMS-states, through the GNS-construction, and sees the Lindbladian as a non-commutative analog of a *Dirichlet form*. Dirichlet forms were introduced by Beurling and Deny [8] and the characterization was completed by Fukushima [20]. The noncommutative Dirichlet forms were defined by Albeverio and Hoegh-Krøhn [2] and they were characterized in full generality by Cipriani [11]. The definition requires some notion of positivity in the Hilbert space of the GNS-representation. Such a positivity is provided by a cone in the Hilbert space, that was identified and characterized in the early seventies by Araki [4] and Connes [13]. In each fiber of the field of Hilbert spaces, it is the cone generated by positive elements of the corresponding fiber of the field of observable algebras. For the sake of the reader, this will be explained in Section 3.4.

Because the formalism required here is so heavy, it seems wiser to restrict the present paper to the description of the dynamics and the return to equilibrium. However, the real goal is to show that this model is liable to account for the Mott prediction concerning the low temperature behavior of the conductivity. An important result was obtained by [19]: by looking at the variable range hopping at very large length scale, as a random walk in a random environment the authors could prove that the Mott prediction was a lower bound to the diffusion constant. However, to get an upper bound is highly non trivial. In order to

do so, using the present model, it will be necessary to face two challenges. The first one is the definition of the local currents. As it turns out, this is not a trivial problem because of possible divergent effects in the infinite volume limit. It requires insight about the physical nature of currents and of the dissipation mechanisms. With the proper concept, though it is possible to prove rigorously the validity of the Kubo formula [1] whenever the charge carriers can be considered as a continuous fluid. The other challenge is the discontinuous nature of the variable range hopping mechanism, forcing the charge carriers to hop at distances ten times larger than the average distance between impurities. In particular, the *continuous fluid picture breaks down!* This is why Miller and Abrahams [31] proposed to see the charge carriers as electric currents in a random network. In the early seventies percolation theory was successfully introduced into this picture [3, 35, 39] to justify the prediction of Mott. However, several approximations, justified by the physical situation, require additional work in order to make this argument rigorous within the scope of the present model. It will be hopefully the subject of a forthcoming publication.

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## 2. PHYSICS OF LIGHTLY DOPED SEMI-CONDUCTORS

**2.1. Orders of Magnitude.** The content of this section can be found in several textbook, in particular the one by Shklovskii and Efros [39].

The two types of semi-conductors that are the most used and studied today are *silicon* and *GaAs*, due to their importance in modern electronics, telecommunication and computer hardware. Silicon is currently obtained, in the industrial process, in a form of cylindrical ingots of about  $2m$  in length and  $25cm$  in diameter. The crystal is perfect with less than  $10^{-10}$  impurity or defect per atom. Because silicon has 4 valence electrons, the atomic orbitals hybridate in the  $sp_3$  form, leading to a diamond crystal, where tetrahedra alternating in a staggered way. *Ga* belongs to the column III of the periodic table, namely it has 3 valence electrons. *As* has 5 valence electrons and thus, it belongs to the column V. Mixed together in equal quantity, *Ga* and *As* exchange one electron to produce pairs of tetrahedra, leading also to a diamond lattice. If the electron-electron interactions are ignored, the band spectrum is similar to the one of graphene, namely two bands touch on a conical point exactly at the Fermi level, (the Fermi level is the maximum energy of an electron at zero temperature).

However, the Coulomb interaction between the valence electrons belonging to the same atom, is strong enough to force the opening of a gap at the Fermi level, which will be called the *main gap*. For *Si*, this gap is about  $1.1eV$ , which is enormous. Correspondingly, the temperature necessary to allow a large number of electrons to jump from the valence band

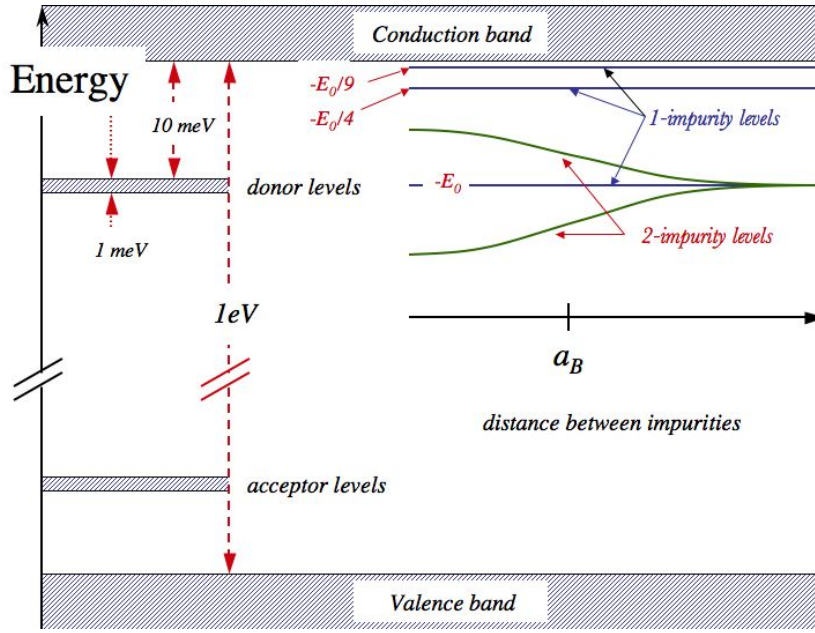


FIGURE 1. (i) Left: energy ranges (ii) Right: 1 & 2 hydrogen atom levels

to the conduction band would be  $1.2 \times 10^4 K$ . In other words, the clean silicon is a perfect insulator. For *GaAs* the gap is  $1.43 eV$  corresponding to a temperature of  $1.6 \times 10^4 K$ .

For the purpose of electronic applications, the doping is about  $10^{-9}$  impurity per atomic site. Such a crystal is called *lightly doped*. For *n*-type doping, the impurities are atoms with one more electron in the valence band than the crystal atoms (such as *P, As, Sb*). Then, an impurity band is created in the gap at a distance of about  $10 meV$  from the conduction band (see Fig. 1). Hence, the gap between the impurity band and the conduction band is 100 times smaller than the main gap. In particular, at room temperature, most electrons of the impurity band jump into the conduction band, which explains why semi-conductors are actually conductors. However, the conduction electron density is controlled by the impurity concentration, namely it is much smaller than in metals. At very low temperatures, say around  $1 K$ , this gap is too large to allow electrons to jump, so that electrons are confined in the impurity band (see Fig. 1). Similarly, for *p*-type doping, the impurities are acceptors instead, namely they have one electron less on their valence band than the crystal atoms (such as *Al, Ga, In*). Then, a hole-impurity band is created near the valence band at approximately the same distance as before. Similar conclusions arise after proceeding to the *hole-particle symmetry*: (i) the charge carriers are holes instead of electrons, (ii) the origin of the energy is the top of the valence band instead of the bottom of the conduction band and (iii) the sign of energy is reversed.

For an impurity concentration  $c \sim 10^{-9}$  per atom, the average distance between impurities is given by  $r_{av} = c^{-1/3} \sim 1000$  atomic spacings. As will be seen in the next Section 2.2,

an isolated impurity behaves like an Hydrogen atom with a re-normalized mass and a re-normalized Coulomb coupling constant. In these materials, the Bohr radius of an impurity is  $a_B \simeq 100\text{\AA}$ . Hence the average distance between impurities corresponds approximately to  $10a_B$ . This gives a band width of approximately  $W \simeq 1\text{ meV}$ , 10 times smaller than the gap between the impurity band and the conduction (or valence) band (see Fig. 1).

In most cases a semi-conductor has the two types of impurities simultaneously. In such a case it is called *compensate*. If *n*-type impurities dominate, then all acceptors will get an extra electron since their energy is lower, creating negative ions, and some proportion of the donors will become ionized creating positive charges in the crystal. The Fermi level, namely the maximum energy of an electron at zero temperature, will be located within the donor impurity band at a position depending upon the relative concentration of donor and acceptor. If the acceptors dominate, some of them will acquire all the electrons coming from the donors and will become negatively charged, while all the donors will be ionized. The Fermi level will then be inside the acceptor impurity band and the conduction will be due to holes instead. In both cases however, the positions of the ions are random, creating a random Coulomb field within the crystal which influence the energy level of the impurity electrons.

**2.2. One Impurity & Hydrogen Atom.** If one donor impurity is inserted in a perfectly periodic crystal, Slater [40] showed in 1949 using a theorem by Wannier [52], that the extra electron behaves like in an hydrogen atom, with re-normalized parameters for its mass and the dielectric constant, provided the origin of energies is taken at the bottom of the conduction band. If the impurity is an acceptor instead, a similar argument can be used for holes instead of electrons, using electron-hole symmetry. If the bottom of the conduction band is an isotropic minimum, the corresponding effective Hamiltonian describing the electron near the impurity is given by

$$H_H = -2E_0 \left( \frac{\Delta_r}{2} + \frac{1}{|r|} \right), \quad r = \frac{x}{a_B}, \quad a_B = \frac{\hbar^2 \kappa}{me^2}, \quad E_0 = \frac{me^4}{2\hbar^2 \kappa^2}. \quad (2.1)$$

Here,  $e$  is the electron charge,  $m$  is the effective mass of the electron in the crystal and  $\kappa$  is the re-normalized Coulomb constant in the medium. The ground state is given by

$$\phi_0(x) = \frac{e^{-|x|/a_B}}{Z_0}, \quad Z_0 = (\pi a_B^3)^{1/2} \quad (2.2)$$

so that  $a_B$ , called the *Bohr radius*, gives the length scale beyond which the wave function becomes negligible. The corresponding eigenvalue is  $-E_0$  given in eq. (2.1). The other energy levels are given by  $E_n = -E_0/n^2$  for  $n = 1, 2, 3, \dots$ . In particular the gap between the ground-state and the first excited state is  $3E_0/4$  (see Fig. 1).

As a result, in the donor case, the lower energy state of the impurity is located in the gap of the crystal, at a distance  $E_0$  from the conduction band. Applied to impurities in a crystal, the Wannier equivalence theorem provides effective values for  $m$  and  $\kappa$ , which are usually different from the values for the electron in the vacuum. For silicon, this Bohr radius is about  $100\text{\AA}$  and  $E_0 \simeq 10\text{ meV}$  [39].

**2.3. The Anderson model.** In a semi-conductor, there is a density of impurities. Even if this density is as small as  $10^{-9}$ , as it is for lightly doped media, the effect on the charge carrier dynamics is not negligible. The first important effect occurs if the semi-conductor is compensated. For indeed since the density of donors and acceptors are not equal, there is a nonzero density of ions in the crystal. The position of these ions is random. Consequently, they create a static electric field within the crystal, which is also random. Since the charge carriers are confined on the impurity site, they cannot move to screen this field. As a consequence, the effective potential seen by each charge carrier at an impurity site, is itself shifted by a random term. Since the distance between ions is very large, the values of potential at each impurity site might be considered as independent random variables. By homogeneity, they should have the same distribution.

On the other hand, since the ground state wave function decays exponentially fast, the tunneling of a charge carrier between two impurity sites is controlled by the overlap of the two wave packets localized on each of the impurities. This overlap is of the order of  $e^{-r/a_B}$  if  $r$  is the distance between the two impurities. If the average distance between impurities is about  $10a_B$ , this term is very small (of the order of  $10^{-4}$ ). This implies that the effective Hamiltonian describing the motion of a charge carrier through the impurity sites is made of two contributions: (i) the contribution of the random potential created by the ions in the crystal and (ii) the hopping term of the order of  $e^{-r/a_B}$ . Since this latter contribution is much smaller than the potential contribution, this effective Hamiltonian is given by an Anderson model in which the kinetic term is much smaller than the potential term. Hence, *paradoxically, lightly doped semi-conductors, at very low temperature, correspond to a strong disordered Anderson model.* In the present paper, the hopping term will be ignored, so that the one-particle Hamiltonian will be a pure on-site potential.

**2.4. Mott's Variable Range Hopping Transport.** In 1968, Mott [32] proposed the following argument liable to explain the conductivity properties of Anderson insulators at low temperature (see also [39]). Mott assumes that the solid is a  $d$ -dimensional Anderson insulator, namely the electrons are strongly localized. In particular, each energy level of the electron energy spectrum is associated with a position in the solid within a ball of diameter given by the localization length  $\xi$  (which can be taken of the order of  $a_B$ ). Moreover, he assumes that the density of states (DOS)  $n_F$  at the Fermi level is non vanishing but small enough so as to avoid overlap between electron states. In particular the mean distance between neighboring electron states is large compared to  $\xi$ . The inverse of  $n_F \cdot \xi^d$  is a measure of the mean level spacing between states within a ball of diameter  $\xi$ . The temperature will be small compared with this scale, namely

$$k_B T n_F \xi^d = \frac{T}{T_0} \ll 1,$$

if  $k_B$  denotes the Boltzmann constant. Then within a small error, all states with energy smaller than the Fermi level  $E_F$  are occupied, whereas the ones with higher energy are empty. The probability that a phonon of energy  $\varepsilon$  is produced is proportional to  $e^{\varepsilon/k_B T}$ . This is true provided  $\varepsilon \gg k_B T$ . Such a phonon can be absorbed by an electron of the

Fermi sea localized at  $x \in \mathcal{L}$ , with energy  $\epsilon_x < E_F$ , and cause this electron to hop into a state localized at  $y \in \mathcal{L}$  with energy  $\epsilon_y > E_F$ . The probability for such an event to happen is controlled by the tunneling effect forcing the electron to move from  $x$  to  $y$ . Let  $r = |x - y|$  be the distance between such states, so that the tunneling probability be proportional to  $e^{-r/\xi}$ . Therefore, the probability  $P$  of transfer of an electron at distance  $r$  from its original location is proportional to

$$P \propto \exp \left[ - \left\{ \frac{\varepsilon}{k_B T} + \frac{r}{\xi} \right\} \right] \quad (2.3)$$

By definition of the DOS, the product  $\varepsilon \cdot n_F \cdot r^d$  represents the average number of states in an energy interval of width  $\varepsilon$  localized in a cube of size  $r$ . Mott then argued that the most likely value of the radius satisfies

$$n_F \cdot \varepsilon \cdot r^d \approx 1. \quad (2.4)$$

The probability of jump  $P$  is then optimized over  $r$  and  $\varepsilon$ . This can be done by giving  $r$  its minimum value compatible with eq. (2.4) and by maximizing *w.r.t.*  $\varepsilon$ . Therefore the conductivity, which is proportional to the sum of contributions of all such jumps, will be controlled by the maximal value of  $P$ . Optimizing over  $\varepsilon$  leads to

$$P \propto \exp \left[ - \left\{ \frac{T_0}{T} \right\}^{1/(d+1)} \right], \quad k_B T_0 = \frac{(d+1)^{d+1}}{d^d} \frac{1}{n_F \xi^d}. \quad (2.5)$$

Then, the phonon energy optimizing  $P$  is given by  $\varepsilon_{opt} \approx T^{d/d+1} \gg T$ , while the average distance of the jump is given by  $r/\xi \approx (T_0/T)^{1/d+1} \gg 1$ .

Clearly, the conductivity is proportional to the probability of transfer of electrons per unit time. Therefore, we expect the conductivity to be proportional to the same factor as a function of the temperature.

For lightly doped 3D-semiconductors like silicon (see [39]), the localization length is given by the Bohr radius of the impurity, which is about  $100\text{\AA}$ . For a concentration of  $10^{-9}$ , the mean distance between impurities is  $1000\text{\AA}$  which is approximately  $10\xi$ . Then, the typical width of the impurity band is about  $1\text{meV}$ . Assuming the DOS to be flat on it gives  $n_f = c/W$  which leads to  $T_0 \approx 1.1 \times 10^5 K$  ! This is huge indeed. At  $T = 1 K$ , this gives  $(T_0/T)^{1/4} \simeq 18$ , namely (i) the electron hops at about  $18\xi \simeq 2r_{av}$ , (ii) the conductivity is multiplied by a factor  $e^{-18} \simeq 1.2 \times 10^{-8}$  due to this mechanism ! It turns out that this is exactly what happens in the quantum Hall effect (QHE) [34, 5]: the Mott variable range hopping controls the fluctuation of the plateaus, leading to the amazing accuracy of this experiment.

Mott's law has been well documented in the experimental physics literature. A review of the results obtained in the seventies on various semi-conductors can be found in [22] (see also [53]) and some results have been reported in [33]. In addition a large part of the book by Efros and Shklovskii [39] is dedicated to this effect (see Chapter 9, in particular and references therein) .



The previous version of Mott's argument is based on a critical assumption summarized by eq. (2.4). This part of the argument has been the focus of attention of several works in the early seventies [54, 18, 3, 25, 27, 35, 7] and it is the main topic of a large fraction of the book of Shklovskii and Efros [39]. Following the description provided by Miller and Abrahams [31], the electron conduction is seen as a random resistor network. In the early sixties, Ziman [54] suggested that percolation theory should be a key technique to investigate hopping transport. Since then it became indeed increasingly clear that percolation gives the right argument to justify eq. (2.4) and Mott's prediction.

**2.5. Description of the Model.** Based upon the argument of Mott, it becomes possible to propose a model. This section will be dedicated to its heuristic description, leaving the rigorous definitions for the Sections that follow. All along the present paper, only  $n$ -type doped semiconductors will be considered. A particle-hole symmetry permits to consider the  $p$ -type case.

Let  $\mathcal{L}$  be the lattice  $\mathbb{Z}^d$ . For any  $x \in \mathcal{L}$  let  $s_x \in \{0, 1\}$  be a random variable with  $s_x = 1$  if and only if an electron state is available at  $x$ . Given a family of random variables  $s = (s_x)_{x \in \mathcal{L}}$  each taking values in  $\{0, 1\}$ , let  $\mathcal{L}(s)$  be the random subset of  $\mathcal{L}$  containing all sites  $x$  where  $s_x = 1$ . Thus  $\mathcal{L}(s)$  is the set of sites in the semiconductor on which an impurity electron state is available. In the tradition of Solid State Physics, and only for the heuristic description of the model, the total number of sites available will be considered as finite. The electron will be described in the second quantization picture, (see [16]), through a pair of Fermion creation-annihilation operators  $(a_x^\dagger, a_x)$ , where  $x \in \mathcal{L}(s)$  is a lattice site, thus obeying the canonical anti-commutation rules which we abbreviate by CAR,

$$a_x a_y + a_y a_x = 0, \quad a_x^* = a_x^\dagger, \quad a_x a_y^\dagger + a_y^\dagger a_x = \delta_{xy} \mathbf{1}. \quad (\text{CAR}) \quad (2.6)$$

Here  $A^*$  denotes the adjoint of the operator  $A$ . In this work, for simplicity, only spinless electrons are considered, because the model proposed will not couple the spin to the motion. The quantum dynamics will be made of two parts: (i) the coherent part, describing the electron motion in absence of dissipation, and (ii) the dissipative part, taking into account the interaction with other degrees of freedom, especially the phonons.

Since the dynamic concerns the electron gas, the system will be considered in the *local equilibrium approximation* [6]. Namely the electron gas is seen as the union of *mesoscopic cells* with size large enough so that within a *mesoscopic time* each such cell has the time to return to equilibrium. On the other hand the size of the cells is small compared to the size of the sample under consideration. In each mesoscopic cell, the electron gas is in an equilibrium state, namely defined by the usual thermodynamic parameters, the temperature  $T$  (or rather the inverse temperature  $\beta = 1/k_B T$ ) and the chemical potential  $\mu$ . Such mesoscopic cells are opened, namely they allow both energy and electrons to be exchanged from cell to cell. Therefore, only the average of the energy and of the particle number is fixed. In addition, if the system is put out of equilibrium, then both  $T$  and  $\mu$  may vary slowly in space and time, so that the relative variation in each mesoscopic cell is negligible.

The coherent part is given by a Hamiltonian and the observables evolve according to the Heisenberg equation. But, because of the local equilibrium approximation and because each mesoscopic cell is opened, the Hamiltonian will represent the *free energy* of the gas in the mesoscopic cell. Therefore it has the form  $F = H - \mu N$  where  $H$  represents the mechanical energy of the gas, while  $N$  is the number of electrons (in the mesoscopic cell). The mechanical part  $H$  of this Hamiltonian will be reduced to the potential energy of the electrons on the impurity sites, ignoring the hopping term between impurities, since the tunneling effect is so small for lightly doped semiconductors. The potential will be represented by a family  $\epsilon = (\epsilon_x)_{x \in \mathcal{L}(s)}$  of independent identically distributed random variables, where  $\epsilon_x$  belongs to the impurity band  $\Delta$  which is a compact interval. For the sake of the present model, the distribution will be assumed to be uniform in  $\Delta$  and it shall be independent of the random variables  $s = (s_x)_{x \in \mathcal{L}}$ .

The family of all random variables  $s = (s_x)$  and  $\epsilon = (\epsilon_x)$  will be simply denoted by a random variable  $\omega = (s, \epsilon)$ . The corresponding probability space consisting of the compact set of all such families will be denoted by  $\Omega$  and the corresponding probability measure on the Borel sets of  $\Omega$  by  $\mathbb{P}$ . Many operators such as the free energy depend on the randomness  $\omega$ , e.g. the free energy becomes

$$F(\omega) = \sum_{x \in \mathcal{L}(s)} (\epsilon_x - \mu) a_x^\dagger a_x \quad (2.7)$$

Since the electron-phonon interaction is weak, the dissipative part will be given in the Markov approximation, in which all degrees of freedom other than the electrons are integrated out, while the time scale for this interaction to rearrange the electron state is considered as negligible. Namely, the dissipative part of the dynamics will be described by the generator of a Markov semigroup, which, thanks to the Theorem by Lindblad [28] is given by a Lindbladian of the form

$$\mathfrak{D}(A) = \sum_i \left( \frac{1}{2} (L_i^* L_i A + A L_i^* L_i) - L_i^* A L_i \right). \quad (2.8)$$

The description of the model will precisely consist in proposing an expression for the  $L_i$ 's. In order to implement the Mott scheme, these operators should describe the jump of an electron from an impurity site  $x \in \mathcal{L}(s)$  to an impurity site  $y \in \mathcal{L}(s)$ . The corresponding jump operator, denoted by  $L_{x \rightarrow y}$ , should be proportional to  $a_y^\dagger a_x$ , since the later annihilates an electron at  $x$  and creates one at  $y$ . If one of the sites,  $x$  or  $y$ , is not in the random lattice  $\mathcal{L}(s)$ , then there can't be a jump from  $x$  to  $y$ . Hence we set  $L_{x \rightarrow y}$  equal to zero in this case. Hence,

$$L_{x \rightarrow y}(\omega) = s_x s_y \sqrt{\Gamma_{x \rightarrow y}} a_y^\dagger a_x. \quad (2.9)$$

where  $\Gamma_{x \rightarrow y}$  is the probability rate at which the jump arises. Following the Mott argument, this jump probability rate needs to take into account the probability of a phonon being absorbed or created by the electrons. The absorption process is dominated by the probability of a phonon of energy  $\epsilon$  to be created by the thermal bath. It is given by

the Boltzmann factor  $e^{-\beta\epsilon}$  with a good approximation<sup>1</sup>. Then the electron located at  $x$  absorbing this energy must find an available site  $y$  with energy  $\epsilon_y = \epsilon + \epsilon_x > \epsilon_x$ . On the other hand, if an electron at  $x$  is in an excited state, it might decrease its energy by spontaneously emitting a phonon or energy  $\epsilon \geq 0$ , provided it finds a site  $y$  at which the energy available is  $\epsilon_y = \epsilon_x - \epsilon < \epsilon_x$ . The spontaneous emission does not require any Boltzmann factor. Hence the probability for absorption and emission is proportional to  $e^{-\beta(\epsilon_y - \epsilon_x)}$  if  $\epsilon_y \geq \epsilon_x$  and to 1 if  $\epsilon_y \leq \epsilon_x$ . Thus a unified formula for the probability of absorption or emission is  $e^{-\beta(\epsilon_y - \epsilon_x)^+}$  where  $\epsilon^+$  denotes the positive part of the real number  $\epsilon$ . This difference between emission and absorption leads to the quotient  $\frac{\Gamma_{x \rightarrow y}}{\Gamma_{y \rightarrow x}} = e^{-\beta(\epsilon_y - \epsilon_x)}$  which is also known as *detailed balance* condition.

In addition, in both cases, the electron must jump from  $x$  to  $y$  through a tunneling effect, namely decreasing the probability by a factor proportional to  $e^{-|x-y|/r}$ , if  $r$  is the localization length. Note that in lightly doped semiconductors,  $r$  is of the order of the Bohr radius  $a_B$ , since the average distance between impurities corresponds approximately to  $10a_B$ . This proportionality factor will be normalized by dividing it by  $Z$ , where

$$Z = \sum_{m \in \mathcal{L}(s)} e^{-|m|/r}. \quad (2.10)$$

This leads to the expression of the jump rate proposed by Mott

$$\Gamma_{x \rightarrow y}(\omega) = \Gamma_0 \frac{e^{-|x-y|/r}}{Z} e^{-\beta(\epsilon_y - \epsilon_x)^+}, \quad x, y \in \mathcal{L}(s) \quad (2.11)$$

which gives the detailed balance condition

$$\frac{\Gamma_{x \rightarrow y}}{\Gamma_{y \rightarrow x}} = e^{-\beta(\epsilon_y - \epsilon_x)}. \quad (2.12)$$

**Remark 2.1.** *In the present paper we assume that  $\Gamma_0$  is a constant. The present results however remain valid if  $\Gamma_0$  is a function of  $\epsilon_x$ ,  $\epsilon_y$  and  $\beta$  satisfying the following properties:  $\Gamma_0(\epsilon_x, \epsilon_y, \beta) = \Gamma_0(\epsilon_y, \epsilon_x, \beta)$ , (hence the detailed balance condition (2.12) will remain valid), and  $\inf\{\Gamma_0(\epsilon_x, \epsilon_y, \beta) : \epsilon_x, \epsilon_y \in \Delta, \beta \in (0, \infty)\} > 0$ .*

The factor  $\Gamma_0$  is a parameter with the dimension of the inverse of a time, fixing the order of magnitude of the effect. We let  $\mathfrak{D}_\omega^{kin}$  denote the Lindblad operator obtained from this family of jump operators,

$$\mathfrak{D}_\omega^{kin}(A) = \sum_{x \neq y \in \mathcal{L}(s)} \left( \frac{1}{2} (L_{x \rightarrow y}^* L_{x \rightarrow y} A + A L_{x \rightarrow y}^* L_{x \rightarrow y}) - L_{x \rightarrow y}^* A L_{x \rightarrow y} \right). \quad (2.13)$$

However, this part of the model is insufficient to describe the return to equilibrium, because  $\mathfrak{D}^{kin}$  leaves the number operator  $N$  invariant, as can be checked easily. Therefore it does not take into account the possibility for an electron to jump out or to jump in the system (thermal bath). An electron can jump out of the system in two ways: (i) either

<sup>1</sup>It should actually be given by the Bose-Einstein distribution  $(e^{\beta\epsilon} - 1)^{-1}$ . But if  $\beta\epsilon \gg 1$  it follows that  $(e^{\beta\epsilon} - 1)^{-1} \approx e^{-\beta\epsilon}$

its energy becomes too large or too small to stay within the impurity band  $\Delta$ , or (ii) the electron is kicked out of the mesoscopic cell under consideration. Similarly the opposite processes arises to allow an electron to jump in the system. A standard way to take this process into account is to create an extra site  $\star$  called the *cemetery* and to describe these processes as a simple jump  $\star \rightarrow x$  or  $x \rightarrow \star$ . In the first case, an electron is created at  $x$ , while in the other it is annihilated at  $x$ . The cemetery really describes the thermal bath and it is only natural to interpret the chemical potential  $\mu$  as the energy associated with this new site. Hence, the corresponding jump operators will be given by

$$L_{x \rightarrow \star}(\omega) = s_x (\Gamma_{x \rightarrow \star})^{1/2} a_x, \quad L_{\star \rightarrow x}(\omega) = s_x (\Gamma_{\star \rightarrow x})^{1/2} a_x^\dagger. \quad (2.14)$$

A convenient way to define the jump rates  $\Gamma_{x \rightarrow \star}$  is to imitate what was done earlier and to consider the sites  $\star$  similar as the sites occupied by electrons, associated with the energy  $\mu$ , leading to

$$\Gamma_{x \rightarrow \star}(\omega) = \Gamma_\star e^{-\beta(\mu - \epsilon_x)^+}, \quad \Gamma_{\star \rightarrow x}(\omega) = \Gamma_\star e^{-\beta(\epsilon_x - \mu)^+}, \quad \frac{\Gamma_{x \rightarrow \star}}{\Gamma_{\star \rightarrow x}} = e^{\beta(\epsilon_x - \mu)} \quad (2.15)$$

where  $x \in \mathcal{L}(s)$ . Here again,  $\Gamma_\star > 0$  is a jump rate probability fixing the time scale for the cemetery process.

**Remark 2.2.** *In the present paper  $\Gamma_\star$  is a constant. The present results however remain valid if  $\Gamma_\star$  is a function of  $\epsilon_x$ , and  $\beta$  satisfying  $\inf\{\Gamma_\star(\epsilon_x, \beta) : \epsilon_x \in \Delta, \beta \in (0, \infty)\} > 0$ . See also the related Remarks 2.1 and 5.9.*

The new jump operators describing the exchange between the thermal bath and the electron gas gives rise to a Lindblad operator denoted by  $\mathfrak{D}_\omega^\star$ . However, due to the anti-commutation rules, the Lindblad operator, acting on an observable  $A$ , takes the form

$$\begin{aligned} \mathfrak{D}_\omega^\star(A) &= \sum_{x \in \mathcal{L}(s)} \left( \frac{1}{2} (L_{x \rightarrow \star}^* L_{x \rightarrow \star} A + A L_{x \rightarrow \star}^* L_{x \rightarrow \star}) - (-1)^{d_A} L_{x \rightarrow \star}^* A L_{x \rightarrow \star} \right) + \\ &\quad \sum_{x \in \mathcal{L}(s)} \left( \frac{1}{2} (L_{\star \rightarrow x}^* L_{\star \rightarrow x} A + A L_{\star \rightarrow x}^* L_{\star \rightarrow x}) - (-1)^{d_A} L_{\star \rightarrow x}^* A L_{\star \rightarrow x} \right) \end{aligned} \quad (2.16)$$

where  $d_A$  is the degree of  $A$  given by the natural  $\mathbb{Z}_2$  grading of the CAR algebra  $\mathfrak{A}(s)$ . More details will be given below. Hence, the total dynamics is described by an operator  $\mathfrak{L}$  acting on the set of observables by

$$\mathfrak{L}_\omega(A) = \iota[F(\omega), A] + \mathfrak{D}_\omega^{kin}(A) + \mathfrak{D}_\omega^\star(A). \quad (2.17)$$

What is left for the mathematician, is to make sure that this description does not produce any hidden effect that could be related with the infinite volume limit. For indeed, the mesoscopic cells have an undefined size, only the order of magnitude of the size is fixed. In addition, the random character of the set  $\mathcal{L}(s)$  of the impurity sites and of the  $\epsilon_x$ 's must be included in the description to make sure that the results obtained are almost surely independent of the configuration of the impurities. These two requirements are the very reason why the mathematical formalism is so demanding.

### 3. THE COHERENT EVOLUTION

In this section the coherent evolution of the quantum motion is studied and the equilibrium state of the unperturbed system is obtained. The coherent evolution is a one parameter group automorphism on the algebra of the observables.

**3.1. Observables.** In this subsection the CAR algebra of the observables is studied. Recall that a CAR algebra is a  $C^*$ -algebra generated by a countable number of creation and annihilation operators which satisfy the rules stated in eq. (2.6). It is well-known, and elementary to show, that the complex algebra generated by the pair  $a_x, a_x^\dagger$  is isomorphic to the set  $M_2(\mathbb{C})$  of  $2 \times 2$  matrices by using the analogy

$$a_x \rightarrow \sigma^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad a_x^\dagger \rightarrow \sigma^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

If a finite family  $\Lambda \subset \mathcal{L}(s)$  of sites is considered instead, the  $\mathbb{Z}_2$ -graded algebra generated by the family  $\{a_x, a_x^\dagger; x \in \Lambda\}$  is isomorphic to the tensor product  $M_2(\mathbb{C})^{\otimes \Lambda}$  but the previous analogy ought to be modified in order to insure that  $a_x$  and  $a_y$  anti-commute. One way to describe such an isomorphism is given by the so-called *Jordan-Wigner transformation*: since  $\Lambda$  is finite let its elements be numbered from  $1, 2, \dots, m = |\Lambda|$ . Then

$$a_k \rightarrow \underbrace{\sigma_3 \otimes \dots \otimes \sigma_3}_{k-1} \otimes \sigma^- \otimes \underbrace{\mathbf{1}_2 \otimes \dots \otimes \mathbf{1}_2}_{m-k}, \quad a_k^\dagger \rightarrow \underbrace{\sigma_3 \otimes \dots \otimes \sigma_3}_{k-1} \otimes \sigma^+ \otimes \underbrace{\mathbf{1}_2 \otimes \dots \otimes \mathbf{1}_2}_{m-k}, \quad (3.1)$$

where

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

As can be seen, the Jordan-Wigner transformation is not canonical in that it requires the choice of an order on  $\Lambda$ . However it can sometimes be convenient for practical uses.

A *canonical* description of the observable algebra has been given in the past and the main references are [9, 10]. Let this construction be summarized here. As shown before, there is a non-canonical isomorphism between  $\mathfrak{A}_\Lambda(s)$  and  $M_2(\mathbb{C})^{\otimes \Lambda \cap \mathcal{L}(s)}$ , so that  $\mathfrak{A}_\Lambda(s)$  has dimension  $2^{|\Lambda \cap \mathcal{L}(s)|}$ . If  $\Lambda \subset \Lambda'$  there is a canonical isometric embedding  $i_{\Lambda, \Lambda'} : \mathfrak{A}_\Lambda(s) \mapsto \mathfrak{A}_{\Lambda'}(s)$  (in the Jordan-Wigner representation (3.1) this embedding consists simply in adding to the  $a_k$ 's one more factor  $\otimes \mathbf{1}_2$  at the end of the chain). The algebras  $\mathfrak{A}_\Lambda(s)$  are called *local*, while the inductive limit

$$\mathfrak{A}(s) = \lim_{\rightarrow} (\mathfrak{A}_\Lambda(s), i_{\Lambda, \Lambda'}) , \quad (3.2)$$

is equal to the *quasi-local* observable algebra. It should be remarked at this point that  $\mathfrak{A}(s)$  is *random* as it depends upon the random variable  $s = (s_x)_{x \in \mathcal{L}}$ . However, since  $s \in \{0, 1\}^{\mathcal{L}} = \Xi$  and since  $\Xi$  can be considered as a compact space, this family of algebras can be seen as a *continuous field* of  $C^*$ -algebras [17, 47, 48, 49].

Recall that a  $C^*$ -algebra algebra  $\mathfrak{A}$  is  $\mathbb{Z}_2$  graded when there exists a  $\star$ -automorphism  $\sigma$  on  $\mathfrak{A}$  which satisfies  $\sigma^2 = \mathbf{1}_{\mathfrak{A}}$ , (in other words, when the group  $\mathbb{Z}_2$  acts on  $\mathfrak{A}$ ). The  $\star$ -automorphism  $\sigma$  is called *grading*. In the case of the CAR algebra, the grading is the canonical  $\star$ -automorphism defined by  $\sigma(a_x) = -a_x$  for all  $x$ . If  $\mathfrak{A}$  is a  $\mathbb{Z}_2$  graded  $C^*$ -algebra

with grading  $\sigma$  then the elements  $A \in \mathfrak{A}$  with  $\sigma(A) = A$  are called even, and the elements  $A \in \mathfrak{A}$  with  $\sigma(A) = -A$  are called odd. The degree of every even element  $A$  is defined to be equal to zero, (denoted by  $d_A = 0$ ), and the degree of every odd element  $A$  is defined to be equal to one, (denoted by  $d_A = 1$ ). In the above Jordan-Wigner representation, the operators of degree zero are represented by diagonal matrices, while the operators of degree one are given by the off diagonal ones.

**Proposition 3.1.** *The family  $\{\mathfrak{A}(s); s \in \Xi = \{0, 1\}^{\mathcal{L}}\}$  can be endowed with the structure of a continuous field of  $\mathbb{Z}_2$ -graded  $C^*$ -algebras.*

**Proof:** Recall first [17] that a section of the field  $(\mathfrak{A}(s))_{s \in \Xi}$  is an element of the Cartesian product  $\prod_{s \in \Xi} \mathfrak{A}(s)$ . To define a structure of a continuous field it is necessary to define first a set  $\mathcal{F}$  of *reference sections* with the property that (i) for each  $s \in \Xi$  the set  $\{\xi(s); \xi \in \mathcal{F}\}$  is dense in  $\mathfrak{A}(s)$  and (ii) if  $\xi \in \mathcal{F}$ , the map  $s \in \Xi \mapsto \|\xi(s)\| \in \mathbb{R}_+$  is continuous. Then a *continuous section* is an element  $\eta \in \prod_{s \in \Xi} \mathfrak{A}(s)$  such that for any  $s \in \Xi$  and any  $\epsilon > 0$  there is a neighborhood  $\mathcal{U}$  of  $s$  in  $\Xi$  and a reference section  $\xi \in \mathcal{F}$  such that  $\|\xi(s') - \eta(s')\| < \epsilon$  for  $s' \in \mathcal{U}$ .

In the present situation it ought to be remarked that the product topology on  $\Xi$  is defined through the set of open sets  $\mathcal{U}_\Lambda(s) = \{s' \in \Xi; s'_x = s_x, \forall x \in \Lambda\}$ . Hence  $s'$  is close to  $s$  if there is some finite subset  $\Lambda \subset \mathcal{L}$  such that  $s'_x = s_x$  for  $x \in \Lambda$ . Then let  $a_x$  be the section defined by  $a_x(s) = s_x a_x \in \mathfrak{A}(s)$ . Let  $\mathcal{F}$  be the  $*$ -algebra defined by the  $a_x$ 's. Namely  $\xi \in \mathcal{F}$  if and only if there is  $\Lambda \subset \mathcal{L}$  finite, an integer  $M \in \mathbb{N}$  and continuous functions  $\{\lambda_{x_1, \dots, x_m}^{\#_1, \dots, \#_m} \in \mathcal{C}(\Xi)\}$  such that

$$\xi(s) = \sum_{m=0}^M \sum_{x_1, \dots, x_m \in \Lambda} \sum_{\#_1, \dots, \#_m} \lambda_{x_1, \dots, x_m}^{\#_1, \dots, \#_m}(s) s_{x_1} \cdots s_{x_m} a_{x_1}^{\#_1} \cdots a_{x_m}^{\#_m} \quad (3.3)$$

where the  $\#_i$ 's  $\in \{., \dagger\}$  label the  $a_x, a_x^\dagger$ 's. Clearly, if  $\Lambda \subset \Lambda'$  and if  $s' \in \mathcal{U}_{\Lambda'}(s)$  the previous expression for  $\xi(s)$  does not change as  $s'$  replace  $s$ , so that  $s \in \Xi \mapsto \|\xi(s)\| \in \mathbb{R}_+$  is continuous. Moreover, the set  $\{\xi(s); \xi \in \mathcal{F}\}$  generate  $\mathfrak{A}(s)$  as a  $C^*$ -algebra. Hence  $\mathcal{F}$  satisfy the conditions.  $\square$

**Proposition 3.2.** *The continuous field  $((\mathfrak{A}(s))_{s \in \Xi}, \mathcal{F})$  is covariant by the translation group, namely, for any  $a \in \mathbb{Z}^d$ , there is a  $\mathbb{Z}_2$ -graded  $*$ -isomorphism  $\eta_a : \mathfrak{A}(s) \mapsto \mathfrak{A}(T^a s)$  leaving  $\mathcal{F}$  invariant.*

**Proof:** It is elementary to check that the  $*$ -isomorphism  $\eta_a$  is defined by  $\eta_a(a_x)(s) = s_x a_{x+a}$  exists and does the job since  $(T^a s)_{x+a} = s_x$ . Moreover it is easy to check that  $\eta_a \circ \eta_b = \eta_{a+b}$ , that  $\eta_0 = id$  and that  $\eta_a$  commutes with the grading.  $\square$

The continuous field  $((\mathfrak{A}(s))_{s \in \Xi}, \mathcal{F})$  contains an important sub-field made of commutative  $C^*$ -algebras. For given  $s \in \Xi$ , let  $\mathfrak{C}(s)$  be the closed sub-algebra of  $\mathfrak{A}(s)$  generated by the family  $\{s_x n_x; x \in \mathcal{L}\}$  where  $n_x = a_x^\dagger a_x$ . It is easy to see that  $\mathfrak{C}(s)$  is commutative, it contains only elements of degree zero, it is generated by projections and generates also a continuous  $\mathbb{Z}^d$ -covariant field of  $C^*$ -algebras.

Two new ingredients will be needed in the rest of the paper. First it will be convenient to consider the field  $\mathfrak{A}$  as a field over the compact space  $\Omega$  instead. The only difference will be that, in the eq. (3.3), the coefficients will be allowed to depend continuously upon  $\omega = (s, \epsilon)$ . When necessary this field will be denoted by  $\mathfrak{A}(\omega)$ , and similarly the corresponding commutative sub-field will be denoted by  $\mathfrak{C}(\omega)$ . The following can be found in [17].

**Corollary 3.3.** *The field  $\mathfrak{A}$  can be continued as a continuous field  $\mathfrak{A} = (\mathfrak{A}(\omega))_{\omega \in \Omega}$  of  $C^*$ -algebras which is covariant with respect to the translation group. Similarly  $\mathfrak{C} = (\mathfrak{C}(\omega))_{\omega \in \Omega}$  will denote the abelian sub-field extending  $(\mathfrak{C}(s))_{s \in \Xi}$ .*

The other ingredient is the use of *groupoids* to describe the  $\mathbb{Z}^d$  action [14, 38]. Here the group  $\mathbb{Z}^d$  acts on the space  $\Omega$  by  $\mathsf{T}^a(\omega) = \mathsf{T}^a(s, \epsilon) = (\mathsf{T}^a(s), \mathsf{T}^a(\epsilon))$  where  $a \in \mathbb{Z}^d$ ,  $(\mathsf{T}^a(s))_x = s_{x-a}$  and  $(\mathsf{T}^a(\epsilon))_x = \epsilon_{x-a}$ . Consequently, the *crossed product*  $\Gamma_\Omega = \Omega \rtimes \mathbb{Z}^d$  is a groupoid described as follows: (i) elements are pairs  $\gamma = (\omega, a) \in \Omega \times \mathbb{Z}^d$ , (ii)  $\Omega$  is the set of *objects* or of *units*, (iii) each element has a *range* and a *source* in  $\Omega$ , here  $r(\omega, a) = \omega$ ,  $s(\omega, a) = \mathsf{T}^{-a}\omega$ , (iv) two elements  $\gamma$  and  $\gamma'$  are composable if  $s(\gamma) = r(\gamma')$ , namely if  $\gamma = (\omega, a)$  then  $\gamma' = (\mathsf{T}^{-a}\omega, a')$  and there is a composition law  $\gamma \circ \gamma'$  which is here given by  $(\omega, a + a')$ , (v) the elements of the form  $(\omega, 0)$  are units and can be identified with the points in  $\Omega$ , (vi) each element  $\gamma$  admits an inverse with exchange of range and source, namely here  $\gamma^{-1} = (\mathsf{T}^{-a}\omega, -a)$ . The groupoid  $\Gamma_\Omega$  will be endowed with the product topology and it is elementary to check that all groupoid maps defined above are continuous.

To express the covariance of the field  $\mathfrak{A} = (\mathfrak{A}(\omega))_{\omega \in \Omega}$ , it is convenient to see the translation  $\eta_a$  as a function of the groupoid variables instead, namely if  $\gamma = (\omega, a)$ , then  $\gamma$  can be seen as sending its source  $s(\gamma) = \mathsf{T}^{-a}\omega$  into its range  $r(\gamma) = \omega$ , so that

$$\eta_{(\omega, a)} : \mathfrak{A}(\mathsf{T}^{-a}\omega) \mapsto \mathfrak{A}(\omega) .$$

**Definition 3.4.** *A field  $(\theta_\omega)_{\omega \in \Omega}$  of  $*$ -homomorphisms from  $\mathfrak{A}(\omega)$  into itself will be called covariant if the following diagram is commutative*

$$\begin{array}{ccc} \mathfrak{A}(\mathsf{T}^{-a}\omega) & \xrightarrow{\eta_{(\omega, a)}} & \mathfrak{A}(\omega) \\ \theta_{\mathsf{T}^{-a}\omega} \downarrow & & \theta_\omega \downarrow \\ \mathfrak{A}(\mathsf{T}^{-a}\omega) & \xrightarrow{\eta_{(\omega, a)}} & \mathfrak{A}(\omega) \end{array} .$$

*This field will be called continuous whenever it transforms every continuous section of  $\mathfrak{A}(\omega)$  into a continuous section.*

**3.2. The coherent dynamics.** Recall that the coherent part of the quantum motion is a group automorphism acting on the CAR algebra  $\mathfrak{A}$ . Let  $\Lambda \subset \mathcal{L}$  be a finite set. Then the particle number  $N_\Lambda$ , the Hamiltonian  $H_\Lambda$  and the free energy  $F_\Lambda$  within  $\Lambda$  are defined as follows:

$$N_\Lambda(\omega) = \sum_{x \in \Lambda} s_x n_x , \quad H_\Lambda(\omega) = \sum_{x \in \Lambda} s_x \epsilon_x n_x , \quad (3.4)$$

$$F_\Lambda(\omega) = H_\Lambda(\omega) - \mu N_\Lambda(\omega) = \sum_{x \in \Lambda} s_x(\epsilon_x - \mu) n_x, \quad (3.5)$$

where

$$n_x = a_x^\dagger a_x \quad \text{and} \quad \omega = (s, \epsilon) \in \Omega.$$

The finite volume coherent dynamics is generated by the free energy operator

$$\alpha_t^{(\omega, \Lambda)}(A) = e^{t(H_\Lambda - \mu N_\Lambda)} A e^{-t(H_\Lambda - \mu N_\Lambda)}, \quad A \in \mathfrak{A}_\Lambda(\omega), \quad t \in \mathbb{R}. \quad (3.6)$$

Usual arguments [10], which become trivial in the present situation, show that the infinite volume limit exists, namely

$$\alpha_t^{(\omega)}(A) = \lim_{\Lambda \uparrow \mathcal{L}} \alpha_t^{(\omega, \Lambda)}(A), \quad A \in \mathfrak{A}_{\Lambda_0}(s), \quad t \in \mathbb{R}. \quad (3.7)$$

Indeed, it is elementary to show that

$$\alpha_t^{(\omega)}(s_x a_x^\dagger) = e^{ts_x(\epsilon_x - \mu)} s_x a_x^\dagger, \quad \alpha_t^{(\omega)}(s_x a_x) = e^{-ts_x(\epsilon_x - \mu)} s_x a_x, \quad (3.8)$$

since it is already true for  $\alpha_t^{(\omega, \Lambda)}$  as soon as  $\Lambda \ni x$ . Hence,  $\alpha_t^{(\omega)}$  can be computed on any monomial, thus on any polynomial, in the creation-annihilation operator. Therefore  $\alpha_t^{(\omega)}$  is defined anywhere in  $\mathfrak{A}(s)$ . It also follows from these formulas that  $s_x n_x$  is invariant by the dynamics, a fact which comes from ignoring the hopping terms due to possible tunneling between impurity sites. As a consequence, the elements of  $\mathfrak{C}$  are left invariant by the dynamics. To summarize, thanks to the Definition 3.4, the following holds (the proof is left to the reader).

**Proposition 3.5.** *The field of equilibrium dynamics  $\alpha^{(\omega)} = \left( \alpha_t^{(\omega)} \right)_{t \in \mathbb{R}}$  induced by the Hamiltonian (3.4) on each  $\mathfrak{A}(\omega)$  is well defined,  $\mathbb{Z}^d$ -covariant and continuous. Moreover its restriction to the sub-field  $\mathfrak{C}$  is trivial.*

The generator of the coherent evolution  $\alpha_t^{(\omega)}$  is given by the commutator with the free energy operator defined in eq. (3.5), *i.e.*

$$\delta_\omega(A) = \left. \frac{d\alpha_t^{(\omega)}(A)}{dt} \right|_{t=0} = \lim_{\Lambda \uparrow \mathcal{L}} \imath[F_\Lambda(\omega), A] \quad \text{for} \quad A \in \mathfrak{A}_{loc}. \quad (3.9)$$

**3.3. Equilibrium State.** Once the dynamics is defined, the next step is to find the possible equilibrium states. There are two ways to do that. The first is to consider the finite volume approximations and establish the quantum version of the DLR equations. The other one, valid in the infinite volume limit, consists in implementing the Kubo-Martin-Schwinger conditions (KMS) which were introduced in [26, 30] and further studied in [21, 44, 10, 46]. In the present situation, due to the extreme simplicity of the dynamics, both approaches can be used and lead to the same explicit result. The KMS condition will be used here.



**Definition 3.6.** Let  $\rho$  be a state on a  $C^*$ -algebra  $\mathfrak{A}$ , and let  $\alpha = (\alpha_t)_{t \in \mathbb{R}}$  be a one-parameter group of  $*$ -automorphisms of  $\mathfrak{A}$ . Then  $\rho$  will be called  $\beta$ -KMS with respect to  $\alpha$  if

- (i) it is invariant by the dynamics, namely  $\rho \circ \alpha_t = \rho$  for all  $t \in \mathbb{R}$ ,
- (ii) if  $A, B \in \mathfrak{A}$ , then

$$\rho(AB) = \rho(\alpha_{-i\beta}(B)A) \quad \text{(KMS condition)} \quad (3.10)$$

Note that (i) is a trivial consequence of (ii) if  $\mathfrak{A}$  is unital, but it is convenient to define this condition in this way. In the present situation the following holds.

**Proposition 3.7.** For  $\mathbb{P}$ -almost every  $\omega = (s, \epsilon) \in \Omega$ , there is a unique  $\beta$ -KMS state  $\rho_\omega$  on  $\mathfrak{A}(\omega)$  with respect to the dynamic  $\alpha^{(\omega)}$ . It is defined by

$$\rho_\omega \left( \prod_{x \in \Lambda} A_x \right) = \prod_{x \in \Lambda} \rho_\omega(A_x), \quad A_x \in \mathfrak{A}_{\{x\}}(s) \quad (3.11)$$

$$\rho_\omega(s_x a_x^\sharp) = 0, \quad (3.12)$$

$$\rho_\omega(s_x n_x) = s_x \frac{1}{1 + e^{\beta(\epsilon_x - \mu)}}. \quad (3.13)$$

Moreover, the field of such states is continuous and covariant with respect to translations, namely  $\rho_\omega \circ \eta_{\omega, a} = \rho_{\tau^{-a}\omega}$ .

**Proof:** (i) In order to prove eq. (3.12) and (3.13), it is sufficient to assume that  $s_x = 1$ . Then, each element of the elementary algebra  $\mathfrak{A}_{\{x\}}$  can be written as a linear combination of  $\mathbf{1}$ ,  $a_x$ ,  $a_x^\dagger$  and  $n_x = a_x^\dagger a_x$ . Thanks to eq. (3.8) and since a KMS-state is time invariant, it follows that  $\rho_\omega(a_x) = \rho_\omega(\alpha_t(a_x)) = e^{-it(\epsilon_x - \mu)} \rho_\omega(a_x)$  for all  $t \in \mathbb{R}$ . Since the distribution of  $\epsilon_x$  is absolutely continuous,  $\epsilon_x \neq \mu$   $\mathbb{P}$ -almost surely. Hence  $\rho_\omega(a_x) = 0$ . In much the same way  $\rho_\omega(a_x^\dagger) = 0$ . Similarly, if  $A, B \in \mathfrak{A}_\Lambda(\omega)$  where  $x \notin \Lambda$ , the same argument shows that  $\rho_\omega(Aa_x B) = \rho_\omega(Aa_x^\dagger B) = 0$ . Consequently if  $A \in \mathfrak{A}(\omega) \setminus \mathfrak{C}(\omega)$  it follows by induction on  $\Lambda$  and by density, that  $\rho_\omega(A) = 0$ . Hence it is sufficient to reduce the analysis on the commutative sub-algebra  $\mathfrak{C}(\omega)$ .

(ii) Since  $n_x$  commutes to any element of  $\mathfrak{C}_\Lambda(\omega)$ , it is sufficient to consider an expression of the form  $\rho_\omega(An_x) = \rho_\omega(Aa_x^\dagger a_x)$  where  $A$  is a local observable in  $\mathfrak{C}(\omega)$  with support not meeting  $x$ . Using the KMS-condition, this gives

$$\rho_\omega(An_x) = \rho_\omega(\alpha_{-i\beta}(a_x)Aa_x^\dagger) = e^{-\beta(\epsilon_x - \mu)} \rho_\omega(a_x A a_x^\dagger).$$

Since the support of  $A$  does not meet  $x$  and  $A \in \mathfrak{C}(\omega)$ , it follows that  $A$  commutes with  $a_x$  so that  $\rho_\omega(a_x A a_x^\dagger) = \rho_\omega(A a_x a_x^\dagger) = \rho_\omega(A(\mathbf{1} - n_x))$ . Hence this gives

$$\rho_\omega(An_x) = \frac{\rho_\omega(A)}{1 + e^{\beta(\epsilon_x - \mu)}}$$

Eliminating points of the support of  $A$  one after another leads to the formulas above.

(iii) It is obvious that the field  $\rho = (\rho_\omega)_{\omega \in \Omega}$  of states is continuous. For indeed it is enough to show that the map  $\omega \in \Omega \mapsto \rho_\omega(\xi(\omega)) \in \mathbb{C}$  is continuous for any continuous section of the field  $\mathfrak{A}$ . By definition of continuous sections, it is sufficient to chose  $\xi \in \mathcal{F}$ . But

this is exactly choosing  $\xi(\omega)$  as a polynomial in the creation-annihilation operators. The formulas (3.11), (3.12) and (3.13) show immediately the continuity with respect to  $\omega$ .

(iv) Similarly, the covariance with respect to translation follows immediately from the formulas (3.11), (3.12) and (3.13).  $\square$

**Proposition 3.8.** *For all  $\omega \in \Omega$  the state  $\rho_\omega$  is faithful.*

**Proof:** First the state  $\rho_\omega$  is defined by the formulas (3.11), (3.12) and (3.13). It is easy to check that such result can also be obtained from (omitting the reference to  $\omega$ )

$$\rho(A) = \frac{\text{Tr} \left( e^{-\beta F_\Lambda} A \right)}{\mathcal{Z}(\Lambda)}, \quad A \in \mathfrak{A}_\Lambda,$$

where  $\mathcal{Z}(\Lambda)$  is a normalization constant and  $F_\Lambda = H_\Lambda - \mu N_\Lambda$ . In particular, the restriction of  $\rho$  to  $\mathfrak{A}_\Lambda$  is faithful. Moreover, there is a conditional expectation  $\mathbb{E}_\Lambda : \mathfrak{A} \mapsto \mathfrak{A}_\Lambda$  defined by

$$\mathbb{E}_\Lambda(A) = \mathbb{E}_{\Lambda', \Lambda}(A) = \frac{\text{Tr}_{\Lambda' \setminus \Lambda} \left( e^{-\beta F_{\Lambda' \setminus \Lambda}} A \right)}{\mathcal{Z}(\Lambda)}, \quad \text{if } A \in \mathfrak{A}_{\Lambda'}.$$

It is elementary to check that  $A \geq 0 \Rightarrow \mathbb{E}_\Lambda(A) \geq 0$ , that  $\mathbb{E}_\Lambda(\mathbf{1}) = \mathbf{1}$ , so that  $\|\mathbb{E}_\Lambda(A)\| \leq \|A\|$  for  $A \in \mathfrak{A}$ . Moreover, it is easy to check that  $\rho \circ \mathbb{E}_\Lambda = \rho$  for all finite  $\Lambda \subset \mathcal{L}$ .

Let now  $A \in \mathfrak{A}$  be positive and such that  $\rho(A) = 0$ . Then  $\rho(A) = \rho(\mathbb{E}_\Lambda(A)) = 0$ . Since  $\rho$  is faithful on  $\mathfrak{A}_\Lambda$  it follows that  $\mathbb{E}_\Lambda(A) = 0$  for all finite  $\Lambda$ 's. Let  $\varepsilon > 0$ , there is  $\Lambda_\varepsilon \subset \mathcal{L}$  finite and  $A_\varepsilon \in \mathfrak{A}_{\Lambda_\varepsilon}$  such that  $\|A - A_\varepsilon\| < \varepsilon/2$ . Hence, whenever  $\Lambda \supset \Lambda_\varepsilon$ ,  $\|\mathbb{E}_\Lambda(A - A_\varepsilon)\| = \|A_\varepsilon\| < \varepsilon/2$ . Therefore  $\|A\| \leq \varepsilon$  for any  $\varepsilon$  implying  $A = 0$ .  $\square$

**3.4. The GNS Representation.** The Gelfand-Naïmark-Segal construction (GNS) is a fundamental tool in the study of  $C^*$ -algebras. Let  $\mathfrak{A}$  be a  $C^*$ -algebra, for convenience it will be assumed to be unital. Let  $\rho$  be a state on  $\mathfrak{A}$ . Then a Hilbertian inner product can be defined through  $\langle A|B \rangle = \rho(A^*B)$  for  $A, B \in \mathfrak{A}$ . The set  $\mathfrak{N}$  of elements  $A \in \mathfrak{A}$  for which  $\rho(A^*A) = 0$  is a closed left  $\mathfrak{A}$ -module. The quotient space  $\mathfrak{A}/\mathfrak{N}$  inherits the structure of a separated pre-Hilbert space. By completion it gives a Hilbert space  $\mathcal{H} = L^2(\mathfrak{A}, \rho)$  and a canonical linear map  $\zeta : A \in \mathfrak{A} \mapsto \zeta(A) \in \mathcal{H}$  such that

$$\langle \zeta(A)|\zeta(B) \rangle = \rho(A^*B). \quad (3.14)$$

If  $\mathfrak{A}$  is abelian, then, by Gelfand's theorem, it is isomorphic to the set of continuous functions on some compact space  $X$ , unique up to homeomorphism, called the spectrum of  $\mathfrak{A}$ . Then a state is just a probability measure on  $X$  and  $\mathcal{H} = L^2(X, \rho)$ .

$\mathcal{H}$  inherits the structure of a left  $\mathfrak{A}$ -module so that the map  $\pi(A) : \zeta(B) \in \mathcal{H} \mapsto \zeta(AB) \in \mathcal{H}$  is well defined and extends to a representation of  $\mathfrak{A}$  in  $\mathcal{H}$ . The vector  $\xi = \zeta(\mathbf{1})$  is cyclic since  $\zeta(A) = \pi(A)\xi$  so that  $\mathcal{H}$  is the closure of the set of  $\pi(A)\xi$  as  $A$  varies in  $\mathfrak{A}$ . The weak closure of  $\pi(\mathfrak{A})$  is a von Neumann algebra denoted by  $L^\infty(\mathfrak{A}, \rho) = \mathfrak{M}$ . Clearly,  $\rho$  extends as a normal state on  $\mathfrak{M}$  since

$$\rho(A) = \langle \xi|\pi(A)\xi \rangle. \quad (3.15)$$

If  $\mathfrak{A}$  is abelian with spectrum  $X$ , the representation is given by point wise multiplication in  $L^2(X, \rho)$ . The vector  $\xi$  is the constant function equal to one and the von Neumann algebra is the space  $L^\infty(X, \rho)$  of essentially bounded  $\rho$ -measurable functions on  $X$ .

If  $\alpha$  is a  $*$ -automorphism of  $\mathfrak{A}$  leaving  $\rho$  invariant, it defines a unitary operator  $U_\alpha : \zeta(A) \mapsto \zeta(\alpha(A)) \in \mathcal{H}$  such that

$$U_\alpha \pi(A) U_\alpha^{-1} = \pi(\alpha(A)), \quad A \in \mathfrak{A}, \quad U_\alpha \xi = \xi. \quad (3.16)$$

The Tomita-Takesaki theory [44] is based upon the densely defined conjugate linear operator  $S$  defined by  $S\zeta(A) = \zeta(A^*)$ . The main result of Tomita and Takesaki is that  $S$  is well defined, closable and that, if  $S$  denotes also the closure,  $\Delta = S^*S$  is a positive self-adjoint operator on  $\mathcal{H}$  called the *modular operator*. The polar decomposition  $S = J\Delta^{1/2}$ , defines a conjugate linear involution  $J$  such that  $J\mathfrak{M}J$  coincides with the commutant of  $\mathfrak{M}$  on  $\mathcal{H}$ . It has been shown [44] that there is a group of  $*$ -automorphism  $\theta$  on  $\mathfrak{M}$  satisfying a KMS-condition, namely

$$\theta_t(M) = \Delta^{it} M \Delta^{-it}, \quad \rho(M_1 M_2) = \rho(\theta_{-i}(M_2) M_1), \quad M, M_1, M_2 \in \mathfrak{M}.$$

In the commutative case the modular operator is trivial while  $JA$  coincides with the complex conjugate of  $A$ . In the early seventies, Araki [4] and Connes [13] realized that the set of  $\zeta(A)$  with  $A \in \mathfrak{A}, A \geq 0$  generates a cone  $\mathcal{H}_+$  defined by

$$\mathcal{H}_+ = \overline{\{\Delta^{1/4} \pi(A) \xi; A \in \mathfrak{A}, A \geq 0\}}, \quad J\mathcal{H}_+ = \mathcal{H}_+.$$

Connes characterized such a positive cone in  $L^2(\mathfrak{A}, \rho)$  as being *self-dual*, *homogeneous* and *oriented*. It is the non-commutative analog of the set of positive square integrable functions.

In the present situation the general theory gives the following result.

**Proposition 3.9.** *Let  $\mathfrak{A} = (\mathfrak{A}(\omega))_{\omega \in \Omega}$  be the continuous field of  $C^*$ -algebras defined in Section 3.1. Let  $\alpha = (\alpha^{(\omega)})_{\omega \in \Omega}$  be the field of dynamics defined in Section 3.2. Then the GNS-construction leads to a continuous field  $\mathcal{H} = (\mathcal{H}_\omega)_{\omega \in \Omega}$  of Hilbert spaces, a continuous section  $\xi = (\xi_\omega)_{\omega \in \Omega}$  of unit vectors, with a  $\mathbb{Z}^d$  action, namely a unitary representation of the groupoid  $\Gamma_\Omega$ , given by unitary maps  $V_{\omega, a} : \mathcal{H}_{\mathbb{T}^{-a}\omega} \mapsto \mathcal{H}_\omega$  satisfying*

$$V_{\omega, a+b} = V_{\omega, a} V_{\mathbb{T}^{-a}\omega, b}, \quad V_{\omega, a} \xi_{\mathbb{T}^{-a}\omega} = \xi_\omega.$$

*The  $\mathbb{Z}_2$ -grading in  $\mathfrak{A}$  is represented by a continuous covariant field  $G = (G_\omega)_{\omega \in \Omega}$  of operators satisfying  $G_\omega = G_\omega^* = G_\omega^{-1}$  and the covariance condition*

$$V_{\omega, a} G_{\mathbb{T}^{-a}\omega} V_{\omega, a}^{-1} = G_\omega, \quad G_\omega \xi_\omega = \xi_\omega.$$

*The field  $\mathfrak{A}$  is represented by a continuous field  $\pi = (\pi_\omega)_{\omega \in \Omega}$  of representations, for which  $\xi$  is a field of cyclic vectors, and satisfying the covariance condition*

$$V_{\omega, a} \pi_{\mathbb{T}^{-a}\omega}(A) V_{\omega, a}^{-1} = \pi_\omega(\eta_{\omega, a}(A)), \quad A \in \mathfrak{A}(\mathbb{T}^{-a}\omega).$$

*The dynamic gives rise to a strongly continuous group of unitary operators on  $\mathcal{H}_\omega$  with generator  $F_\omega$ . The latter defines a continuous covariant field of self-adjoint operators*

satisfying

$$V_{\omega,a} F_{T^{-a}\omega} V_{\omega,a}^{-1} = F_\omega, \quad F_\omega \xi_\omega = 0.$$

The family of Tomita-Takesaki modular operators define the covariant continuous field of KMS-dynamics as follows,

$$\theta_t = \alpha_{t\beta}^{(\omega)}, \quad (t \in \mathbb{R}), \quad \Delta_\omega \prod_{x \in \Lambda} \pi_\omega(s_x a_x^{\sharp x}) \xi_\omega = e^{\beta \sum_{x \in \Lambda} \sharp x s_x (\epsilon_x - \mu)} \prod_{x \in \Lambda} \pi_\omega(s_x a_x^{\sharp x}) \xi_\omega,$$

where  $\sharp$  denotes  $\{\cdot, \dagger\}$  in the exponent of the  $a$ 's, while it corresponds to  $\{-1, +1\}$  respectively in the exponential. The Araki-Connes cones  $\mathcal{H}_{\omega,+}$  gives also a continuous covariant field of self-dual, homogeneous, oriented cones generated by vectors of the form

$$\pi_\omega \left( e^{\beta/4(H_\Lambda(\omega) - \mu N_\Lambda(\omega))} A e^{-\beta/4(H_\Lambda(\omega) - \mu N_\Lambda(\omega))} \right) \xi_\omega, \quad \Lambda \subset \mathcal{L}, A \in \mathfrak{A}_\Lambda(\omega), A \geq 0.$$

Since the proof is straightforward it will be left to the reader.

**Remark 3.10.** The notation  $F_\omega$  for the generator of the dynamics in  $\mathcal{H}_\omega$  is justified, because it corresponds to the free energy and can be seen as an infinite volume limit of  $H_\Lambda - \mu N_\Lambda$ .

#### 4. DISSIPATIVE DYNAMICS

The general dissipative operator  $\mathfrak{D}_\omega = \mathfrak{D}_\omega^{kin} + \mathfrak{D}_\omega^*$  defined in Subsection 2.5 is the main object of study of this section.  $\mathfrak{D}$  can be considered as a field of operators  $(\mathfrak{D}_\omega)_{\omega \in \Omega}$  on the field of  $C^*$ -algebras  $(\mathfrak{A}(\omega))_{\omega \in \Omega}$ . Theorem 4.9 is the main result of the section where the Friedrich extension Theorem is used to prove that the closure of the operator  $\mathfrak{D}_\omega$  is a positive self-adjoint operator on the Hilbert space of the GNS representation of the algebra of the observables using the equilibrium state.

**4.1. Complete Positivity.** If  $\mathfrak{A}$  is a unital  $C^*$ -algebra then a map  $\eta : \mathfrak{A} \rightarrow \mathfrak{A}$  is positive whenever  $A \geq 0 \Rightarrow \eta(A) \geq 0$ . Then  $\eta$  extends to  $\mathfrak{A} \otimes M_n(\mathbb{C})$  by  $\eta_n = \eta \otimes id$ . Then  $\eta$  is called completely positive, (a term which was introduced in [43]), if  $\eta_n$  is positive for all  $n$ 's.  $\eta$  is called normalized if  $\eta(\mathbf{1}_{\mathfrak{A}}) = \mathbf{1}_{\mathfrak{A}}$ , whenever  $\mathbf{1}_{\mathfrak{A}}$  denotes the unit of  $\mathfrak{A}$ . By  $CP(\mathfrak{A})$  we will denote the set of completely positive maps and by  $CP_1(\mathfrak{A})$  the subset of normalized CP-maps. Examples of CP-maps are:

- (1) Any  $*$ -homomorphism is CP. It is normalized if it sends  $\mathbf{1}$  to itself.
- (2) If  $L \in \mathfrak{A}$  then the map  $A \mapsto L^* A L$  is CP. It is normalized if and only if  $L$  is an isometry.
- (3) If  $\Phi$  is CP and  $\Phi(\mathbf{1})$  is invertible, then the map  $\Phi'(A) = \Phi(\mathbf{1})^{-1/2} \Phi(A) \Phi(\mathbf{1})^{-1/2}$  is CP and normalized.
- (4) Any convex combination of CP-maps is CP and the same holds for  $CP_1$ .
- (5) The composition of two CP-maps is CP and the same holds for  $CP_1$ .
- (6) If  $(\Phi_n)_{n \in \mathbb{N}}$  is a sequence of CP-maps such that  $\Phi(A) = \lim_{n \rightarrow \infty} \Phi_n(A)$  exists for all  $A \in \mathfrak{A}$ , then  $\Phi$  is CP, the same holds for  $CP_1$ .
- (7) If  $L \in \mathfrak{A}$  let  $\Psi_L(A) = L^* A + A L$ . Then  $e^{-t\Psi_L}(A) = e^{-tL^*} A e^{-tL}$  is CP for  $t > 0$ .

As a result we get the following.

**Proposition 4.1.** *Let  $\mathfrak{A}$  be a  $\mathbb{Z}_2$ -graded  $C^*$ -algebra with grading automorphism  $\sigma$ . Let  $\Psi : \mathfrak{A} \mapsto \mathfrak{A}$  be a linear map of the form*

$$\Psi(A) = \sum_{i=1}^N \left\{ \frac{1}{2} (L_i^* L_i A + A L_i^* L_i) - (-1)^{d_{L_i} d_A} L_i^* A L_i \right\}, \quad (4.1)$$

where  $L_i \in \mathfrak{A}$  for all  $i$ 's and all of them are either even or odd elements. Then the map  $\Phi_t = e^{-t\Psi}$  commutes with the grading and is  $\text{CP}_1$  for  $t \geq 0$ .

**Proof:** Since  $\Psi$  is a bounded operator, the exponential does exist. Moreover  $\Psi(\mathbf{1}) = 0$  since the degree of  $\mathbf{1}$  is zero, so that  $\Phi_t(\mathbf{1}) = \mathbf{1}$  for all  $t$ 's. Since  $\Psi$  is a finite sum of operators, the Trotter product formula [50, 24] will prove that  $\Phi_t \in \text{CP}$  as soon as each pieces of the sum gives a CP-map. Since the grading is given by a  $*$ -automorphism, the map  $A \mapsto (-1)^{d_{L_i} d_A} L_i^* A L_i$  can be seen as the composition of two CP-maps namely  $a \mapsto \sigma^d(A)$  (with  $d$  the degree of  $L_i$ ) and  $A \mapsto L_i^* A L_i$ . Exponentiation is given by a limit of polynomials in these maps with positive coefficients, namely it is CP. The other maps have the form  $A \mapsto L_i^* L_i A + A L_i^* L_i$ , the exponential of which is CP as well (see the last example in the list above).  $\square$

The proposition above justifies the form of the Lindbladian in eq. (2.8) for the generator of a Markov semigroup acting on a finite  $C^*$ -algebra. It also gives the extension to  $\mathbb{Z}_2$ -graded  $C^*$ -algebras .

Let now  $\rho$  be a grading invariant state on  $\mathfrak{A}$ . Then the GNS-construction gives a Hilbert space  $\mathcal{H} = L^2(\mathfrak{A}, \rho)$ , a representation of  $\mathfrak{A}$  and a cyclic vector  $\xi_0$ . Moreover, the state  $\rho$  is  $\sigma$ -invariant, so that  $\sigma$  defines a unitary operator  $G$  on  $\mathcal{H}$ , called the *degree*, such that  $G\xi_0 = \xi_0$ . Then clearly  $G\pi(a)G^{-1} = \pi(\sigma(a))$ . In addition, since  $\sigma$  is an involution,  $G^2 = \mathbf{1}_{\mathcal{H}}$ ,  $G^* = G$ . Let  $\Delta, J$  be the modular operator and the modular involution (see Section 3.4) and let  $\mathcal{H}_+$  be the Araki-Connes homogeneous self-dual oriented cone in  $\mathcal{H}$ .

If  $\mathfrak{A} = M_n(\mathbb{C})$  and if  $\rho$  is the normalized trace  $\text{tr}_n$  then the Hilbert space  $\mathcal{H}$  is the space  $L^2(M_n)$  of Hilbert-Schmidt operators on  $\mathbb{C}^n$ . It can be seen as the set of families  $(x_{ij})_{1 \leq i, j \leq n}$  with inner product  $\langle x|y \rangle = (1/n) \sum_{i, j} \bar{x}_{ij} y_{ij}$ . The corresponding modular operator is trivial while the conjugacy is defined by  $j_n(x)_{ij} = \bar{x}_{ji}$ . The positive cone is the set  $\text{HS}_+(n)$  of positive  $n \times n$  Hilbert-Schmidt matrices. Consequently,  $M_n(\mathfrak{A}) = \mathfrak{A} \otimes M_n(\mathbb{C})$  can be endowed with the states  $\rho \otimes \text{tr}_n$ , leading to the Hilbert space  $\mathcal{H} \otimes L^2(M_n)$ , made of families  $(x_{ij})_{1 \leq i, j \leq n}$  with  $x_{ij} \in \mathcal{H}$ . The inner product is  $\langle x|y \rangle_n = (1/n) \sum_{i, j} \langle x_{ij}|y_{ij} \rangle$ . The modular operator is then  $\Delta_n = \Delta \otimes \mathbf{1}_n$ , with modular conjugacy  $J \otimes j_n$ , namely  $J_n(x)_{ij} = J(x_{ji})$ . The corresponding positive cone will be denoted by  $\mathcal{H}_{n+}$ .

**Definition 4.2.** *A bounded linear map  $F$  on  $\mathcal{H}$  is positivity preserving if  $F(\mathcal{H}_+) \subset \mathcal{H}_+$ . It will be called completely positive if  $F_n = F \otimes \mathbf{1}_n$  is positivity preserving on  $\mathcal{H}_n$  for any  $n$ .*

It follows from this definition that if  $(F_m)_{m \in \mathbb{N}}$  is a sequence of completely positive maps on  $\mathcal{H}$  converging weakly to  $F$ , then  $F$  is completely positive as well.

**4.2. Markov Semi-Groups.** One important property of the Lindbladian is the analog of the Leibniz formula for the second derivative, namely  $(fg)'' - f''g - fg'' = 2f'g'$ . The following formula shows that a Lindbladian as defined in eq. (4.1) behaves like the analog of  $-\Delta$  if  $\Delta$  is a Laplacian. For the statement of this proposition, if  $\mathfrak{A}$  is a  $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ -graded CAR algebra, define the *graded commutator* by

$$[A, B]_g = AB - (-1)^{d_A d_B} BA \quad \text{for } A, B \in \mathfrak{A}. \quad (4.2)$$

A *graded derivation* of degree  $d \in \mathbb{Z}_2$ , is a linear operator on the CAR-algebra such that

$$\delta(AB) = \delta(A)B + (-1)^{d d_A} A \delta(B). \quad (4.3)$$

In particular, if  $X$  is an element of the CAR-algebra, the map  $\delta_X : A \mapsto [X, A]_g$  is a *graded derivation* with degree  $d_X$ .

**Proposition 4.3.** *Let  $\mathfrak{L}$  be the Lindblad operator given by  $\mathfrak{L}(A) = \iota[F, A] + \Psi$  where  $F = F^*$  and  $\Psi$  is given as in eq. (4.1). Then for any pair  $A, B \in \mathfrak{A}$*

$$(i) \mathfrak{L}(A^*) = (\mathfrak{L}(A))^*$$

$$(ii) \mathfrak{L}(A^*B) - A^*\mathfrak{L}(B) - \mathfrak{L}(A^*)B = -\sum_{i=1}^N [L_i, A]_g^* [L_i, B]_g \quad (\text{Leibniz formula}).$$

**Proof:** (i) Since  $F = F^*$ , the first claim is obvious by inspection.

(ii) The map  $\delta : A \in \mathfrak{A} \mapsto \iota[F, A]$  is a  $*$ -derivation and therefore it satisfies the Leibniz formula,  $\delta(A^*B) = \delta(A)^*B + A^*\delta(B)$ . Hence this part does not contribute to the *r.h.s.*. It is enough then to consider the case  $N = 1$ . Let  $J$  denote the left hand side of the Leibniz formula for  $N = 1$ . Then

$$\begin{aligned} 2J &= L^*LA^*B + A^*BL^*L - A^*L^*LB - A^*BL^*L - L^*LA^*B - A^*L^*LB \\ &\quad - 2(-1)^{d_L(d_A+d_B)}L^*A^*BL + 2(-1)^{d_L d_B}A^*L^*BL + 2(-1)^{d_L d_A}L^*A^*LB. \end{aligned}$$

After cancellation this gives

$$2J = -2A^*L^*\left(LB - (-1)^{d_L d_B}BL\right) + 2(-1)^{d_L d_A}L^*A^*\left(LB - (-1)^{d_L d_B}BL\right),$$

leading to

$$2J = -2\left(LA - (-1)^{d_L d_A}AL\right)^*\left(LB - (-1)^{d_L d_B}BL\right) = -2[L, A]_g^*[L, B]_g.$$

□

**4.3. A Convergence Theorem.** In this section a convergence result will be obtained for Lindblad operators with an infinite number of jump operators acting on the algebra  $\mathfrak{A}(\omega)$ . It is worth noticing that the proof of this result is modeled on a similar result found in [10]. For each finite set  $X \subset \mathcal{L}$  let  $L_X$  be a continuous section of the field  $\mathfrak{A}$  such that

- (1)  $L_X(\omega) \in \mathfrak{A}_X(\omega)$  and has degree  $d_X$ ,
- (2) the section is covariant, namely  $\eta_{\omega, a}(L_{X-a}(\mathbb{T}^{-a}\omega)) = L_X(\omega)$ ,

(3) for all natural numbers  $N \in \mathbb{N}$  there exists  $p > 0$  such that

$$\sum_{m \in \mathbb{N}} e^{pm} \sum_{n=0}^N \sup_{\omega \in \Omega} \sum_{0 \in X; \text{diam}(X)=m; |X|=n+1} \|L_X(\omega)\|^2 =: C_L < \infty,$$

where  $\text{diam}(X)$  denotes the diameter of the set  $X$  and  $|X|$  the number of points in  $X$ . For convenience, the diameter will be calculate w.r.t. the metric in  $\mathbb{R}^d$  given by  $|x| = \max |x_i|$ ;  $1 \leq i \leq d$ . Hence, a ball of diameter  $R$  is a cube of side  $R$  with sides parallel to the canonical basis of  $\mathbb{R}^d$ . Such a cube has a volume  $R^d$ . On the other hand, only the nonempty  $X$ 's matter, so that  $|X| = n + 1$  for some natural integer  $n$ .

Then  $\mathcal{D}_{X,\omega}$  denotes the operator acting on  $\mathfrak{A}(\omega)$  defined by

$$\mathcal{D}_{X,\omega}(A) = \frac{1}{2} \{L_X(\omega)^* L_X(\omega), A\} - (-1)^{d_X d_A} L_X(\omega)^* A L_X(\omega),$$

where  $\{A, B\} = AB + BA$  is the anti-commutator. Then the following result holds.

**Theorem 4.4.** *Let  $\mathcal{F}$  denote the set of finite subsets of  $\mathcal{L}$ . Then, the operator  $\mathfrak{D}_\omega = \sum_{X \in \mathcal{F}} \mathcal{D}_{X,\omega}$  is well defined on  $\mathfrak{A}_{loc}(\omega)$ . It is covariant and, if  $\xi$  is a continuous section of  $\mathfrak{A}_{loc}$ , its image  $\mathfrak{D}(\xi)$  is a continuous section of  $\mathfrak{A}$ . In addition its exponential  $e^{-t\mathfrak{D}_\omega}$  defines a continuous covariant field of Markov semi-groups.*

**Proof:** 1)- Let  $A \in \mathfrak{A}_\Lambda(\omega)$ . Then  $\mathcal{D}_{X,\omega}(A) = 0$  whenever  $X \cap \Lambda = \emptyset$ . Hence the sum over  $X$  is restricted to those subsets  $X$  intersecting  $\Lambda$ . From the estimate given in eq. (3), it follows easily that the sum defining  $\mathfrak{D}_\omega(A)$  converges in norm, uniformly w.r.t.  $\omega$ . Then the continuity and the covariance are straightforward to check.

2)- Iterating the definition of  $\mathfrak{D}$  gives (omitting  $\omega$ )

$$\mathfrak{D}^k(A) = \sum_{X_1, \dots, X_k} \mathcal{D}_{X_k} \circ \dots \circ \mathcal{D}_{X_1}(A),$$

where the family  $(X_1, \dots, X_k)$  of finite subsets of  $\mathcal{L}$  is submitted to satisfy the *compatibility condition*  $X_j \cap \Lambda_{j-1} \neq \emptyset$  for  $1 \leq j \leq k$ , whenever  $\Lambda_0 = \Lambda$  and  $\Lambda_j = \Lambda_{j-1} \cup X_j$ . It is easy to check that

$$\|\mathcal{D}_X(A)\| \leq 2\|L_X\|^2 \|A\|.$$

Therefore, the *l.h.s.* can be estimated by

$$\|\mathfrak{D}^k(A)\| \leq \|A\| \sum_{X_1, \dots, X_k} 2^k \prod_{j=1}^k \|L_{X_j}\|^2.$$

By assumption,  $\|L_{X_j}\|^2 \leq C_L e^{-p \text{diam}(X_j)}$ . Let  $N(m_1, n_1; \dots; m_k, n_k)$  denote the number of compatible families  $(X_1, \dots, X_k)$  such that  $\text{diam}(X_j) = m_j$  and  $|X_j| = n_j + 1$ . This gives

$$\|\mathfrak{D}^k(A)\| \leq (2C_L)^k \sum_{m_1, \dots, m_k} e^{-p(m_1 + \dots + m_k)} \sum_{n_1, \dots, n_k} N(m_1, n_1; \dots; m_k, n_k).$$

In order to estimate  $N(m_1, n_1; \dots; m_k, n_k)$ , it ought to be remarked that

- (i)  $|X| \leq \text{diam}(X)^d$ , so that  $n_j < m_j^d$  for all  $j$ ,
- (ii) the maximal number of choices for  $X_j$  is obtained by choosing a point in  $\Lambda_{j-1}$  then by choosing  $n_j$  points in a hypercube of side at most  $m_j$ . There is at most  $m_j^{dn_j} \times |\Lambda_{j-1}|$  ways of making this choice.
- (iii) the number of points in  $\Lambda_{j-1}$  is at most  $\{|\Lambda| + n_1 + n_2 + \dots + n_{j-1}\} \leq \{|\Lambda| + N(j-1)\} \leq Nj(1 + (|\Lambda|/N - 1)/j)$ , by construction. In particular this gives

$$N(m_1, n_1; \dots; m_k, n_k) \leq N^k k! \prod_{j=1}^k m_j^{dn_j} \prod_{j=1}^k \left(1 + \frac{|\Lambda| - N}{Nj}\right).$$

Then using  $(1 + u) \leq e^u$  and also  $1 + 1/2 + \dots + 1/k \leq 1 + \ln(k)$  gives

$$N(m_1, n_1; \dots; m_k, n_k) \leq N^k k! e^{|\Lambda|/N-1} k^{|\Lambda|/N-1} \prod_{j=1}^k m_j^{dn_j}$$

It follows that

$$\sum_{k=0}^{\infty} |t|^k \frac{\|\mathfrak{D}^k(A)\|}{k!} \leq \|A\| \sum_{k=0}^{\infty} k^\kappa (C_1 |t|)^k$$

with  $\kappa = |\Lambda|/N - 1$  and  $C_1 = 2NC_L \sum_{m \geq 1} m^{dN} e^{-pm} < \infty$ . Hence, if  $C_1 |t| < 1$ , the sum defining  $e^{-t\mathfrak{D}}(A)$  converges absolutely and uniformly in  $\omega \in \Omega$ . Since  $C_1$  does not depend on the volume  $\Lambda$  it follows that  $e^{-t\mathfrak{D}}$  is well defined on  $\mathfrak{A}$  for  $t \in \mathbb{C}$  such that  $|t| < C_1^{-1}$ .

3)- Using the Proposition 4.1, and the previous convergence, it follows that  $e^{-t\mathfrak{D}}$  is CP for  $0 \leq t < C_1^{-1}$ . Moreover, since  $\mathfrak{D}(\mathbf{1}) = 0$ , it is actually  $\text{CP}_1$ . In particular, it is a contraction semi-group. Moreover, by construction of the exponential, if  $s, t \in \mathbb{C}$  are such that  $|s| + |t| < C_1^{-1}$ , then  $e^{-(s+t)\mathfrak{D}} = e^{-s\mathfrak{D}} e^{-t\mathfrak{D}}$ . Therefore if  $t \in \mathbb{R}_+$  let  $n$  be an integer such that  $t/n < C_1^{-1}$ . Then  $e^{-t\mathfrak{D}} = (e^{-t/n\mathfrak{D}})^n$  is well defined and does not depend upon which  $n$  has been chosen. And for the same reason it defines a Markov semi-group.

4)- The continuity and the covariance follow from the definition and the proof will be left to the reader.  $\square$

**4.4. Jump Dynamics.** The general dissipation operator  $\mathfrak{D}_\omega$  defined in Section 2.5 is the main object of study in this subsection. The main result is Theorem 4.9.

The model defined in Section 2.5 is a specific example of a larger class of models of the form

$$\mathfrak{D}_\omega(A) = \sum_{\gamma \in \mathcal{J}} \left( \frac{1}{2} \{L_\gamma^*(\omega) L_\gamma(\omega), A\} - (-1)^{d_{L_\gamma} d_A} L_\gamma^*(\omega) A L_\gamma(\omega) \right), \quad (4.4)$$

where the following axioms are satisfied.

- J1- The index set  $\mathcal{J}$ , called the set of *jumps*, is countable and admits a bijective involution  $\gamma \in \mathcal{J} \mapsto \bar{\gamma} \in \mathcal{J}$  called *time-reversal*. Moreover, the translation group  $\mathbb{Z}^d$  acts on  $\mathcal{J}$  in a bijective way and the action is denoted by  $\gamma \mapsto \gamma + a$  by mappings commuting with the involution.



- J2- For each  $\gamma \in \mathcal{J}$  there is a continuous covariant field of local observables  $L_\gamma$ , called the *jump operators*. In particular
- (i)  $\exists \Lambda \subset \mathcal{L}$  finite, depending on  $\gamma$ , so that  $L_\gamma(\omega) \in \mathfrak{A}_\Lambda(\omega)$ ,  $\forall \omega \in \Omega$ ,
  - (ii) the smallest such  $\Lambda$  is called the *support* of  $\gamma$  and is denoted  $\text{supp}\{\gamma\}$  and satisfies  $\text{supp}\{\gamma + a\} = \text{supp}\{\gamma\} + a$ ,
  - (iii)  $\eta_{\omega,a}\{L_{\gamma-a}(\mathbb{T}^{-a}\omega)\} = L_\gamma(\omega)$  and  $L_\gamma, L_{\gamma-a}$  have the same degree.
- J3- Under the time evolution the jump operators satisfy  $\alpha_t(L_\gamma) = e^{t\varepsilon_\gamma} L_\gamma$  for all  $t \in \mathbb{R}$ . where the  $\varepsilon_\gamma : \Omega \mapsto \mathbb{R}$ 's are continuous functions.
- J4- The jump operators satisfy a  $\beta$ -KMS condition, namely  $L_\gamma^* = e^{-\beta\varepsilon_\gamma/2} L_{\bar{\gamma}}$ , in particular  $\varepsilon_{\bar{\gamma}} = -\varepsilon_\gamma$  and  $L_{\bar{\gamma}}$  has the same degree as  $L_\gamma$ .
- J5- The following sum converges  $\sum_{\gamma; 0 \in \text{supp}\{\gamma\}} L_\gamma^*(\omega) L_\gamma(\omega) \in \mathfrak{A}(\omega)$  uniformly with respect to  $\omega$ .

**Remark 4.5.** *The  $\beta$ -KMS condition is also called detailed balance in the Physics literature (see for instance [42]). This axiom is essential to describe the dissipative evolution at equilibrium.*

**Proposition 4.6.** *If the jump operators satisfy the assumptions [J1-J5] (except possibly the axiom [J4]), the operator  $\mathfrak{D}_\omega$  is well defined by eq. (4.4) on the set of local observables. In addition*

$$\alpha_t^{(\omega)} \circ \mathfrak{D}_\omega \circ \alpha_{-t}^{(\omega)} = \mathfrak{D}_\omega, \quad \text{(time-invariance)} \quad (4.5)$$

$$\eta_{\omega,a} \circ \mathfrak{D}_{\mathbb{T}^{-a}\omega} \circ \eta_{\omega,a}^{-1} = \mathfrak{D}_\omega, \quad \text{(space-covariance)} \quad (4.6)$$

and, if  $\xi \in \mathcal{F}$ , the field  $\omega \in \Omega \mapsto \mathfrak{D}_\omega(\xi(\omega))$  is continuous.

**Proof:** Let  $A$  be localized in  $\Lambda_0$ . Each  $L_\gamma$  belongs to the local algebra  $\mathfrak{A}_{\text{supp}\{\gamma\}}$ . In particular if  $\text{supp}\{\gamma\} \cap \Lambda_0 = \emptyset$ , it follows that  $A$  commutes with  $L_\gamma^* L_\gamma$  and that  $(-1)^{d_{L_\gamma} d_A} L_\gamma^* A L_\gamma = L_\gamma^* L_\gamma A$ . Thus, the term  $\frac{1}{2}\{L_\gamma^* L_\gamma, A\} - (-1)^{d_{L_\gamma} d_A} L_\gamma^* A L_\gamma$  just vanishes. Hence,

$$\mathfrak{D}_\omega(A) = \sum_{\gamma \in \mathcal{J}; \text{supp}\{\gamma\} \cap \Lambda_0 \neq \emptyset} \left( \frac{1}{2}\{L_\gamma^* L_\gamma, A\} - (-1)^{d_{L_\gamma} d_A} L_\gamma^* A L_\gamma \right).$$

Thanks to the axiom [J5], this sum converges in  $\mathfrak{A}$ . Thanks to axiom [J2-J3] the time-invariance, the covariance condition and the continuity are satisfied.  $\square$

**Proposition 4.7.** *Let  $\rho$  be the continuous field of  $\beta$ -KMS states over  $\mathfrak{A}$  defined in Prop. 3.7. If the jump operators satisfy [J1-J5], then  $\rho$  is  $\mathfrak{D}$ -invariant and satisfies for all finite  $\Lambda \subset \mathcal{L}$  and all  $A, B \in \mathfrak{A}_\Lambda(\omega)$*

$$\rho_\omega(A^* \mathfrak{D}_\omega(B)) = \rho_\omega(\mathfrak{D}_\omega(A)^* B) = \frac{1}{2} \sum_{\gamma \in \mathcal{J}} \rho_\omega([L_\gamma(\omega), A]_g^* [L_\gamma(\omega), B]_g). \quad (4.7)$$

**Remark 4.8.** *This shows that  $\mathfrak{D}$  acts on the GNS-representation of the ground state as a positive self-adjoint operator. Moreover it can be seen as a generalization of a Laplacian. For indeed, the family of (graded) commutators by  $L_\gamma$ , indexed by  $\gamma \in \mathcal{J}$ , can be seen as a*

gradient and the right hand side of the previous equation looks like a Sobolev norm of the type  $\int |\nabla A|^2$  if  $A = B$ .

**Proof:** Since there is no confusion, the reference to  $\omega \in \Omega$  will be omitted. Thanks to the Leibniz formula (Proposition 4.3), it is sufficient to show that (i)  $\rho \circ \mathfrak{D} = 0$  ( $\mathfrak{D}$ -invariance of  $\rho$ ) and (ii)  $\mathfrak{D}$  is symmetric.

1)- Let  $A \in \mathfrak{A}_\Lambda$ . Since  $\mathfrak{D}(A)$  has the same degree as  $A$ , it is sufficient to assume that  $d_A = 0$ , because  $\rho(B) = 0$  whenever  $B$  has degree one. Then  $\rho(\mathfrak{D}(A))$  is the sum of three types of terms, namely  $(1/2)\rho(L_\gamma^* L_\gamma A)$ ,  $(1/2)\rho(AL_\gamma^* L_\gamma)$  and  $(-1)\rho(L_\gamma^* AL_\gamma)$ . Thanks to [J3], it follows that  $L_\gamma^* L_\gamma$  is invariant by the dynamics  $\alpha_t$ . In particular  $\alpha_{-i\beta}(L_\gamma^* L_\gamma) = L_\gamma^* L_\gamma$ . Thus, thanks to the  $\beta$ -KMS condition (eq. (3.10))

$$\rho(AL_\gamma^* L_\gamma) = \rho(\alpha_{-i\beta}(L_\gamma^* L_\gamma)A) = \rho(L_\gamma^* L_\gamma A) .$$

In particular the first two terms are equal. Moreover, the axioms [J3-J4] imply that  $\alpha_{-i\beta}(L_\gamma) = e^{\beta\varepsilon_\gamma} L_\gamma = e^{\beta\varepsilon_\gamma/2} L_\gamma^*$  and, similarly,  $e^{\beta\varepsilon_\gamma/2} L_\gamma^* = L_{\bar{\gamma}}$ . Hence

$$\rho(L_\gamma^* AL_\gamma) = \rho(\alpha_{-i\beta}(L_\gamma) L_\gamma^* A) = \rho(L_{\bar{\gamma}}^* L_{\bar{\gamma}} A) .$$

Therefore, since the time-reversal  $\gamma \mapsto \bar{\gamma}$  is a bijection, the sum of these last terms compensate the sum of the other terms to give zero.

2)- To prove that  $\mathfrak{D}$  defines a symmetric operator, let  $A, B$  be elements in  $\mathfrak{A}_\Lambda$ . Without loss of generality, it can be assumed that  $A$  and  $B$  have the same degree, otherwise the  $\rho$ -average vanishes and the identity becomes trivial. Then  $\rho(A^* \mathfrak{D}(B))$  is a sum of three types of terms. The first ones are

$$\rho(A^* L_\gamma^* L_\gamma B) = \rho((L_\gamma^* L_\gamma A)^* B) .$$

The next terms have the form

$$\rho(A^* B L_\gamma^* L_\gamma) = \rho(L_\gamma^* L_\gamma A^* B) = \rho((AL_\gamma^* L_\gamma)^* B) ,$$

where the middle identity comes from the  $\beta$ -KMS condition and the invariance of  $L_\gamma^* L_\gamma$  under the time evolution (axiom [J3]). The last terms are more involved. Using again the  $\beta$ -KMS condition, the identities  $d_A = d_B$ ,  $d_\gamma = d_{\bar{\gamma}}$  and the axiom [J4], leads to

$$(-1)^{d_\gamma d_B} \rho(A^* L_\gamma^* B L_\gamma) = (-1)^{d_\gamma d_A} e^{\beta\varepsilon_\gamma} \rho(L_\gamma A^* L_\gamma^* B) = (-1)^{d_{\bar{\gamma}} d_A} \rho((L_{\bar{\gamma}}^* A L_{\bar{\gamma}})^* B) ,$$

Since the map  $\gamma \mapsto \bar{\gamma}$  is a bijection the sum of all these terms gives  $\rho(A^* \mathfrak{D}(B)) = \rho(\mathfrak{D}(A)^* B)$ .  $\square$

Through the GNS-representation, the field of ground-states  $\rho$  defines a continuous translation covariant field of Hilbert spaces  $\mathcal{H}_\omega = L^2(\mathfrak{A}(\omega), \rho_\omega)$  [23, 45].  $\mathcal{H}_\omega$  is obtained from  $\mathfrak{A}(\omega)$  through the inner product  $\langle A|B \rangle_\omega = \rho_\omega(A^* B)$ , after taking the quotient by the subspace of elements of zero norms and completing. The canonical map from  $\mathfrak{A}(\omega)$  into  $\mathcal{H}_\omega$  will be denoted by  $\zeta_\omega$ . Hence

$$\rho_\omega(A^* B) = \langle \zeta_\omega(A) | \zeta_\omega(B) \rangle_\omega , \quad \rho_\omega(A^* A) = \|\zeta_\omega(A)\|_\omega^2 , \quad A, B \in \mathfrak{A}(\omega) .$$

On each element of this field the Proposition 4.7 defines a densely defined field of positive quadratic forms as follows

$$\mathcal{Q}_\omega(A, B) = \frac{1}{2} \sum_{\gamma \in \mathcal{J}} \rho_\omega([L_\gamma(\omega), A]_g^* [L_\gamma(\omega), B]_g). \quad (4.8)$$

The following result shows that, as a consequence of the Friedrich extension theorem [37], this form defines a positive self adjoint operator  $\mathcal{H}_\omega$ .

**Theorem 4.9.** *If the jump operators satisfy [J1-J5], the quadratic form  $\mathcal{Q}_\omega$ , which is densely defined on  $\mathcal{H}_\omega$ , is closable. Its closure defines a positive self-adjoint operator, denoted by  $\mathcal{D}_\omega$ , on  $\mathcal{H}_\omega$ . The contraction semi-group  $e^{-t\mathcal{D}_\omega}$  is completely positive. The corresponding field of contraction semi-groups is continuous, time-invariant and covariant.*

**Proof:** (i) *Closability.* By abuse of notation  $A$  will represent here either an element of  $\mathfrak{A}(\omega)$  or its image in  $\mathcal{H}_\omega$ . Let  $\|A\|_{\omega, Q}$  denote the norm

$$\|A\|_{\omega, Q}^2 = \|A\|_\omega^2 + \mathcal{Q}_\omega(A, A)$$

To prove that  $\mathcal{Q}_\omega$  is closable, let  $(A_n)_{n \in \mathbb{N}}$  be a  $\|\cdot\|_{\omega, Q}$ -Cauchy sequence in  $\mathfrak{A}_{loc}(\omega)$  such that  $\lim_{n \rightarrow \infty} \|A_n\|_\omega = 0$ . It should be proved that  $\lim_{n \rightarrow \infty} \|A_n\|_{\omega, Q} = 0$ . Since this sequence is  $\|\cdot\|_{\omega, Q}$ -Cauchy, it follows that  $\mathcal{Q}_\omega(A_n, A_n)$  converges and is therefore uniformly bounded in  $n$ . In particular, it follows that  $\delta(A_n) = ([L_\gamma(\omega), A_n]_g)_{\gamma \in \mathcal{J}}$  is Cauchy if seen as an element of the Hilbert space  $\mathcal{H}_\omega \otimes \ell^2(\mathcal{J})$ . Therefore there is  $\xi = (\xi_\gamma)_{\gamma \in \mathcal{J}} \in \mathcal{H}_\omega \otimes \ell^2(\mathcal{J})$ , such that  $\delta(A_n) \rightarrow \xi$ . Hence, given  $\epsilon > 0$  there is a finite subset  $I \subset \mathcal{J}$ , such that  $\sum_{\gamma \notin I} \|\xi_\gamma\|_\omega^2 \leq \epsilon$ . Now, thanks to [J1-J5], and omitting  $\omega$  inside  $L_\gamma$

$$\begin{aligned} \rho_\omega(|[L_\gamma, A_n]_g|^2) &= \rho_\omega(A_n^* L_\gamma^* L_\gamma A_n) + \rho_\omega(L_\gamma^* A_n^* A_n L_\gamma) \\ &\quad - (-1)^{d_{L_\gamma} d_{A_n}} \left( \rho_\omega(A_n^* L_\gamma^* A_n L_\gamma) + \rho_\omega(L_\gamma^* A_n^* L_\gamma A_n) \right). \end{aligned}$$

The last two terms of the *r.h.s.* can be estimated in terms of the first two. Since  $L_\gamma(\omega)$  is bounded, the first term is bounded by  $\|L_\gamma(\omega)\|^2 \|A_n\|_\omega^2$  which converges to zero. Using the axiom [J3-J4], the second term can be written (omitting  $\omega$ ), as

$$\rho(L_\gamma^* A_n^* A_n L_\gamma) = e^{\beta \varepsilon_\gamma} \rho(L_\gamma L_\gamma^* A_n^* A_n) = \rho(L_{\bar{\gamma}}^* L_{\bar{\gamma}} A_n^* A_n).$$

Thanks to [J3], it follows that  $L_{\bar{\gamma}}^* L_{\bar{\gamma}}$  is invariant by the modular automorphism, so that, using the Cauchy-Schwartz inequality,

$$\rho(L_{\bar{\gamma}}^* L_{\bar{\gamma}} A_n^* A_n) \leq \|A_n\| \rho(L_{\bar{\gamma}}^* L_{\bar{\gamma}} A_n^* A_n L_{\bar{\gamma}}^* L_{\bar{\gamma}})^{1/2} = \|A_n\| \rho\left((L_{\bar{\gamma}}^* L_{\bar{\gamma}})^2 A_n^* A_n\right)^{1/2}.$$

Iterating  $m$ -times, leads to

$$\rho(L_{\bar{\gamma}}^* L_{\bar{\gamma}} A_n^* A_n) \leq \|A_n\|^{2-2^{-m}} \rho\left((L_{\bar{\gamma}}^* L_{\bar{\gamma}})^{2^{m-1}} A_n^* A_n\right)^{2^{1-m}} \leq \|A_n\|^{2-2^{-m}} \|L_{\bar{\gamma}}\|^2 \|A_n\|_{\mathfrak{A}}^{2^{-m}}.$$

Consequently, as  $m \rightarrow \infty$ ,

$$\rho(L_{\bar{\gamma}}^* L_{\bar{\gamma}} A_n^* A_n) \leq \|A_n\|^2 \|L_{\bar{\gamma}}\|^2,$$

which is also converging to zero as  $n \rightarrow \infty$ . Hence, the finite sum  $\sum_{\gamma \in I} \rho(\|L_\gamma, A\|^2)$  vanishes, so that  $\|\xi\| \leq \epsilon$ . Since  $\epsilon$  can be taken arbitrarily small, it follows that  $\xi = 0$ , proving that  $\mathcal{Q}_\omega$  is closable.

It follows from the Friedrich extension method that there exists a positive self-adjoint operator  $\mathcal{D}_\omega$ , with domain  $\mathcal{D} = \{A \in \mathcal{H}_\omega; \exists C > 0, \|\mathcal{Q}_\omega\|_\omega(B, A) \leq C\|B\|_\omega, \forall B \in \mathcal{H}_\omega\}$ , defined by

$$\mathcal{Q}_\omega(B, A) = \langle B | \mathcal{D}_\omega A \rangle_\omega, \quad A \in \mathcal{D}. \quad (4.9)$$

(ii) *Complete Positivity.* Let  $\mathcal{J}$  be finite to begin with. Thanks to the Propositions 4.7 and 4.1, it follows that the complete positivity holds if  $\mathcal{J}$  is finite in the algebra  $\mathfrak{A}(\omega)$ . Moreover, since  $\mathfrak{D}$  commutes with the dynamics (Proposition 4.6), the operator  $\mathcal{D}_\omega$  commutes with the Modular operator  $\Delta_\omega$ . In particular  $e^{-t\mathcal{D}_\omega}$  will be completely positive on  $\mathcal{H}_\omega$  (see Section 4.1). If  $\mathcal{J}$  is not finite, then  $\mathcal{Q}_\omega$  can be seen as the supremum of a countable family of similar forms with  $\mathcal{J}$  finite. Correspondingly there is a non decreasing sequence of positive self-adjoint operators  $\mathcal{D}_{n,\omega}$  converging weakly to  $\mathcal{D}_\omega$ . The Lemma 4.10 belows shows then that  $\mathcal{D}_{n,\omega}$  converges to  $\mathcal{D}_\omega$  in the strong resolvent sense. This, in turns proves that the semi-group  $e^{-t\mathcal{D}_{n,\omega}}$  converges strongly to  $e^{-t\mathcal{D}_\omega}$ . In particular,  $e^{-t\mathcal{D}_\omega}$  is completely positive.

(iii) *Covariance and Continuity.* The same argument as before shows that the continuity of the field  $\omega \in \Omega \mapsto \mathcal{Q}_\omega$  implies the strong resolvent continuity of the field  $\omega \in \Omega \mapsto \mathcal{D}_\omega$ . Hence the semi-group  $e^{-t\mathcal{D}_\omega}$  is also continuous in  $\omega$ . The covariance is a simple consequence of Proposition 4.6.  $\square$

**Lemma 4.10.** *Let  $A$  be a positive self-adjoint operator on the Hilbert space  $\mathcal{H}$ , with dense domain  $\mathcal{D}$ . Let  $(A_n)_{n \in \mathbb{N}}$  be a non decreasing sequence of bounded positive operators converging weakly to  $A$  on  $\mathcal{D}$ . Then the resolvent of  $A_n$  converges strongly to the resolvent of  $A$ .*

**Proof:** Since  $0 \leq A_1 \leq A_2 \leq \dots \leq A_n \leq \dots \leq A$ , it follows that the sequence of bounded operators  $(1 + A_n)^{-1}$  converges weakly. Let  $R$  be the weak limit. Since the sequence is bounded the sequence converges strongly as well. Then the inequality  $1 \geq (1 + A_n)^{-1} \geq (1 + A)^{-1}$  implies  $1 \geq R \geq (1 + A)^{-1}$ . In particular,

$$1 + A \geq (1 + A)^{1/2}(1 + A_n)^{-1}(1 + A)^{1/2} \geq (1 + A)^{1/2}R(1 + A)^{1/2} \geq 1.$$

Hence,  $(1 + A)^{1/2}R(1 + A)^{1/2}$  is invertible and its inverse satisfies

$$(1 + A)^{-1/2}(1 + A_n)(1 + A)^{-1/2} \leq (1 + A)^{-1/2}R^{-1}(1 + A)^{-1/2} \leq 1.$$

Since the left hand side of this inequality converges weakly to 1, thus also strongly, it follows that  $R^{-1} = (1 + A)$ . Thus  $s - \lim_{n \rightarrow \infty} (1 + A_n)^{-1} = (1 + A)^{-1}$ . By standard arguments, this shows that for  $z \in \mathbb{C} \setminus \mathbb{R}_+$ ,  $s - \lim_{n \rightarrow \infty} (A_n - z)^{-1} = (A - z)^{-1}$ .  $\square$

## 5. THE RETURN TO EQUILIBRIUM

In this section we first justify the introduction of the thermal bath part of the dissipation operator in eq. (2.16). Then it will be shown that the dissipation operator  $\mathfrak{L}_\omega$ , given in

eq. (2.17), defines a semigroup on the state space. Furthermore we obtain that the point wise limit of the semigroup at any initial state is equal to the equilibrium state as the time parameter tends to infinity (Theorem 5.4). This justifies the title of the present Section. Finally, the spectrum of the dissipation operator  $\mathfrak{D}_\omega$  is examined which allows us to conclude that the return to equilibrium is exponentially fast in time.

If  $\mathfrak{D}$  is the generator of a semigroup  $(e^{-t\mathfrak{D}})_{t \geq 0}$  on a  $C^*$ -algebra  $\mathfrak{A}$ , then an element  $A \in \mathfrak{A}$  is called invariant for the semigroup if  $A$  is an eigenvector of  $e^{-t\mathfrak{D}}$  corresponding to the eigenvalue 1 for every  $t \geq 0$ . Also, if  $\rho$  is any state on the  $C^*$ -algebra  $\mathfrak{A}$  then  $\rho$  is called  $\mathfrak{D}$ -invariant if

$$\rho(e^{-t\mathfrak{D}}(B)) = \rho(B) \quad \forall B \in \mathfrak{A}, \forall t \geq 0.$$

The following result states elementary properties of the jump operators which are defined by eq. (2.9). It can be obtained by inspection and the proof will be left to the reader.

**Proposition 5.1.** *The jump operators  $L_{x \rightarrow y}(\omega)$  given by eq. (2.9) & (2.11) have degree zero and satisfy the axiom [J1-J5] with*

- (i)  $\text{supp}\{x \rightarrow y\} = \{x, y\}$ ,
- (ii)  $\{x \rightarrow y\} + a = \{x + a \rightarrow y + a\}$
- (iii) the time-reversal corresponds to  $\{y \rightarrow x\}$ ,
- (iv)  $\varepsilon_{x \rightarrow y} = \epsilon_y - \epsilon_x$ .

In order to justify the need for the thermal bath part of the dissipation operator, it is enough to observe that the kinetic part  $\mathfrak{D}_\omega^{kin}$  suffers from an annoying disease: since it leaves the number operator invariant, it does not have a unique invariant state. In fact, varying the chemical potential  $\mu$  in (3.13) produces infinitely many invariant states. Thus, the thermal bath fixes the chemical potential and the kinetic part of the dissipation operator alone cannot drive the electron gas towards equilibrium. This is the content of the following result.

**Proposition 5.2.** *The kinetic part of the dissipation operator, given in eq. (2.13), commutes with the number operator. In particular its set of invariant state is not reduced to a point.*

**Proof:** The number operator has been defined in eq. (3.4) for finite volume. As for the Hamiltonian dynamics it generates an automorphism group which is defined, in the infinite volume limit by

$$\nu_t^{(\omega)}(s_x a_x) = e^{-it} s_x a_x, \quad \nu_t^{(\omega)}(s_x a_x^\dagger) = e^{it} s_x a_x^\dagger.$$

In particular,  $L_{x \rightarrow y}(\omega)$  is invariant by this automorphism group. Therefore, like in Proposition 4.6, it follows that

$$\nu_t^{(\omega)} \circ \mathfrak{D}_\omega^{kin} \circ \nu_{-t}^{(\omega)} = \mathfrak{D}_\omega^{kin}.$$

Hence any state generated by a finite volume Hamiltonian of the form

$$F_\Lambda(\omega) = P \left( \sum_{x \in \Lambda} \epsilon_x n_x \right) + Q \left( \sum_{x \in \Lambda} n_x \right),$$

where  $P$  and  $Q$  are polynomials, is invariant by  $\mathfrak{D}_\omega^{kin}$ .  $\square$

The following result states elementary properties of the jump operators which are defined by eq. (2.14). It can be obtained by inspection and the proof will be left to the reader.

**Proposition 5.3.** *The jump operators  $L_{x \rightarrow \star}(\omega)$  and  $L_{\star \rightarrow x}(\omega)$ , given by eq. (2.14) & (2.15), have degree one and satisfy the axiom [J1-J5] with*

- (i)  $\text{supp}\{x \rightarrow \star\} = \{x\} = \text{supp}\{\star \rightarrow x\}$
- (ii)  $\{x \rightarrow \star\} + a = \{x + a \rightarrow \star\}$  and similarly for  $\{\star \rightarrow x\}$
- (iii)  $x \rightarrow \star$  is time reversed from  $\star \rightarrow x$ ,
- (iv)  $\varepsilon_{\star \rightarrow x} = \varepsilon_x - \mu$ .

The following theorem shows the return to equilibrium  $\rho_\omega$  as defined in Proposition 3.7 for the dynamical system  $(e^{-t\mathfrak{D}_\omega})_{t \geq 0}$ .

**Theorem 5.4.** *The operators  $\mathfrak{D}_\omega^*$  and  $\mathfrak{D}_\omega$  have a unique invariant state given by  $\rho_\omega$ . Also, for every state  $\tilde{\rho}$  on the  $C^*$ -algebra  $\mathfrak{A}(\omega)$ , and for every observable  $A \in \mathfrak{A}(\omega)$ ,*

$$\lim_{t \rightarrow \infty} \tilde{\rho}(e^{-t\mathfrak{D}} A) = \rho_\omega(A). \quad (5.1)$$

**Proof:** 1)- Assume that  $\mathfrak{D}_\omega^*(A) = 0$ . Then thanks to Proposition 4.7, it follows that  $0 = \rho_\omega(A^* \mathfrak{D}_\omega^*(A)) = \sum_x \Gamma_{x \rightarrow \star} \rho_\omega(|[a_x, A]_g|^2) + \sum_x \Gamma_{\star \rightarrow x} \rho_\omega(|[a_x^\dagger, A]_g|^2)$ . Thanks to Proposition 3.8,  $\rho_\omega$  is faithful, so that

$$[a_x, A]_g = 0 = [a_x^\dagger, A]_g \quad \forall x \in \mathcal{L}(s).$$

Then the only elements of  $\mathfrak{A}(\omega)$  with this property are the multiples of  $\mathbf{1}$ . To prove this let  $A$  be a local observable first, namely  $A \in \mathfrak{A}_\Lambda(\omega)$ . Then, using the decomposition into monomials  $A$  can be written as  $A = n_x B + (1 - n_x) B' + a_x C + a_x^\dagger C'$  where  $B, B', C, C' \in \mathfrak{A}_{\Lambda \setminus \{x\}}(\omega)$ . Thus  $[a_x, A]_g = C + a_x(B - B') = 0$  implies that  $C = 0$  and  $B = B'$ . In much the same way  $[a_x^\dagger, A]_g = 0$  implies  $C' = 0$ . Thus  $A = B \in \mathfrak{A}_{\Lambda \setminus \{x\}}(\omega)$ . Using this argument inductively on every point of  $\Lambda$  shows that  $A$  is a multiple of  $\mathbf{1}$ .

If now  $A \in \mathfrak{A}(\omega)$ , then for all  $\forall \varepsilon > 0$  there is a finite set  $\Lambda_\varepsilon$  and  $A_\varepsilon \in \mathfrak{A}_{\Lambda_\varepsilon}(\omega)$  such that  $\|A - A_\varepsilon\| < \varepsilon/3$ . For any finite subset  $\Lambda$  of  $\mathcal{L}$  let  $\mathbb{E}_\Lambda : \mathfrak{A}(\omega) \rightarrow \mathfrak{A}_\Lambda(\omega)$  be the canonical projection. The commutation rule above implies that  $\mathbb{E}_\Lambda(A)$  also commutes with all the  $a_x, a_x^\dagger$  for  $x \in \Lambda$ , showing that there is  $c(\Lambda) \in \mathbb{C}$  such that  $\mathbb{E}_\Lambda(A) = c(\Lambda)\mathbf{1}$ . For  $\Lambda \supset \Lambda_\varepsilon$ ,  $\|\mathbb{E}_\Lambda(A - A_\varepsilon)\| = \|c(\Lambda)\mathbf{1} - A_\varepsilon\| < \varepsilon/3$ . It follows that (i)  $|c(\Lambda)| \leq \|A\|$  and (ii)  $|c(\Lambda) - c(\Lambda')| < 2\varepsilon/3$  for  $\Lambda, \Lambda' \supset \Lambda_\varepsilon$ . Thus it is a Cauchy sequence which converges to  $c \in \mathbb{C}$  as the volume tends to infinity. Hence  $\|A - c\mathbf{1}\| \leq \|A - A_\varepsilon\| + |c - c(\Lambda)| + \|A_\varepsilon - c(\Lambda)\mathbf{1}\| < \varepsilon$ . This shows that  $A = c\mathbf{1}$ .

2)- Assume that  $\mathfrak{D}(A) = 0$ . Then

$$0 = \rho_\omega(A^* \mathfrak{D}_\omega(A)) = \rho_\omega(A^* \mathfrak{D}_\omega^{kin}(A)) + \rho_\omega(A^* \mathfrak{D}_\omega^*(A)).$$

By Proposition 4.7 applied to  $\mathfrak{D}_\omega^{kin}$  we obtain that  $\rho_\omega(A^* \mathfrak{D}_\omega^{kin}(A)) \geq 0$ . Thus

$$0 = \rho_\omega(A^* \mathfrak{D}_\omega^{kin}(A)) = \rho_\omega(A^* \mathfrak{D}_\omega^*(A)).$$

Hence by 1),  $A$  is a multiple of the identity.

3)- By construction, both  $\mathfrak{D}_\omega$  and  $\mathfrak{D}_\omega^*$  leave each  $\mathfrak{A}_\Lambda(\omega)$  invariant. Therefore  $e^{-t\mathfrak{D}_\omega}$  and  $e^{-t\mathfrak{D}_\omega^*}$  are well defined on each of the  $\mathfrak{A}_\Lambda(\omega)$ 's and define Markov semi-groups (Proposition 4.1). Since the multiples of the identity are the only invariant observables of the semigroups, 1 is a simple eigenvalue of these semi-groups. Consequently  $\lim_{t \rightarrow \infty} e^{-t\mathfrak{D}_\omega}(A)$  and  $\lim_{t \rightarrow \infty} e^{-t\mathfrak{D}_\omega^*}(A)$  exist for each  $A \in \mathfrak{A}_{loc}(\omega)$  and, (since these limits are invariant observable for the corresponding semigroups), these limits are multiples  $m_\omega(A)$  and  $m_\omega^*(A)$  respectively of the identity. Since the semi-groups are contractions the result extends by continuity to all elements of  $\mathfrak{A}(\omega)$ .

Theorem 4.4 allows us to define  $e^{-t\mathfrak{D}_\omega}$ . For every  $A \in \mathfrak{A}_\omega$ , this leads to

$$\lim_{t \rightarrow \infty} e^{-t\mathfrak{D}_\omega}(A) = m_\omega(A)\mathbf{1} \quad \text{and} \quad \lim_{t \rightarrow \infty} e^{-t\mathfrak{D}_\omega^*}(A) = m_\omega^*(A)\mathbf{1}. \quad (5.2)$$

4)- By eq. (5.2) it follows that  $m_\omega$  and  $m_\omega^*$  are states on  $\mathfrak{A}(\omega)$ . If now  $\rho$  is a  $\mathfrak{D}_\omega$ -invariant state and  $A \in \mathfrak{A}_\omega$ , then

$$m_\omega(A) = \lim_{t \rightarrow \infty} \rho(e^{-t\mathfrak{D}_\omega}(A)) = \rho(A).$$

Similarly, if  $\rho$  is a  $\mathfrak{D}_\omega^*$ -invariant state then  $m_\omega^* = \rho$ .

5)- Thanks to Proposition 4.7, it follows that  $\rho_\omega$  is both  $\mathfrak{D}^*$  and  $\mathfrak{D}$ -invariant, thus  $\rho_\omega = m_\omega$  and  $\rho_\omega = m_\omega^*$ . Therefore  $\rho_\omega$  is the unique invariant state for  $\mathfrak{D}_\omega$  and  $\mathfrak{D}_\omega^*$ .

6)- Now let  $\tilde{\rho}$  be any state on  $\mathfrak{A}_\omega$ . By eq. (5.2),

$$\lim_{t \rightarrow \infty} \tilde{\rho}(e^{-t\mathfrak{D}_\omega}(A)) = \tilde{\rho}(m_\omega(A)\mathbf{1}) = m_\omega(A) = \rho_\omega(A)$$

which shows the return to equilibrium  $\rho_\omega$ . □

As explained in Section 4.4 the dissipation operators, defined previously, define self-adjoint positive operators on the Hilbert space of the GNS representation. Some spectral properties are given by the following result

**Theorem 5.5.** (i) If  $\mathfrak{D}_\omega^*$  and  $\mathfrak{D}_\omega$  denote the corresponding positive self-adjoint operators acting on the GNS representation, then both admit 0 as a simple eigenvalue and both have a positive gap separating zero from the rest of the spectrum bounded from below by  $\Gamma_\star/2$ .  
(ii) Let  $\mathcal{K}_\omega$  be the closed subspace of  $\mathcal{H}_\omega$  generated by vectors in  $\pi_\omega(\mathfrak{C}_\omega)\xi_\omega$  which are orthogonal to  $\xi_\omega$ . Then both  $\mathfrak{D}_\omega^*$  and  $\mathfrak{D}_\omega$  leave  $\mathcal{K}_\omega$  invariant and their restriction to  $\mathcal{K}_\omega$  is bounded below by  $\Gamma_\star$ .  
(iii) The operator  $\mathfrak{D}_\omega^*$  has a pure point spectrum.

**Remark 5.6.** This result shows that the Markov semi-group generated by  $\mathfrak{D}_\omega$  converges to equilibrium exponentially fast with a lifetime given by the inverse of  $\Gamma_\star$ .

**Proof:** (i) In the Hilbert space  $\mathcal{H}_\omega$  of the GNS representation, monomials in the annihilation-creation operators make up a total set. This helps creating an orthonormal basis in this Hilbert space. First, for  $x \in \mathcal{L}$  let  $b_x(\omega), b_x^\dagger(\omega)$  be defined by

$$b_x(\omega) = e^{-\beta(\mu - \epsilon_x)^+/2} s_x a_x, \quad b_x^\dagger(\omega) = e^{-\beta(\epsilon_x - \mu)^+/2} s_x a_x^\dagger. \quad (5.3)$$

It is easy to check that both elements define unit vectors in  $\mathcal{H}_\omega$ , and that they are orthogonal to each other. For  $X = (x_1, x_2 \cdots, x_m) \in \mathcal{L}(\omega)^{\times m}$  let  $b_X(\omega), b_X^\dagger(\omega) \in \mathcal{H}_\omega$  be defined by (omitting the reference to  $\omega$ )

$$b_X = b_{x_1} b_{x_2} \cdots b_{x_m}, \quad b_X^\dagger = b_{x_m}^\dagger b_{x_{m-1}}^\dagger \cdots b_{x_1}^\dagger. \quad (5.4)$$

If two components of  $X$  are equal, then  $b_X = 0$ . Thus the  $X$ 's will be restricted to the set of elements in  $\mathcal{L}(\omega)^{\times n}$  made of distinct points. If the order in presenting the points of  $X$  is changed, then  $b_X$  changes sign according to the signature of the permutation. Hence, modulo a sign,  $b_X$  depends only upon the set  $\{x_1, x_2, \cdots, x_m\} \subset \mathcal{L}(\omega)$ . In much the same way the vectors  $\sigma_X(\omega)$  will be defined as follows

$$\sigma_x(\omega) = s_x \left( e^{\beta/2(\epsilon_x - \mu)} n_x - e^{-\beta/2(\epsilon_x - \mu)} (1 - n_x) \right), \quad \sigma_X = \sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_m} \quad (5.5)$$

Here, the ordering of points is irrelevant since the  $n_x$ 's commute. Before continuing the proof we need the following lemma and corollary.

**Lemma 5.7.** *The vectors in  $\mathcal{H}_\omega$  given by  $\zeta_\omega(b_X^\dagger b_Y \sigma_Z)$ , where  $X, Y, Z$  vary among the set of triplets of three disjoint finite subsets of  $\mathcal{L}(\omega)$ , including the empty set, make up an orthonormal basis of  $\mathcal{H}_\omega$ .*

**Proof:** First, these elements define all possible monomials in  $\mathfrak{A}(\omega)$  up to a scalar multiplication. Consequently they generate a dense subspace. If it is proved that these vectors make up an orthonormal family, then they make up a Hilbert basis. Second, due to the factorization property of the equilibrium state  $\rho_\omega$  and since  $X, Y, Z$  are disjoint, it follows that

$$\rho_\omega \left( (b_X^\dagger b_Y \sigma_Z)^* b_X^\dagger b_Y \sigma_Z \right) = \rho_\omega(b_X b_X^\dagger) \rho_\omega(b_Y^\dagger b_Y) \rho_\omega(\sigma_Z^2).$$

Again the factorization property of  $\rho_\omega$  and the commutation rules gives

$$\rho_\omega(b_Y^\dagger b_Y) = \prod_{i=1}^n e^{-\beta|\epsilon_{y_i} - \mu|/2} \rho_\omega(n_{y_i}) = 1.$$

A similar result applies for  $\rho_\omega(b_X b_X^\dagger)$ . It also applies to  $\rho_\omega(\sigma_Z^2)$  once it is remarked that  $\sigma_x^2 = s_x \left( e^{\beta(\epsilon_x - \mu)} n_x + e^{-\beta(\epsilon_x - \mu)} (1 - n_x) \right)$ . Hence each of these vectors is normalized.

Let now  $(X, Y, Z)$  be three disjoint finite subsets of  $\mathcal{L}$ , and let  $(X', Y', Z')$  be three other ones, such that these triples are distinct. Then without loss of generality, it can be assumed that  $X \cup Y \cup Z \neq \emptyset$ . Assume first that there is a point  $x \in (X \cup Y \cup Z) \setminus (X' \cup Y' \cup Z')$ . If  $x \in X$ , then the inner product  $\rho_\omega \left( (b_X^\dagger b_Y \sigma_Z)^* b_{X'}^\dagger b_{Y'} \sigma_{Z'} \right)$  will have a factor  $\rho_\omega(b_x) = 0$ , if  $x \in Y$ , the factor will be  $\rho_\omega(b_x^\dagger) = 0$ , whereas if  $x \in Z$  it will be  $\rho_\omega(\sigma_x) = 0$ . Hence the two vectors are orthogonal. If  $X \cup Y \cup Z = X' \cup Y' \cup Z'$ , then either  $X \neq X'$ ,  $Y \neq Y'$  or  $Z \neq Z'$ . In the first case there is  $x \in X \setminus X'$ . Then  $x \in Y' \cup Z'$ . If  $x \in Y'$ , then the inner product get a factor  $\rho_\omega(b_x b_x) = 0$ , whereas if  $x \in Z'$  the factor is  $\rho_\omega(b_x \sigma_x)$ , which vanishes as well. The same argument holds in all other cases leading to the result.  $\square$



**Corollary 5.8.** *The family  $\{\zeta_\omega(\sigma_Z); \emptyset \neq Z \subset \mathcal{L}(\omega), Z \text{ finite}\}$  is an orthonormal basis of  $\mathcal{K}_\omega$ .*

**Proof:** Since elements of  $\mathcal{K}_\omega$  are orthogonal to  $\xi_\omega = \zeta_\omega(\mathbf{1})$  they are generated by linear combination of the  $b_X^\dagger b_Y \sigma_Z$ 's with  $X \cup Y \cup Z \neq \emptyset$ . Since they come from the commutative sub-algebra  $\mathfrak{C}_\omega$  they are generated by the operators  $s_x n_x$ , namely by the  $\sigma_X$ 's only.  $\square$

**Proof of Theorem 5.5 (continued):** Let  $\mathfrak{D}_{x,\star}$  be the operator on  $\mathfrak{A}(\omega)$  defined by

$$\mathfrak{D}_{x,\star}(A) = \frac{1}{2}(a_x^\dagger a_x A + A a_x^\dagger a_x) - a_x^\dagger \sigma(A) a_x. \quad (5.6)$$

In much the same way, let  $\mathfrak{D}_{\star,x}$  be defined by

$$\mathfrak{D}_{\star,x}(A) = \frac{1}{2}(a_x a_x^\dagger A + A a_x a_x^\dagger) - a_x \sigma(A) a_x^\dagger. \quad (5.7)$$

It follows that if  $A = A_x B$  with  $A_x \in \mathfrak{A}_{\{x\}}$  and  $B \notin \mathfrak{A}_{\{x\}}$  then  $\mathfrak{D}_{x,\star}(A) = \mathfrak{D}_{x,\star}(A_x)B$  and the same holds for  $\mathfrak{D}_{\star,x}$ . Since  $A_x$  can only be a linear combination of  $\mathbf{1}$ ,  $b_x$ ,  $b_x^\dagger$ ,  $\sigma_x$ , it is enough to consider each of these cases. This leads to

$$\mathfrak{D}_{x,\star}(\mathbf{1}) = 0, \quad \mathfrak{D}_{x,\star}(b_x) = \frac{b_x}{2}, \quad \mathfrak{D}_{x,\star}(b_x^\dagger) = \frac{b_x^\dagger}{2}, \quad \mathfrak{D}_{x,\star}(\sigma_x) = 2 \cosh\left(\frac{\beta}{2}(\epsilon_x - \mu)\right) n_x,$$

$$\mathfrak{D}_{\star,x}(\mathbf{1}) = 0, \quad \mathfrak{D}_{\star,x}(b_x) = \frac{b_x}{2}, \quad \mathfrak{D}_{\star,x}(b_x^\dagger) = \frac{b_x^\dagger}{2}, \quad \mathfrak{D}_{\star,x}(\sigma_x) = -2 \cosh\left(\frac{\beta}{2}(\epsilon_x - \mu)\right) (\mathbf{1} - n_x).$$

Since  $\mathfrak{D}_\omega^\star = \sum_{x \in \mathcal{L}(s)} \Gamma_{x \rightarrow \star} \mathfrak{D}_{x,\star} + \Gamma_{\star \rightarrow x} \mathfrak{D}_{\star,x}$  it follows immediately that  $b_x$  and  $b_x^\dagger$  are both eigenvectors of  $\mathfrak{D}_\omega^\star$  for the common eigenvalue

$$\gamma_x = \frac{\Gamma_{x \rightarrow \star}}{2} + \frac{\Gamma_{\star \rightarrow x}}{2} = \frac{\Gamma_\star}{2} (1 + e^{-\beta|\epsilon_x - \mu|}). \quad (5.8)$$

This means  $\zeta_\omega(b_x)$  and  $\zeta_\omega(b_x^\dagger)$  are eigenvectors of  $\mathfrak{D}_\omega^\star$  with eigenvalue  $\gamma_x$ . Similarly, an elementary calculation shows that

$$\mathfrak{D}_\omega^\star(\zeta_\omega(b_X^\dagger b_Y \sigma_Z)) = \gamma_{X,Y,Z} \zeta_\omega(b_X^\dagger b_Y \sigma_Z), \quad (5.9)$$

where

$$\gamma_{X,Y,Z} = \sum_{x \in X \cup Y} \gamma_x + 2 \sum_{z \in Z} \gamma_z. \quad (5.10)$$

Hence,  $\mathfrak{D}_\omega^\star$  has pure point spectrum. In addition,  $\gamma_{X,Y,Z}$  vanishes if and only if  $X = Y = Z = \emptyset$ . This shows that  $\xi_\omega = \zeta_\omega(\mathbf{1})$  is the only eigenvector with eigenvalues 0. It shows also that  $\mathcal{K}_\omega$  is invariant by  $\mathfrak{D}_\omega^\star$  and, thanks to the Corollary 5.8, the restriction of  $\mathfrak{D}_\omega^\star$  on  $\mathcal{K}_\omega$  is bounded below by  $\Gamma_\star$  (uniformly in  $\beta, \mu$  and  $\omega$ ).

Let  $\mathfrak{D}_{x,y}(A) = \frac{1}{2}(L_{x \rightarrow y}^* L_{x \rightarrow y} A + A L_{x \rightarrow y}^* L_{x \rightarrow y}) - L_{x \rightarrow y}^* A L_{x \rightarrow y}$ , then one has  $\mathfrak{D}_{x,y}(n_x) = \Gamma_{x \rightarrow y} n_x (\mathbf{1} - n_y)$ ,  $\mathfrak{D}_{x,y}(n_y) = -\Gamma_{x \rightarrow y} n_x (\mathbf{1} - n_y)$  and  $\mathfrak{D}_{x,y}(n_z) = 0$  for  $z \notin \{x, y\}$ . Hence,  $\mathfrak{D}_\omega$  preserves the sub-algebra  $\mathfrak{C}_\omega$ . Moreover  $\mathfrak{D}_\omega(\mathbf{1}) = 0$ , showing that  $\mathfrak{D}_\omega$  leaves the subspace  $\mathcal{K}_\omega$  invariant. As  $\mathfrak{D}_\omega \geq \mathfrak{D}_\omega^\star$  it follows that 0 is a simple eigenvalue of  $\mathfrak{D}_\omega$  and that  $\mathfrak{D}_\omega$  has at least the same gap as  $\mathfrak{D}_\omega^\star$ .  $\square$

**Remark 5.9.** *If  $\Gamma_\star$  is not constant but depends on  $\epsilon_x$  and  $\beta$  as specified by Remark 2.2 then (5.9) and (5.10) are still correct. Only the values of  $\gamma_x$  would change correspondingly as one has to replace  $\Gamma_\star$  by  $\Gamma_\star(\epsilon_x, \beta)$  in (5.8).*

*By the assumptions of Remark 2.2, the infimum of this constant for all  $\epsilon_x \in \Delta$  and all  $\beta \in (0, \infty)$  is positive, and therefore  $\mathcal{D}_\omega^\star$  and  $\mathcal{D}_\omega$  will still have a spectral gap (uniformly in  $\beta$ ,  $\mu$  and  $\omega$ ) as asserted in Theorem 5.5.*

## REFERENCES

- [1] G. Androulakis, J. Bellissard, C. Sadel.: Dissipative dynamics and transport in semiconductors. Work in progress.
- [2] S. Albeverio, R. Höegh-Krøhn.: Dirichlet Forms and Markovian semigroups on  $C^*$ -algebras, *Comm. Math. Phys.*, **56**, (1977), 173-187.
- [3] V. Ambegaokar, B. I. Halperin, J. S. Langer.: Hopping Conductivity in Disordered Systems. *Phys. Rev. B*, **4**, (1971), 2612-2620.
- [4] H. Araki.: Some properties of modular conjugation operator of von Neumann algebras and a non commutative Radon-Nykodim theorem with a chain rule. *Pacific Journ. Math.*, **50**, (1974), 309-354.
- [5] J. Bellissard, H. Schulz-Baldes, A. van Elst.: The Non Commutative Geometry of the Quantum Hall Effect. *J. Math. Phys.*, **35**, (1994), 5373-5471.
- [6] J. Bellissard.: Coherent and dissipative transport in aperiodic solids. *Dynamics of Dissipation*, Garbaczewski, P.; Olkiewicz, R. (Eds.), *Lecture Notes in Physics*, **597**, (2003), 413-486.
- [7] J. Bernasconi.: Electric Conductivity in Disordered Systems. *Phys. Rev. B*, **7**, (1973), 2252-2260.
- [8] A. Beurling, J. Deny.: Dirichlet Spaces, *Proc. Nat. Acad. Sci.*, **45**, (1959), 208-215.
- [9] O. Bratteli, D. W. Robinson.: Operator algebras and quantum statistical mechanics, I:  $C^*$ - and  $W^*$ -algebras, algebras, symmetry groups, decomposition of states, *Texts and Monographs in Physics*, Springer-Verlag, New York-Heidelberg, 1979.
- [10] O. Bratteli, D. W. Robinson.: Operator algebras and quantum-statistical mechanics, II: Equilibrium states. *Models in quantum-statistical mechanics*, *Texts and Monographs in Physics*, Springer-Verlag, New York-Berlin, 1981.
- [11] F. Cipriani.: Dirichlet Forms and Markovian Semigroups on Standard Forms of von Neumann Algebras. *J. Funct. Anal.*, **147**, (1997), 259-300.
- [12] C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg.: Photon and atoms: introduction to quantum electrodynamics, Wiley-VCH Verlag GMBH & Co KGa, 2004.
- [13] A. Connes.: Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann. *Ann. Inst. Fourier, Grenoble*, **24**, (1974), 121-155.
- [14] A. Connes.: Sur la théorie non commutative de l'intégration, (French). *Algèbres d'opérateurs (Séminaire Les Plans-sur-Bex, 1978)*, pp. 19-143, *Lecture Notes in Math.*, **725**, Springer, Berlin, (1979).
- [15] A. Connes.: *Noncommutative Geometry*. Acad. Press., San Diego (1994).
- [16] E. B. Davies.: *Quantum theory of open systems*. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976.
- [17] J. Dixmier.: *Les  $C^*$ -algèbres et leurs représentations*, (French) Deuxième édition. *Cahiers Scientifiques*, Fasc. XXIX. Gauthier-Villars Éditeur, Paris 1969.
- [18] T. P. Eggarter, M. H. Cohen.: Simple Model for Density of States and Mobility of an Electron in a Gas of Hard-Core Scatterers. *Phys. Rev. Lett.*, **25**, (1971), 807-810.
- [19] A. Faggionato, H. Schulz-Baldes, D. Spehner.: Mott law as lower bound for a random walk in a random environment. *Commun. Math. Phys.*, **263**, (2006), 21-64.
- [20] M. Fukushima.: *Dirichlet Forms and Markov Processes*. North-Holland 1980.

- [21] R. Haag, N.M. Hugenholtz, M. Winning.: On the equilibrium states in Quantum Statistical Mechanics. *Commun. Math. Phys.* **5** (1967), 215-236.
- [22] R. M. Hill.: On the Observation of variable Range Hopping. *Phys. Stat. Sol.*, **A35**, (1976), K29-K34.
- [23] R. V. Kadison, J. R. Ringrose.: *Fundamentals of the theory of operator algebras. Vol. I. Elementary theory. Pure and Applied Mathematics, 100.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.
- [24] T. Kato.: *Perturbation theory for linear operators.* Reprint of the 1980 edition, *Classics in Mathematics*, Springer-Verlag, Berlin, 1995.
- [25] S. Kirkpatrick.: *Classical Transport in Disordered Media: Scaling and Effective-Medium Theories.* *Phys. Rev. Lett.*, **27**, (1971), 1722-1725.
- [26] R. Kubo.: *Statistical-Mechanical Theory of Irreversible Processes, I.* *J. Phys. Soc. Japan*, **12**, (1957), 570-586.
- [27] B. J. Last, D. J. Thouless.: *Percolation Theory and Electrical Conductivity.* *Phys. Rev. Lett.*, **27**, (1971), 1719-1721.
- [28] G. Lindblad.: *On the generators of quantum dynamical semigroups.* *Comm. Math. Phys.*, **48**, (1976), 119-130.
- [29] G. Mahan.: *Many Particles Physics*, 2nd Printing, Plenum, (1990).
- [30] P.C. Martin, J. Schwinger.: *Theory of many-particle systems, I* *Phys. Rev.* **115** (1959), 1342-1373.
- [31] A. Miller, E. Abrahams.: *Impurity Conduction at Low Concentration.* *Phys. Rev.*, **120**, (1960), 745-755.
- [32] N. F. Mott.: *J. Non-Crystal. Solids* **1**, 1 (1968). See also N. F. Mott.: *Metal-Insulator Transitions*, Taylor and Francis, London, 1974.
- [33] N. F. Mott, M. Pepper, S. Pollitt, R. H. Wallis & C. J. Adkins.: *The Anderson transition.* *Proc. R. Soc. Lond.*, **A345**, (1975), 169-205.
- [34] R. Prange, S. Girvin (editors): *The quantum Hall effect*, (Springer-Verlag, Berlin, 1990).
- [35] M. Pollak.: *A percolation treatment of dc-hopping conduction.* *J. Non-Crystalline Solids*, **11**, (1972), 1-24.
- [36] D. G. Polyakov, B. I. Shklovskii.: *Variable range hopping as the mechanism of the conductivity peak broadening in the quantum Hall regime.* *Phys. Rev. Lett.*, **70**, (1993), 3796-3799.
- [37] M. Reed, B. Simon.: *Methods of modern mathematical physics, Vol. I-IV.* Academic Press, London, (1975).
- [38] J. Renault.: *A Groupoid Approach to  $C^*$ -Algebras.* *Lecture Notes in Math.* **793**, Springer, Berlin (1980).
- [39] B. Shklovskii, A. Efros.: *Electronic properties of doped semiconductors.* Springer, Berlin, 1984.
- [40] J. C. Slater.: *Electrons in Perturbed Periodic Potentials.* *Phys. Rev.*, **76**, (1949), 1592-1601.
- [41] D. Spehner.: *Contributions à la théorie du transport électronique dissipatif dans les solides aperiódiques (in French).* accessible at <http://www-fourier.ujf-grenoble.fr/~spehner>
- [42] H. Spohn.: *Kinetic equations from Hamiltonian dynamics: Markovian limits.* *Rev. Mod. Phys.*, **53**, (1980), 569-615.
- [43] W. F. Stinespring.: *Positive Functions on  $C^*$ -Algebras.* *Proc. Am. Math. Soc.*, **6**, (1955), 211-216.
- [44] M. Takesaki.: *Tomita's theory of modular Hilbert algebras and its applications*, *Lecture Notes in Mathematics*, Vol. **128** Springer-Verlag, Berlin-New York, 1970.
- [45] M. Takesaki.: *Theory of operator algebras. I*, Springer-Verlag, New York-Heidelberg, 1979.
- [46] M. Takesaki.: *Theory of operator algebras. II*, *Encyclopaedia of Mathematical Sciences*, 125, *Operator Algebras and Non-commutative Geometry*, 6, Springer-Verlag, Berlin, 2003.
- [47] J. Tomiyama.: *On the projection of norm one in  $W^*$ -algebras. II.* *Tôhoku Math. J.*, **10**, (1958), 204-209.
- [48] J. Tomiyama.: *Topological representation of  $C^*$ -algebras.* *Tôhoku Math. J.*, **14**, (1962), 187-204.

- [49] J. Tomiyama.: A characterization of  $C^*$ -algebras whose conjugate spaces are separable. Tôhoku Math. J., **15**, (1963), 96-102.
- [50] H. F. Trotter.: On the product of semi-groups of operators. Proc. Amer. Math. Soc., **10**, (1959), 545-551.
- [51] in Quotations by John von Neumann at <http://www-groups.dcs.st-and.ac.uk/~history>
- [52] G. H. Wannier.: The Structure of Electronic Excitation Levels in Insulating Crystals. Phys. Rev., **52**, (1937), 191-197.
- [53] A. G. Zabrodskii.: Hopping conduction and density of localized states near the Fermi level. Fiz. Tekh. Poluprov., **11**, (1977), 595 (English translation in Sov. Phys.-Semicond., **11**, (1977), 345).
- [54] J. M. Ziman.: Hopping Conductivity in Disordered Systems. J. of Phys. C, **1**, (1968), 1532-1538.

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