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Gap-Labelling for three dimensional aperiodic solids

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Abstract. Let X be totally disconnected compact Hausdorff space with an action α of Z^3 by commuting homeomorphisms and with an invariant probability measure μ. We show that $\tau_* K_0(C(X) \rtimes_\alpha Z^3) = \mu(C(X, Z))$ where τ is the trace induced on the crossed product $C(X) \rtimes_\alpha Z^3$. © Académie des Sciences/Elsevier, Paris

Le théorème de l'étiquetage des gaps pour les solides apériodiques de dimension 3.

Résumé. Soit X un espace topologique compact séparé, totalement discontinu, muni d'une action α de Z^3 par trois homéomorphismes commutant mutuellement, pour lesquels μ est une mesure de probabilité invariante. Il est alors démontré que $\tau_* K_0(C(X) \rtimes_\alpha Z^3) = \mu\,(C(X, Z))$, expression dans laquelle τ est la trace sur le produit croisé $C(X) \rtimes_\alpha Z^3$ induite par μ. © Académie des Sciences/Elsevier, Paris

Version Française abrégée

Il a été proposé de décrire un solide apériodique, situé dans R^n, à partir de l'ensemble S des positions d'équilibre de ses atomes [2]. La fermeture, dans une topologie convenable, de la famille $\{S+a; a \in R^n\}$ des translatés de S, est un espace topologique compact séparé Ω. Dans de nombreux cas, le système dynamique $\Omega \rtimes R^n$ peut se construire comme la suspension d'une action α de Z^n

Note présentée par .

sur un espace topologique compact séparé X, qui est de plus totalement discontinu. Les théorèmes classiques [3] impliquent que les groupes de K-théorie de ces deux systèmes coïncident. Il a été de plus montré dans [2] que les conditions physiques définissent naturellement sur X une mesure de probabilité invariante μ, le plus souvent ergodique, donc induisant canoniquement une trace τ sur la C^*-algèbre du produit croisé $C(X) \rtimes_\alpha Z^n$. Le théorème *d'étiquetage des bandes interdites* (the gap labelling theorem), fournit une numérotation des lacunes spectrales d'un élément auto-adjoint quelconque de $C(X) \rtimes_\alpha Z^n$ en terme de l'image par τ de l'élément de K-théorie représenté par le projecteur spectral sur la partie du spectre située à gauche de la lacune. L'ensemble des nombres ainsi obtenus est le morceau du sous groupe $\tau_*(K_0(C(X) \rtimes_\alpha Z^n))$ de R contenu dans l'intervalle $[0,1]$. L'identification de ce sous-groupe a été réalisé dans de nombreux cas pour $n = 1, 2$ [1,10]. Il a été conjecturé que si X est totalement discontinu (*i.e.* si X est homéomorphe à l'ensemble de Cantor), alors:

Conjecture. – *Soit X un espace topologique compact séparé, totalement discontinu, muni d'une action α de Z^n par n homéomorphismes commutant mutuellement, pour lesquels μ est une mesure de probabilité invariante. Alors, si τ est la trace sur la C^*-algèbre du produit croisé $C(X) \rtimes_\alpha Z^n$ induite par μ,*

$$\tau_*\left(K_0(C(X) \rtimes_\alpha Z^n)\right) = \mu\left(C(X, Z)\right) .$$

Si X provient de l'enveloppe d'un ensemble de positions atomiques, $\mu\left(C(X,Z)\right)$ peut s'interpréter comme l'ensemble des probabilités d'apparition des configurations atomiques locales.

Pour $n = 1$ cette conjecture est une conséquence immédiate de la suite exacte de Pimsner et Voiculescu [9,1], tandis que pour $n = 2$ elle a pu être démontrée en utilisant deux fois cette suite exacte [11]. Le présent article la démontre pour $n = 3$, qui représente la situation physique réaliste. La méthode utilisée fournit une autre démonstration du cas $n = 2$ et devrait pouvoir s'étendre au cas général $n \geq 2$.

La première étape de la construction consiste à ramener la C^*-algèbre du produit croisé à celle du *mapping torus* $M_\alpha A$, si $A = C(X)$, à savoir la C^*-algèbre des fonctions $\{f : x \in R^n \mapsto A | f(x+a) = \alpha_a(f(x)), a \in Z^n\}$. Dans ce cas,

$$K_*(A \rtimes_\alpha Z^n) \cong K_{*-n}(M_\alpha A) ,$$

isomorphisme qui peut être obtenu à partir de l'isomorphisme de Thom-Connes.

En second lieu on utilise [4]. La classe transverse du feuilletage défini par l'action de R^n sur l'espace sous-jacent à M_α détermine canoniquement une n-trace τ_α sur $M_\alpha A$ telle que :

$$\langle [\tau_\alpha], K_n(M_\alpha A)\rangle = \langle [\tau], K_0(A \rtimes_\alpha Z^n)\rangle ,$$

ou τ est la trace induite sur $A \rtimes_\alpha Z^n$ (*cf.* théorème 1).

En troisième lieu, nous définissons une suite spectrale, due à Pimsner, à partir de la décomposition de $M_\alpha A$ en $0 = I_n \subset I_{n-1} \subset \cdots \subset I_0 \subset A$, dans laquelle I_l est l'ideal des fonctions de $M_\alpha A \subset C([0,1]^n \otimes A)$ s'annulant sur les faces de dimension l du cube $[0,1]^n$. Cette filtration définit une co-filtration en posant $F_l = M_\alpha A / I_l$ conduisant à une suite exacte $0 \mapsto Q_l \overset{i}{\mapsto} F_l \overset{\pi}{\mapsto} F_{l-1} \mapsto 0$, si $Q_l = \left(S^l A\right)^{\binom{n}{l}}$. Ici, $S^l A = C_0((0,1)^l) \otimes A$. L'application du foncteur K conduit alors à une *paire exacte* [3] donc à une suite spectrale. Forrest et Hunton [6] ont montré que cette suite converge en page deux et fournit un isomorphisme de groupes entre $K_s(M_\alpha A)$ et la somme directe, sur j, des groupes d'homologie $H^{2j+s}(Z^n, C(X, Z))$, dont ils démontrent qu'ils sont libres.

Comme pour l'injection canonique $I_{n-1} = S^n A \stackrel{j}{\hookrightarrow} M_\alpha A$, on a

$$\langle [\tau_\alpha], j_* K_n(S^n A)\rangle \;=\; \langle [\tau], K_0(A)\rangle \;=\; \mu\left(C(X,Z)\right) \,,$$

la conjecture sera vraie si nous montrons que remplacer $S^n A$ par $M_\alpha A$ dans cette égalité, ne change pas le résultat. C'est l'objet du résultat original de cet article (*cf.* théorème 2) que de montrer en effet que, dans la suite spectrale précédente, il est possible de relever tout élément de l'image par i_* de la cohomologie du complexe $(K(Q_l), d_1)$ en un élément de $K(M_\alpha A)$ situé dans le noyau de $[\tau_\alpha]$, pourvu que $l = 0, 1$. En conséquence, pour $n \leq 3$ on peut remplacer $S^n A$ par $M_\alpha A$, sans changer le résultat ce qui prouve la conjecture.

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1. Introduction

Among the basic input data for the description of an aperiodic solid is the point set S of its atomic positions. The algebra of observables is derived from this set. Although S need not be periodic it allows quite often for an action of Z^n by n (the dimension of the solid) independent translations. In this case, which includes e.g. quasicrystals, the C^*-algebra of observables is strongly Morita equivalent to a crossed product $C(X) \rtimes_\alpha Z^n$. Here X is a reduced version of the canonical transversal (also called discrete hull of S in [8]) of the hull of S [2] and consists of point sets which look locally like S, and the action α of Z^n is induced by the above translations (we do not distinguish notationally between the action on X and that induced on $C(X)$). An additional input from physics is an invariant probability measure μ on X. The gap-labelling theorem in its K-theoretic formulation [2] connects the non-commutative topology of S with the values the integrated density of states may take on the gaps in the spectrum of the Hamiltonian. It states that these values belong to $\tau_* K_0(C(X) \rtimes_\alpha Z^n) \cap [0,1]$ where $\tau : C(X) \rtimes_\alpha Z^n \to C$ is the trace induced by μ. This is why we are interested in computing $\tau_* K_0(C(X) \rtimes_\alpha Z^n)$. For more background on this subject we refer the reader to [1,2,7] and references therein.

The topology of X depends on the repetivity properties of S. If S is periodic X is simply a finite discrete set. In many aperiodic cases, including quasicrystals, X is homeomorphic to the Cantor set. These cases fall under the vicinity of the following conjecture [2]:

Conjecture. – *Let X be a totally disconnected compact Hausdorff space with an action of Z^n by n commuting homeomorphisms and invariant probability measure μ. Write τ for the trace induced on the crossed product $C(X) \rtimes_\alpha Z^n$. Then*

$$\tau_* K_0(C(X) \rtimes_\alpha Z^n) = \mu(C(X,Z)).$$

In the case where X is the discrete hull of a point set, $\mu(C(X,Z))$ can be interpreted as the subgroup of R generated by the relative frequencies of finite point configurations in the set.

For $n = 1$ the conjecture follows from the Pimsner-Voiculescu sequence [9,8,1]. Iteration of the Pimsner-Voiculescu sequence has, however, only been succesful for $n = 2$ [11]. We provide a proof for $n = 3$ thus covering the case of interest for real materials.

2. The spectral sequence and the associated n-trace.

Given a C^*-dynamical system (A, α, Z^n), the K-groups of the crossed product $A \rtimes_\alpha Z^n$ are isomorphic to those of the mapping torus $M_\alpha := \{f : R^n \to A | f(x+a) = \alpha_a(f(x)) | a \in Z\}$,

$$K_*(A \rtimes_\alpha Z^n) \cong K_{*-n}(M_\alpha) \tag{1}$$

An essential ingredient of this isomorphism is Connes' Thom isomorphism [4,3]. A cofiltration of M_α gives rise to a spectral sequence whose E_2-term is isomorphic to $H^*(Z^n, K_*(A))$, the cohomology of the group Z^n with coefficients in $K_*(A)$. Forrest and Hunton [6] have established that, for Z^n actions on the Cantor set X, the spectral sequence collapses at the E_2-term and

$$K_i(M_\alpha C(X)) \cong \bigoplus_j H^{2j+i}(Z^n, C(X,Z)). \tag{2}$$

The machinery of spectral sequences proves to be quite useful in computing the K-groups–a refined analysis allows even to compute them explicitely for canonical projection method tilings [7]–but it fails to give information about the order on K_0 and the ranges of tracial states.

The first difficulty is that Connes' Thom isomorphism does not preserve order. Nevertheless the image of a tracial state can be expressed using higher traces (cyclic cohomology). Let ∂t_i be the derivation associated to the coordinate t_i of the R^n-action on M_α, and $\partial = \sum_1^n \partial t_i$. The following result is in in Connes original work [4], (see also Theorem 13 of [9])

Theorem 1. – *Let (A, α, Z^n) be a C^*-dynamical system and τ an invariant trace on A. Then τ extends to a trace τ on $A \rtimes_\alpha Z^n$ and to a densely defined n-trace $\tau_\alpha : M_\alpha A \to C$,*

$$\tau_\alpha(a_0, \cdots, a_n) = \int \tau(a_0 \partial a_1 \cdots \partial a_n) dt_1 \cdots dt_n, \tag{3}$$

satisfying

$$\langle [\tau_\alpha], K_n(M_\alpha A) \rangle = \langle [\tau], K_0(A \rtimes_\alpha Z^n) \rangle. \tag{4}$$

Here we denoted as usual cyclic cohomology classes by $[\ \cdot\ ]$ and Connes pairing [5] between cyclic cohomology and K-theory by $\langle \cdot, \cdot \rangle$. It is important to note that, if one restricts τ_α to the intersection of its domain with the n-fold suspension $S^n A \stackrel{j}{\hookrightarrow} M_\alpha A$ then it corresponds to the cup product of τ with the nth power of the canonical 1-cocycle on $C(S^1)$ and satisfies

$$\langle [\tau_\alpha], j_* K_n(S^n A) \rangle = \langle [\tau], K_0(A) \rangle = \tau_* K_0(A). \tag{5}$$

With Thm. 1 and (5) at hand the conjecture comes down to understanding the difference between $\langle [\tau_\alpha], K_n(M_\alpha C(X)) \rangle$ and $\langle [\tau_\alpha], j_* K_n(S^n C(X)) \rangle$. The rotation algebra is an example where these two groups are not equal. To analyse this we take a closer look at the spectral sequence underlying (2). View $M_\alpha A$ as a subalgebra of $C([0,1]^n, A)$, the algebra of continuous functions from the cube $[0,1]^n$ to A. Let I_l be the ideal of functions $f : [0,1]^n \to A$ which vanish on all l-dimensional faces of the cube. The quotients $F_l := M_\alpha A / I_l$ yield a cofiltration $M_\alpha A = F_n \stackrel{\pi}{\to} F_{n-1} \stackrel{\pi}{\to} \cdots F_0 \cong A$ and exact sequences

$$0 \to Q_l \stackrel{\imath}{\to} F_l \stackrel{\pi}{\to} F_{l-1} \to 0 \tag{6}$$

where $Q_l := (S^l A)^{\binom{n}{l}}$ (direct sum of $\binom{n}{l}$ copies) and $\imath : Q_l \to F_l$ is defined as follows: we number the l-dimensional faces of the cube which contain $(0, \cdots, 0)$ and assign to $(f_1, \cdots, f_{\binom{n}{l}})$ the class of a function $F \in M_\alpha A$ which coincides on the rth l-dimensional face with f_r (a little care is needed to keep track of the identification of variables of the cube).

Applying the functor K to (6) gives rise to the exact pair

$$K(Q) \partial \nearrow\!\searrow \imath_* K(F) \stackrel{\pi_*}{\longleftarrow} K(F)$$

underlying the spectral sequence. Here $F = \bigoplus_l F_l$, $Q = \bigoplus_l Q_l$ and $K(A) = \bigoplus_i K_i(A)$. The lth degree cohomology group $H_{d_1}(K(Q_l))$ of the differential complex $(K(Q), d_1 = \partial \circ i_*)$ is isomorphic to $H^l(Z^n, K(S^l A))$. The vanishing of the higher differentials, i.e. the collapsing of the spectral sequence at the E_2-term, is equivalent to the fact that each element in $\imath_* H_{d_1}(K(Q_l)) = \imath_*(\ker d_1 \cap K(Q_l))$ can be lifted to an element in $K(M_\alpha A)$. Forrest and Hunton have shown this indirectly for Cantor dynamical systems by establishing that $H^l(Z^n, K(C(X)))$ is torsion free. But for determining the difference between $\langle[\tau_\alpha], K_n(M_\alpha C(X))\rangle$ and $\langle[\tau_\alpha], j_* K_n(S^n C(X))\rangle$ we need explicit lifts of the groups $\imath_* H_{d_1}(K(Q_l))$ for $l < n$ and below we construct them for $l = 0, 1$.

3. Explicit lifts

The conjecture would follow if we could find lifts for the elements of $\imath_* H_{d_1}(K(Q_l))$, $l < n$, which lie in the kernel of $\langle[\tau_\alpha], \cdot\rangle$.

Theorem 2. – *Let X be a totally disconnected compact space with an action of Z^n by n commuting homeomorphisms. For each element in $\imath_* H_{d_1}(K_i(Q_l))$, $l = 0, 1$, exists a lift in $K_i(M_\alpha)$ which lies in the kernel of $\langle[\tau_\alpha], \cdot\rangle$.*

Proof. Consider $l = 0$. $H_{d_1}(K_i(Q_0)) = K_i(C(X))^\alpha$, which is trivial for odd i. For even i the map $C(X, Z) \to K_0(C(X)) : 1_U \mapsto [1_U]$ (1_U the indicator function on a clopen set U) extends to a group isomorphism. This implies that any projection whose K_0-class is α-invariant is equivalent to a projection which itself is α-invariant. Given an α-invariant projection p, $P(t_1, \cdots, t_n) := p$ is a projection in (the stabilization of) $M_\alpha C(X)$ which satisfies $\pi_*^n([P]) = \imath_*([p])$. Since P is constant its class lies in the kernel of $\langle[\tau_\alpha], \cdot\rangle$.

Now consider $l = 1$. $H_{d_1}(K_i(Q_1))$ is trivial for even i. Using the isomorphism $C(X, Z) \to K_1(SC(X)) : 1_U \mapsto [e^{2\pi i t 1_U}]$ one finds that the elements of $\ker d_1 \cap K_1(Q_1)$ are given by vectors $([e^{2\pi i t_1 f_1}], \cdots, [e^{2\pi i t_n f_n}])$, $f_k \in C(X, Z)$ which satisfy $\delta_j f_k = \delta_k f_j$ for all j, k, where $\delta_k = \alpha(e_k) - \mathrm{id}$. It is straighforwardly checked that

$$U(t_1, \cdots, t_n) := \exp 2\pi i \left(\sum_{\emptyset \neq J \subset \{1, \cdots, n\}} f_J \prod_{j \in J} t_j \right)$$

with $f_{\{i_1, \cdots, i_k\}} = \delta_{i_1} \cdots \delta_{i_{k-1}} f_{i_k}$ is a unitary in $M_\alpha C(X)$. The class of this unitary is a lift of $i_*([e^{2\pi i t_1 f_1}], \cdots, [e^{2\pi i t_n f_n}])$. Since $U^{-1} \frac{\partial U}{\partial t_i}$ commute for different i the class of $U(t_1, \cdots, t_n)$ lies in the kernel of $\langle[\tau_\alpha], \cdot\rangle$.

This theorem implies the conjecture for $n \leq 3$ since $H_{d_1}(K_1(Q_2))$ is trivial.

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