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Ground state of the Fermi gas on 2D lattices with a magnetic field

R. Rammal ([1]) and J. Bellissard ([2])

([1]) Centre de Recherches sur les Très Basses Températures, CNRS, BP 166 X, 38042 Grenoble Cedex 05, France
([2]) Centre de Physique Théorique, CNRS, Luminy, Case 907, 13288 Marseille Cedex 09, France

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Résumé. — On étudie l'énergie des électrons de Bloch sous champ magnétique en fonction du flux magnétique ϕ et du taux de remplissage ν. On calcule l'énergie totale $E(\phi, \nu)$ à l'aide d'une nouvelle méthode de quantification semi-classique. Le minimum absolu est atteint pour un choix optimal d'un quantum de flux par particule. Ce phénomène se produit pour différents réseaux et sous des conditions assez générales. Pour ν fixé, l'énergie est une ligne brisée en fonction de ϕ. Toutefois, l'énergie du fondamental est une fonction régulière de ν. Les implications de ces résultats pour la stabilisation des Anyons et les phases de flux sont discutées.

Abstract. — The energy of 2D Bloch electrons in a magnetic field is studied as a function of the filling fraction ν and the magnetic flux ϕ. Using a new semi-classical quantization method the total energy $E(\phi, \nu)$ is calculated and shown to have an absolute minimum which corresponds to one flux quantum per particle. This optimal flux phenomenon is shown to occur under large conditions and for different lattices. An explicit cusp-like behavior of E *vs.* ϕ at fixed ν is found both for the absolute and for the relative minima of E. Furthermore, the ground state energy is shown to be a smooth function of ν. The implications of our results for the stabilization of Anyons and the flux states are discussed.

1. Introduction.

Very recently, the search for a possible failure of the Fermi liquid theory in high T_c superconductors was at the origin of a new piece of physics for the Landau levels of Bloch electrons [1]. One of the consequences of the convoluted structure of the energy spectrum is the possible stabilization of the Fermi sea by the gaps which appear at every rational magnetic flux ϕ. Indeed, Anderson [1, 2] was the first to realize the possibility of an optimal flux corresponding to one flux quantum per particle. More precisely, for a given filling fraction ν of fermions in a 2D lattice, the actual ground state is achieved for $\phi = \nu$. It turns out that this relation is at the heart of the Anyon superconductivity [3], where it is called the « fundamental relation ». Since, this conjecture has received a rather strong numerical support from various authors and for different lattice structures [4, 5]. Furthermore, the plot of the total energy at fixed ν exhibits clearly a cusp-like structure with an absolute minimum at

$\phi = \nu$ [5]. This numerical support has been facilitated by the present knowledge of the energy spectrum. Indeed, the spectrum of tight-binding electrons on a 2D lattice in a magnetic field has an extremely rich structure. The generic features of the single particle spectrum (self-similarity, nesting properties, gap-labelling, etc.) are by now well studied [6, 7] for various 2D lattices.

Actually, the complete absence of analytical or rigorous results for this very important conjecture motivated the work presented in this paper. A summary of our results has already been announced elsewhere [8]. A detailed exposition is the main purpose of this paper. The method we used is a semi-classical approach for Bloch electrons in a magnetic field. A recent powerful formulation of this problem is the object of reference [9]. Within the same algebraic framework, we investigated the zero field limit as well the case of an arbitrary rational flux. The present paper is a direct illustration of the results obtained in reference [9].

The paper is organized as follows. In section 2, we work out the case of zero magnetic flux, where the property of optimal flux at $\phi = \nu$ is explicitely obtained. Section 3 is devoted to the case of an arbitrary rational flux ϕ. The detailed structure of the energy $E(\phi, \nu)$ as a function of ϕ and ν is worked out in section 4 : cusps, symmetry, analyticity of the ground state energy, etc. Concluding remarks are the object of section 5, where the possible extensions of our results are briefly indicated.

2. Fermi sea energy : weak field limit.

In this section, we consider the Fermi sea energy of N spinless fermions on a 2D lattice (N_s sites, N_p plaquettes). At very weak magnetic field, the lowest Landau levels are given by the standard expression : $E_n = \varepsilon_0 + \beta H \left(n + \frac{1}{2}\right)$ to first order in the magnetic flux. Here ε_0 denotes the lower edge of the zero field band (e. a. $\varepsilon_0 = -4t$ for a square lattice) and β is defined *via* $\beta H = 2t\gamma$, $\gamma = 2\pi \frac{\phi}{\phi_0}$. Each level admits a degeneracy D, which is equal to the ratio of the total flux (magnetic field $\times$ system area) to the flux quantum ϕ_0. It turns out that for $\nu N_s \equiv N$ fermions, the appropriate variable for the energetics is

$$X = \frac{N}{D} = \frac{N_s}{N_p} \frac{\nu}{\phi},$$

where ν is the lattice filling factor and ϕ is the reduced flux through each of the N_p plaquettes ($\phi_0 = 1$ is the sequel). The ratio N_p/N_s is a geometrical factor equal to 1 (resp. 2 and 1/2) for a square (resp. triangular and honeycomb) lattice. Omitting the origin of energy ε_0, the total energy E is given by

$$\begin{aligned} 2E/N\zeta_0 &= \frac{1}{X}\left[(2n+1) - n(n+1)\frac{1}{X}\right], \quad n \leqslant X \leqslant n+1 \\ &= 1 - \left(1 - \frac{n+1}{X}\right)\left(1 - \frac{n}{X}\right) \end{aligned} \tag{2.1}$$

where $\zeta_0 \equiv \beta HX$ is the chemical potential at zero field.

As a function of X, E exhibits oscillations (Fig. 1) and assumes the same value $2E/N\zeta_0 = 1$ at $X = 1, 2, 3, \ldots$ Each of these degenerate states corresponds to the complete filling of only the first $X = 1, 2, 3, \ldots$ Landau levels. Close to these degenerate minima, E has a down cusp-like behavior. The cusps are symmetric and the oscillation of the corresponding magnetization $M = -\mathrm{d}E/\mathrm{d}H$ appears to correspond to the number of fermions in the partially occupied levels, each having the same magnetic moment.

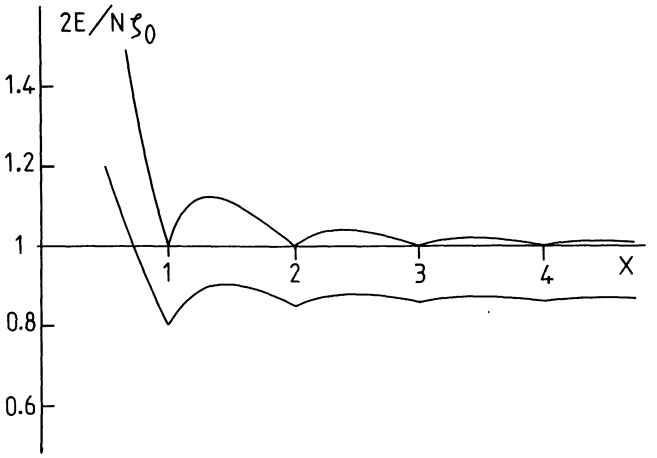

Fig. 1. — Cusp-like structure of the Fermi sea energy E of Bloch electrons, with (a) and without (b) the underlying lattice potential. Here ν is the lattice filling fraction, ϕ/ϕ_0 the reduced flux per plaquette and ζ_0 is the chemical potential at zero magnetic field.

The infinite degeneracy of the ground state energy E is lifted if the crystal potential is considered. To see this consider the case of the simple square lattice, where to the second order in γ:

$$E_n = -4t + \beta H\left(n + \frac{1}{2}\right) - \frac{\beta^2 H^2}{\Delta}[1 + (1+2n)^2], \quad \Delta \equiv 64\, t, \quad \beta H \equiv 2\,\gamma t\,. \tag{2.2}$$

The same calculation can now be repeated, with the final result ($n \leq X \leq n+1$):

$$2\,E/N\zeta_0 = \frac{1}{X}\left[(2n+1) - n(n+1)\frac{1}{X}\right] - \frac{2}{\Delta\zeta_0 X}\left(\frac{\zeta_0}{X}\right)^2 \times \\ \times \left[X(1 + (1+2n)^2) - \frac{4}{3}n(n+1)(2n+1)\right] \tag{2.3}$$

In particular, for $X = n$ filled levels:

$$2\,E/N\zeta_0 = 1 - \frac{\zeta_0}{\Delta}\frac{4}{3}\left(2 + \frac{1}{n^2}\right).$$

This implies that $X = 1$ is selected as the absolute minimum of E, the other minima are reached at $X = 2, 3, \ldots$ etc. Therefore the infinite degeneracy is lifted by the crystal potential. Furthermore the symmetric cusp-like structure is preserved at $X = n$:

$$\frac{1}{N_s}\,\mathrm{d}E/\mathrm{d}\phi = \pm\frac{1}{2}\zeta_0\left(1 - 8\frac{\zeta_0}{\Delta}\right) \text{ independent of } n\,.$$

The above result thus provides a perturbative proof of the conjecture $\phi = \nu$. However, two questions are raised by this finding: how general is the behavior of E found here for the square lattice? Is the result thus obtained stable when further terms in equation (2.2) are included?

In order to answer the first question, we consider the general case worked out in reference [9], and corresponding to :

$$E_n = \varepsilon(0) + \frac{\omega}{2}(2n+1)\gamma + \frac{\gamma^2}{64}\Delta^2\varepsilon(0)(1+(1+2n)^2) - \\ - \frac{\gamma^2}{288\,\omega}[9(3n^2+3n+1)|\Delta\partial\,\varepsilon(0)|^2 + (3n^2+3n+2)|\partial^3\varepsilon(0)|^2] + O(\gamma^3). \quad (2.4)$$

Here $\varepsilon(\mathbf{k})$ is the zero field dispersion relation, $\partial \equiv \partial/\partial k_1 - i\partial/\partial k_2$, $\Delta = \partial^2/\partial k_1^2 + \partial^2/\partial k_2^2$. Using an obvious notation, we get as before a simple expression for E. In particular for $X = n$:

$$2\,E/N\zeta_0 = 1 - \frac{2\,\zeta_0}{n^3}\left[n\left(\frac{1}{\Delta_3} + \frac{2/3}{\Delta_1}\right) + n^3\left(\frac{1}{\Delta_2} + \frac{1}{\Delta_3} + \frac{4/3}{\Delta_1}\right)\right] \quad (2.5)$$

where $-1/\Delta_1 = \Delta^2\varepsilon(0)/64\,\omega^2$, $1/\Delta_2 = \frac{9}{288\,\omega^3}|\Delta\partial\varepsilon(0)|^2$ and $1/\Delta_3 = \frac{1}{288\,\omega^3}|\partial^3\varepsilon(0)|^2$. The previous conclusion holds also in this general case. In particular, for the cases where $\varepsilon(k) = \varepsilon(-k)$ holds (i.e. lattice with inversion center symmetry), the last term in equation (2.4) vanishes and one is reduced to the previous situation of the square lattice, the only difference being the value of $\omega(>0)$ and $\Delta^2\varepsilon(0)$. Therefore, for usual lattice structures (square, triangular, honeycomb, generalized square, etc), the absolute minimum of the ground state energy is reached for $X = 1$, i.e. for $\phi = \frac{N_s}{N_p}\nu$. It is important to notice that this conclusion is true providing $\Delta_1 > 0$. For inversion-symmetric lattices, this appears as the only condition for the validity of the obtained result. The geometrical factor N_s/N_p assumes the values 1, 1/2 and 2 for square, triangular and honeycomb lattice respectively, and then the numerical finding of reference [4] is recovered.

In order to answer the second question, we have considered the effect of the next order perturbation calculation on E_n. In the case of the simple square lattice, we have [9]

$$E_n = -4t + \left(n + \frac{1}{2}\right)\beta H - \frac{\beta^2 H^2}{\Delta}(1+(1+2n)^2) + \frac{\beta^3 H^3}{\Delta'}(n^3 + (n+1)^3) \quad (2.6)$$

valid to third order in γ, where $\Delta = 64\,t$, $\Delta' = 24 \times 64\,t^2$ and $\beta H = 2\,t\gamma$. A simple calculation leads to :

$$2\,E/N\zeta_0 = 1 - \frac{8}{3}\frac{\zeta_0}{\Delta}\left(1 + \frac{1}{2n^2}\right) + \frac{\zeta_0^2}{\Delta'}\left(1 + \frac{1}{n^2}\right) \text{ at } X = n\,.$$

This shows that the absolute minimum is realized at $n = 1$. Furthermore, the cusp-like behavior is preserved. However, at this order in γ, the symmetry of the cusp is broken. Indeed, at $X = n$ the slopes are given by :

$$\frac{1}{N_s}\frac{dE}{d\phi} = \pm\frac{\zeta_0}{2}\left[1 - 8\frac{\zeta_0}{\Delta} + \frac{2\,\zeta_0^2}{n^2\Delta'}(3n^2 \pm n + 1)\right]. \quad (2.7)$$

Therefore, in addition to the asymmetry of the cusp, an explicit dependence on n takes place in equation (2.7). This general behavior of an asymmetric cusp is expected to be a generic feature. Its validity is implied by the following general statement. Consider a discrete set of energy levels : $E_n = H\alpha_n(H)$, each having a degeneracy $D_n = g_n H$, $n \geq 0$. A necessary and

sufficient condition for a symmetric cusp of the total energy of N fermions at $H = H_n^*$ corresponding to the filling of the first n levels, is

$$\frac{E_n + E_{n-1}}{2} = \sum_{r=0}^{n-1} D_r \frac{\partial (HE_r)}{\partial H} \Big/ \sum_{r=0}^{n-1} D_r \,. \tag{2.8}$$

This condition, which is satisfied to second order in γ, is violated at the next order, as can be seen in equation (2.6).

Example 1 : Consider first the case of a filling factor ν on the simple square lattice. To second order in γ, one has $\left(\nu = X \frac{\gamma}{2\pi}, X \geq 0 \right)$:

$$E/N_s = -4\nu + 2\pi\nu^2 - \frac{\pi^2}{3}\nu^3 - \frac{\pi^2}{6}\frac{\nu^3}{X^2}. \tag{2.9}$$

If the next order corrections $O(\gamma^3)$ to Landau levels are taken into account, one obtains :

$$E/N_s = -4\nu + 2\pi\nu^2 - \frac{\pi^2}{3}\nu^3 + \frac{\pi^3}{48}\nu^4 - \frac{\pi^2}{6}\frac{\nu^3}{X^2}\left(1 - \frac{\pi}{8}\nu\right). \tag{2.10}$$

Clearly, the minimum of E is always reached at $X = 1$. It is interesting to notice that the free fermions limit is obtained from equation (2.9) by taking $X = \infty$. Furthermore, we mention the very high accuracy of our result (Eq. (2.9)) in comparison with available numerical data [5].

Example 2 : Let us mention the case of a generalized square lattice with long range hopping : first (t_1), second (t_2) and third (t_3) n.n. interactions. For this model, the Landau levels are given by

$$E_n = \varepsilon_0 + \gamma t(2n+1) - \frac{\gamma^2}{16} t'(1 + (1+2n)^2) + O(\gamma^3),$$

where $t = t_1 + 2t_2 + 4t_3$ and $t' = t_1 + 4t_2 + 16t_3$. For the total energy, and a filling factor $\nu = X\frac{\gamma}{2\pi}$, one obtains $(n \leq X \leq n+1)$:

$$E/N_s = \varepsilon_0 \nu + 2\pi t \nu^2 \left[\left(\frac{n+1}{X} - 1 \right) \left(1 - \frac{n}{X} \right) + 1 \right] - \\ - \frac{\pi^2}{3} t' \nu^3 \frac{n^3 + (X-n)((n+1)^3 - n^3)}{X^3} - \frac{\pi^2}{6} t' \frac{\nu^3}{X^2} \tag{2.11}$$

This result generalizes the previous one (Eq. (2.10)) and can be used to check numerical results.

3. Fermi sea energy : arbitrary flux.

The previous result can actually be generalized to an arbitrary flux. Starting from the weak field limit $\gamma \sim 0$, a periodic potential splits the Landau levels into subbands and this results into broadening and a very convoluted rich spectrum [7]. The situation becomes simpler at rational flux $\phi = p/q$, where q subbands appear in the spectrum. Each subband has the same spectral weight $1/q$. For instance, if the filling factor is $\nu = 1/q$, then the lowest subband can be completely filled. Unless accidental degeneracy, the q subbands are separated by $q-1$ gaps. Within each gap, the integrated density of states (IDS) is given by $r\phi + s$ where r

and s are relative integers labelling the different gaps in the spectrum (gap labelling theorem [10]). The couple $r = 1$, $s = 0$ corresponds to the so-called main gap, where the IDS is exactly equal to ϕ. The stabilization of the Fermi sea at the flux $\phi = \nu$ is actually a consequence of the following argument. Let us start with simple Landau levels ε_ℓ, each having the degeneracy D. For a system of N fermions, if the flux assumes the value ϕ_n such that $N = nD$, then the ground state of the system will have n lower levels completely filled and all higher ones empty. For our purpose, it is important to realize that, when $\phi = \phi_n$, the ground state is *isolated*, separated from the lowest excited states by a finite gap. At $\phi = \phi_n$, when the highest occupied single particle level is not completely occupied, the ground state is *degenerate*.

Assume now $\phi = \phi_n$ for some n, and turns on the interaction with the periodic potential of the lattice. We claim that, to any order in perturbation theory, the ground state remains *non-degenerate*. The non-degeneracy of the ground state cannot be altered by the interactions, even if the radius of convergence of the perturbation theory is zero. Having established that the ground state is realized at $n = 1$ for weak flux, this result remains valid at on arbitrary finite value of the flux.

Our argument is mainly based on the continuity as a function of the strength of the interaction. The situation is somewhat similar to the stability of the shell structure of atoms and nuclei. Another important ingredient of the argument is the special feature of 2D Landau levels in the sense that their degeneracy is a topological data : the total flux through the lattice. A possible improvement of our argument, using homotopy concepts, is under consideration.

4. Analyticity and cusp-like structure.

As outlined in reference [9], the semi-classical approach allows for the calculation of the Landau level structure near an arbitrary rational flux p/q. The expansion parameter is $\phi - p/q$ and explicit results have been obtained. In this section, we discuss some general properties of the Fermi sea energy $E(\phi, \nu)$ as a function of the flux ϕ and the filling fraction ν.

4.1 ANALYTICITY. — The following statement can be proved rigorously [11]. If $E_0(\nu) \equiv E(\phi = \nu, \nu)$ is the ground state energy, then E_0 is a smooth function of ν except possibly at $\nu = 0$ and at every rational value $\nu = p/q$ for which the gap at the Fermi level closes.

By smooth we mean differentiable at every order. This result does not exclude the possibility that $E_0(\nu)$ be analytic with respect to ν. The proof of the analyticity property, without being difficult, requires however an elaborate machinary of non-commutative geometry. A detailed account will be published in a separate paper [11]. Let us give a hint on the proof. The fundamental idea is the analytic form of the Hamiltonian when viewed as an element of the rotation algebra $\mathcal{A}(\alpha)$, where $\alpha \equiv \phi/\phi_0$ in the notation of the present paper. As recalled in reference [9], any element of $\mathcal{A}(\alpha)$ can be expanded as a Fourier series :

$$a = \sum_{m \in \mathbb{Z}^2} a_m(\alpha)\, U^{m_1} V^{-m_2} e^{i\pi\alpha m_1 m_2}, \equiv \sum_m a_m(\alpha)\, W(m) .$$

In the case of the Hamiltonian H, $a \to a_m(\alpha)$ is a continuous application and a_m^* corresponds to $a_{-m}(\alpha)$. Furthermore, $\tau(a) = a_0(\alpha)$ where τ denotes the trace per unit volume. Take for instance the case of the Harper problem : $H = U + V + U^* + V^*$. In this case, any continuous function $f(H)$ of H admits Fourier coefficients which are continuous in α.

Let us denote by $P_F = \chi(H \leq E_F)$ the eigenprojector over energies below the Fermi

energy E_F. If E_F belongs to a gap, then P_F can be written (Cauchy formula) as $P_F = \oint_\gamma \frac{dz}{2\pi i} \frac{1}{z - H}$. But $z - H$ is analytic in α. Therefore the Fourier coefficients of P_F are analytic and the Fermi sea energy $\tau(P_F H)$ is also analytic.

4.2 EXISTENCE OF THE CUSPS (Figs 1 and 2). — In general, the following proposition holds. For a rational value p/q of ν, the total energy $E(\phi, \nu)$ exhibits a cusp (as a function of ϕ) in the vicinity of a dense set $\phi = \frac{mq - p}{nq}$ of flux values $(0 < \phi < 1)$, m and n being the gaps labels. Furthermore, for $\phi = \nu$ the slope difference $(\partial E^+/\partial\phi) - (\partial E^-/\partial\phi)$ is given by the width of the gap labelled by $m = 0$, $n = -1$. Here we adopted the notation $m - n\phi$ for the IDS with the gap (m, n).

This proposition can be proved rigorously [11]. The following argument, valid at $\phi = \nu$, illustrates the heart of the proof (the extension to other cusps is immediate). The local behavior of $E(\phi, \nu)$ at $\phi = \nu$ is controlled by the first Landau level at the subband edge we consider. For $\phi \gtrsim \nu$, the IDS is bigger than ν above the 1st Landau level, whereas it is smaller below. In order that the IDS be equal to ν (at $\phi \gtrsim \nu$), the first Landau level at the top e^+ of the last filled subband (« valence subband ») must be partially filled. Similarly, at $\phi \lesssim \nu$, the same argument implies that the first Landau level at the bottom $e^- > e^+$ of the next subband (« conduction subband ») is partially filled. So the local behaviour of $E(\phi, \nu)$ is controlled by the formula giving the levels as a function of $\phi - \frac{p}{q}$. In particular, the slopes $\partial E^\pm/\partial\phi$ at $\phi = \nu \pm 0$ are governed by the edge of the corresponding partially filled subbands. This argument holds for every cusp due to a jump of the Fermi level (Fig. 1).

The calculation of the slopes can be performed as follows. Let us first assume that $\phi = \nu + \beta$, where ν is a fixed fraction and $\beta \ll \nu$. $\beta = 0$ corresponds to a Fermi level located

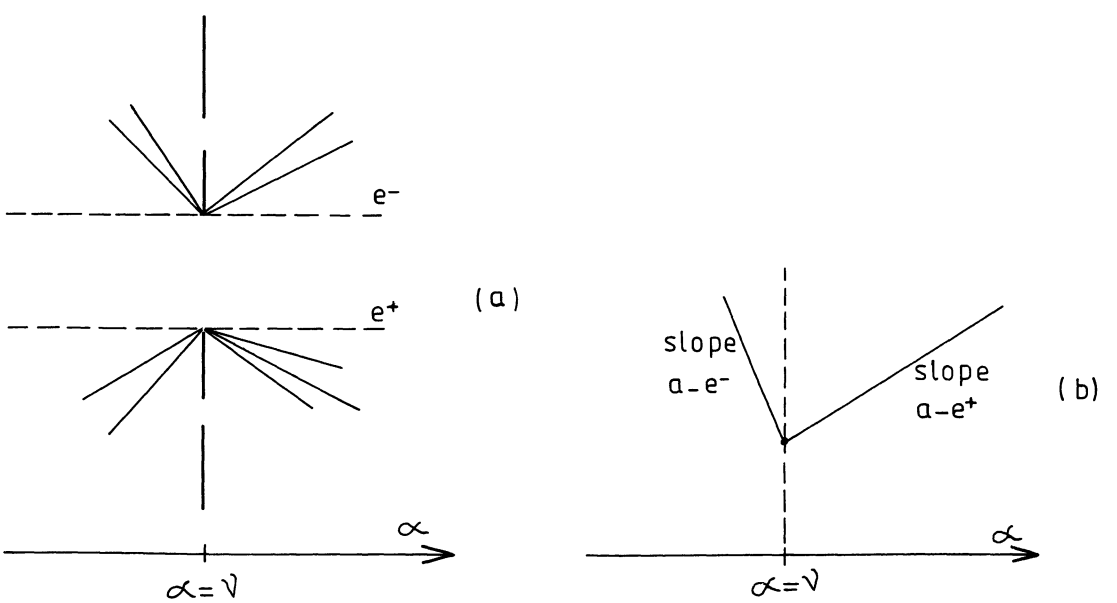

Fig. 2. — Picture of the Fermi level jump (a) at $\alpha = \nu$ between the filled subbands and the empty ones. The associated cusp of the Fermi sea energy (b) is shown. Here a denotes the « magnetization » of the filled subbands at $\alpha = \nu$.

at $e^+(\phi)$, i.e. the top of the last filled subband. For $\beta \gtrsim 0$, the Fermi level moves to the right and the total energy becomes :

$$E(\phi, \nu) = (E(\nu, \nu) + a \cdot \beta) + \left(e^+ + \beta\left(\frac{\omega}{2} + \sigma\right)\right)(-\beta). \tag{4.1}$$

The first term in equation (4.1) corresponds to the energy if the « valence » subbands are completely filled at $\phi = \nu + \beta$. The second contribution describes the deficit $(-\beta)$ in energy due to the partial filling of the first Landau level at the right of e^+. The first order terms in β have been retained and then

$$E(\phi, \nu) \simeq E(\nu, \nu) + (a - e^+)\beta + O(\beta^2), \quad \beta \gtrsim 0. \tag{4.2}$$

The physical meaning of the coefficient a is that of the magnetization (to the right) of the « valence » subbands at $\phi = \nu$ (see below). Therefore,

$$\partial E/\partial\phi\,|_{\phi = \nu^+} = (a - e^+). \tag{4.3}$$

Similarly, the behaviour of $E(\phi, \nu)$ at $\beta \lesssim 0$ is given by :

$$E(\phi, \nu) = (E(\nu, \nu) + a \cdot \beta) + \left(e^- + \beta\left(\frac{\omega'}{2} + \sigma'\right)\right)(-\beta) \tag{4.4}$$

$$\simeq E(\nu, \nu) + (a - e_-) \cdot \beta + 0(\beta^2) \tag{4.5}$$

and then

$$\partial E/\partial\phi\,|_{\phi = \nu^-} = (a - e^-). \tag{4.6}$$

This implies in particular that $E(\phi, \nu)$ admits a cusp at $\phi = \nu$. The difference between the two slopes is given by : $\partial E^+/\partial\phi - \partial E^-/\partial\phi = e_- - e_+$ which is the energy gap between the « valence » and « conduction » subbands. This result is actually a particular case of a more general formula. For a given ν, assume ϕ chosen such that the n lowest subbands are filled (e.a. local minima of E). Therefore, $\phi = \nu/m$ and :

$$\frac{\partial E^+}{\partial\phi} - \frac{\partial E^-}{\partial\phi} = \left[a\left(\frac{\nu}{n}\right) - ne^+\left(\frac{\nu}{n}\right)\right] - \left[a\left(\frac{\nu}{n}\right) - ne^-\left(\frac{\nu}{n}\right)\right] = n \cdot \left(e^-\left(\frac{\nu}{n}\right) - e^+\left(\frac{\nu}{n}\right)\right). \tag{4.7}$$

(A further generalization of Eq. (4.7) will be found below).

More generally, if the ground state is defined by $\phi = \nu = p/q$, then the other singularities of $E(\phi, \nu)$ are located at ϕ given by $p/q = m - n\phi$. Here m and n are the gap labels. Therefore, the cusps also occur at :

$$\phi = \frac{m}{n} - \frac{1}{n}\frac{p}{q}.$$

The constant $0 < \phi < 1$ restricts the allowed values of m to : $0 < m < n + \frac{p}{q}$ for $n > 0$ and $p/q > m > n + p/q$ for $n < 0$. This leaves us with a dense set of flux values, where E admits a cusp behavior.

At this point two remarks are of order. First, the generic cusp is asymmetric : $\partial E^+/\partial\phi + \partial E^-/\partial\phi \neq 0$. Explicit calculations at low fields (Eq. (2.7)) agree with this

conclusion. Secondly, our results have been obtained for a fixed number N of fermions and a fixed unit cell area. This has to be compared with the pertinent discussion of reference [12] relative to the analyticity of the energy under different conditions and for different problems.

4.3 EXPLICIT FORMULA FOR THE CUSPS. — The following formula is a generalization of equation (4.7) (see proof below)

$$\frac{\partial E}{\partial \phi}\bigg|_{\nu^{\pm}} = -\int_{-\infty}^{e^{\pm}(\phi)} \mathrm{d}\varepsilon \, \frac{\partial N(\varepsilon, \phi)}{\partial \phi} \tag{4.8}$$

where $N(\varepsilon, \phi)$ refers to the IDS at flux ϕ. An immediate consequence of equation (4.8) is the following :

$$\frac{\partial E^{+}}{\partial \phi} - \frac{\partial E^{-}}{\partial \phi} = \int_{e^{+}}^{e^{-}} \mathrm{d}\varepsilon \, \frac{\partial N(\varepsilon, \phi)}{\partial \phi} . \tag{4.9}$$

However, for $e^{+} \leq \varepsilon \leq e^{-}$, i.e. within the gap, the gap labelling theorem implies that $\frac{\partial N}{\partial \phi}$ = integer independent of ϕ or ε. So,

$$\frac{\partial E^{+}}{\partial \phi} - \frac{\partial E^{-}}{\partial \phi} = \frac{\mathrm{d}N}{\mathrm{d}\phi}(e^{-} - e^{+}) = \frac{\mathrm{d}\nu}{\mathrm{d}\phi} \times \text{gap} . \tag{4.10}$$

The result, equation (4.8) leads to an explicit formula for the slopes, and can be used for practical purposes. The proof is a simple matter of calculation. For this, starting from equation (4.1) giving the definition of a :

$$a = \frac{\partial}{\partial \phi} \int_{-\infty}^{e^{+}(\phi)} \varepsilon \, \mathrm{d}N(\varepsilon)$$

a partial integration gives :

$$a = \frac{\partial}{\partial \phi} \left\{ e^{+} N(e^{+}) - \int_{-\infty}^{e^{+}} \mathrm{d}\varepsilon \, N(\varepsilon) \right\} = e^{+} \frac{\mathrm{d}N}{\mathrm{d}\phi} + N \frac{\mathrm{d}e^{+}}{\mathrm{d}\phi} - \frac{\mathrm{d}e^{+}}{\mathrm{d}\phi} N(e^{+}) - \int_{-\infty}^{e^{+}} \frac{\partial N}{\partial \phi} \mathrm{d}\varepsilon$$

and then

$$\frac{\partial E^{+}}{\partial \phi} = a - \frac{\mathrm{d}N}{\mathrm{d}\phi} e^{+} = -\int_{-\infty}^{e^{+}} \mathrm{d}\varepsilon \, \frac{\partial N(\varepsilon, \phi)}{\partial \phi} .$$

The same calculation holds for $\partial E^{-}/\partial \phi$ and this implies equation (4.8).

It is interesting to observe that equation (4.10) is somewhat the « dual » of the formula [13] giving the compressibility of interacting fermions, restricted to the lowest Landau level :

$$\frac{\partial E^{+}}{\partial \nu} - \frac{\partial E^{-}}{\partial \nu} = \frac{\alpha}{\nu} \times \text{gap} . \tag{4.11}$$

The only difference is that, ν in equation (4.11) refers to the filling the lowest Landau level, whereas ν in equation (4.10) is the lattice filling factor.

Remark. The analysis leading to the cusp structure can actually be repeated in the cases where the gap closes. A typical example is provided by the simple square lattice at $\phi = 1/2$. In this case the gap closure occurs at $e = 0$, and the (relativistic) Landau levels are : $E_n = \pm (8 n\gamma)^{1/2}$ with $\gamma = 2\pi(\phi - 1/2) \equiv 2\pi\beta$. The same reasoning leads to ($\beta \gtrless 0$) :

$$E(\phi, \nu = 1/2) = E(1/2, 1/2) + 4\pi^{1/2}\beta^{3/2} . \tag{4.12}$$

Thus, there is no more cusp and this is consistent with the absence of the gap.

4.4 MAGNETIZATION OF THE FILLED SUBBANDS. — In the calculation of the slopes, a new quantity appeared : a. It turns out that a can be expressed simply, using the differential calculus outlined in the Appendix. For the sake of simplicity, we consider the case where the Hamiltonian does not depend explicitly on the flux $\alpha = \phi / \phi_0$ (e.a. Harper problem). In this case $\partial H = 0$. We are interested by $a \equiv \mathcal{M} = \partial \tau (PH) \equiv \dfrac{\partial}{\partial \alpha} \tau (PH)$ where P is the eigen-projector on energies below the Fermi level and τ is the trace per unit volum. Applying equation (A.1) of the Appendix with $A = P$ and $B = H$, one obtains

$$\mathcal{M} = \tau (\partial P H) + \frac{i}{4\pi} \tau (\partial_1 P \, \partial_2 H - \partial_2 P \, \partial_1 H) . \tag{4.13}$$

The first term can be calculated as shown in the appendix. Indeed, one has ($C \equiv [\partial_1 P, \partial_2 P]$) :

$$\tau [(1 - P) . \partial P . H] = \frac{i}{4\pi} \tau [CH(1 - P)]$$

and

$$\tau (P . \partial P . H) = \frac{-i}{4\pi} \tau (CHP) . \tag{4.14}$$

This implies :

$$\mathcal{M} = \partial \tau (PH) = \frac{i}{4\pi} \tau ([\partial_1 P, \partial_2 P] \, H(1 - 2P) + (\partial_1 P \, \partial_2 H - \partial_2 P \, \partial_1 H)) . \tag{4.15}$$

Further simplifications are obtained from :

$$\tau (\partial_1 P . \partial_2 H) = - \tau (P . \partial_1 \partial_2 H)$$

$$\tau (\partial_2 P . \partial_1 H) = - \tau (P . \partial_2 \partial_1 H)$$

and this leads to the final expression

$$\mathcal{M} = \frac{i}{4\pi} \tau (CH(1 - 2P)) = \frac{1}{4i\pi} (\tau (CHP) - \tau (CH(1 - P))) . \tag{4.16}$$

This original expression for $\mathcal{M}$ can be interpreted as the difference

$$\mathcal{M} = M^- - M^+ \tag{4.17}$$

between the « magnetization » M^- of the filled subbands (fermions) and that M^+ of holes occupying the empty subbands :

$$M^{\pm} = \frac{1}{4i\pi} \tau ([\partial_1 P^{\pm}, \partial_2 P^{\pm}] \, P^{\pm} \, H) \tag{4.18}$$

with $P^- = P$ and $P^+ = 1 - P$.

5. Conclusion.

Let us first summarize our main results. In this paper, we have presented a detailed study of the energetics of a Fermi gas $E(\phi, \nu)$ as a function of flux ϕ and filling ν. A non-regular

behavior of $E(\phi, \nu)$ at fixed ν has been shown to take place. In particular, a cusp-like structure is exhibited. We have calculated the associated slopes at $\phi = \nu$ as well as the relative minima of $E(\phi, \nu)$. In general the cusps occur on a dense set of values of ϕ. The procedure we used is based on semi-classical ideas, which we believe are the appropriate tools to handle the singular perturbation problem at hand : selection of *the* lowest Landau level at weak field in the infinitely degenerate case of free fermions.

The compressibility property of the Fermi gas in the ground state (i.e. at $\phi = \nu$) results from general considerations leading to the analyticity of $E(\phi = \nu, \nu)$ as a function of ν. Finally, the asymmetry of the cusp at absolute or local minima of $E(\phi, \nu)$ *vs.* ϕ is due to the finite magnetization of Landau subbands at rational flux. An explicit and original expression for this quantity has been derived here.

We conclude this paper with three remarks :

i) the possible analogy between the stabilization at $\phi = \nu$ by the gaps and Peierls instability has been noticed in reference [14]. In both cases, one is reduced to a 1D problem with a one particle spectrum having an infinite number of gaps. The interactions between fermions mediated by the lattice are at the origin of the stabilization phenomenon in both cases. Numerical support [15] for this picture in the 1D Peierls problem is well known. However, we believe that the Fermi gas problem in a magnetic field is more transparent because of the absence of mean-field approximation for instance. Furthermore, beside a very simplified model [16], there is no general proof for the existence of Peierls instability. For this reason, we believe that some deep relation should be found between these two problems ;

ii) a more rigorous justification of our argument in favor of $\phi = \nu$ as absolute minimum, is called for. In some sense the situation is reminiscent of the proof of the labelling-gap theorem [10]. Therefore, we believe that homotopy considerations are probably behind this general result ;

iii) through this paper we have considered only the case of a uniform magnetic field. A natural question arises : what happens if the flux distribution is not so regular ? Partial answers in the case of a periodic flux are now available and will be published elsewhere.

Appendix.

In reference [17] a new type of differential calculus has been used. The purpose of this appendix is to recall the derivation rules, for the elements of the family $\{\mathcal{A}(\alpha)\}$ of rotation algebras. This allows us to manipulate algebraic quantities as a function of α ($\equiv \phi / \phi_0$ in the notation of the present paper). Two principal rules are of importance for us :

(1) For A and B in the universal rotation algebra :

$$\partial(AB) = \partial A \,.\, B + A \,.\, \partial B + \frac{i}{4\pi} \{\partial_1 A \,.\, \partial_2 B - \delta_2 A \,.\, \partial_1 B\} \tag{A1}$$

where $\partial = \partial/\partial\alpha$ and $\partial\mu = \partial/\partial k_\mu$. The last term in equation (A1) stems from the non-cummutative aspect of $\mathcal{A}(\alpha)$.

(2) If A^{-1} exists, then

$$\partial(A^{-1}) = -A^{-1}\left\{\partial A + \frac{i}{4\pi} (\partial_1 A \,.\, A^{-1} \,.\, \partial_2 A - \partial_2 A \,.\, A^{-1} \,.\, \partial_1 A)\right\} A^{-1}. \tag{A2}$$

These two formulas are the key point in proving important results, such as Streda's formula,

the Wilkinson-Rammal formula, etc. Here, we illustrate (A1) in the case : $A = B = P$, where P is an eigenprojector. Using $P^2 = P$, one deduces from (A1) :

$$\partial P \,.\, P + \frac{i}{4\pi} [\partial_1 P, \partial_2 P] = 0 \,.$$

Multiplying by P and taking the trace per unit volume leads to :

$$\tau(P \,.\, \partial P) = \frac{-i}{4\pi} \tau(P \,.\, C)$$

where $C \equiv [\partial_1 P, \partial_2 P]$ is the commutator of $\partial_1 P$ and $\partial_2 P$. A similar result holds for $Q = 1 - P$:

$$\tau((1-P)(-\partial P)) = \frac{-i}{4\pi} \tau((1-P)\,C) \,.$$

Then, using $\tau(C) = 0$, one deduces :

$$\tau(\partial P) = \partial \tau(P) = \frac{1}{2\pi i} \tau\{P[\partial_1 P, \partial_2 P]\} \,. \tag{A3}$$

Stated otherwise,

$$\frac{\partial}{\partial \alpha} \tau(P) = \frac{1}{2i\pi} \mathrm{Ch}\,(P) \tag{A4}$$

where Ch (P) is the Chern class of the projection P. Equation (A4) is the so-called Streda formula [13].

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