J. Phys. France **51** (1990) 2167-2185 1er OCTOBRE **1990,** PAGE 2167

Classification
Physics Abstracts
03.65 — 71.25 — 75.20

Spectrum of 2D Bloch electrons in a periodic magnetic field : algebraic approach

A. Barelli, J. Bellissard and R. Rammal (*)

Centre de Physique Théorique, CNRS Luminy, Case 907, 13288 Marseille Cedex 09, France

(*Received on May 2, 1990, accepted on June 13, 1990*)

Résumé. — Les méthodes algébriques que l'on a introduites récemment pour l'étude des électrons de Bloch sous champ magnétique uniforme sont généralisées au cas des champs périodiques. En utilisant une approche semi-classique, on étudie le cas où la maille magnétique est commensurable avec celle du réseau. En général et selon la valeur ϕ du flux magnétique moyen à travers la cellule élémentaire, deux cas distincts semblent se distinguer. Le premier cas est $\phi \neq 0$, où la structure en niveaux de Landau est retrouvée (cas non commutatif). Dans le second cas $\phi = 0$, on obtient une structure de bandes non triviale (cas commutatif). Nos résultats sont illustrés avec des exemples simples. En particulier on montre, sous certaines conditions, que le mécanisme de stabilisation de la mer de Fermi, avec un quantum de flux par fermion, se généralise au cas d'un champ magnétique périodique.

Abstract. — Algebraic methods recently introduced for 2D Bloch electrons in a uniform magnetic field are extended to the case of periodic magnetic fields. Using a semiclassical approach, we investigate the case where the magnetic unit cell is commensurate with the lattice unit cell. In general and according to the value ϕ of the average flux through the magnetic unit cell, two distinct cases take place. The first one corresponds to finite values of ϕ, where the usual structure of Landau levels is recovered (non commutative case). In the second case where $\phi = 0$, a non trivial band structure is obtained (commutative case). Our results are illustrated by simple examples. In particular we show that, under certain conditions, the mechanism of stabilization of the Fermi sea by the gaps (with one quantum flux per fermion) holds in the general case of periodic magnetic fields.

1. Introduction.

In view of the fascinating behavior of two dimensional (2D) electron systems in a strong magnetic field, it is natural to consider more general problems where the magnetic field is no longer uniform. In this paper, the third in a series, we consider the case of periodic magnetic fields acting on fermions moving on a discrete 2D lattice. The particular case of a uniform

(*) *Permanent address :* Centre de Recherches sur les Très Basses Températures, C.N.R.S., 25 avenue des Martyrs, BP 166X, 38042 Grenoble Cedex 05, France.

magnetic field has been the subject of various contributions during the past. Recent progress and the state of the art have been summarized in references [1, 2]. Part of our motivation for studying this general case comes from the recently developed methods in non commutative geometry to investigate some properties of 2D electrons in a uniform magnetic field [1-5]. Among various aspects, let us mention some basic results obtained so far by this approach : justification of Peierls substitution, new differential calculus which allowed the investigation of the fine structure of the energy spectrum, semiclassical methods for the calculation of the energy gap boundaries, rigorous proof of the Quantum Hall Effect, etc. On general physical grounds, some of these results are expected to survive in more complicated situations such as the periodic magnetic flux case. Another reason for working this kind of problem is the spectacular result obtained by Novikov *et al.* [4] for the free-lattice case, where the ground state has been fully characterized [6].

The questions we try to answer in this paper are two interrelated problems :

1) What are the appropriate algebraic tools to be used in the periodic flux case (generalized rotation algebra, etc.) ?

2) How much different is the energy spectrum in that case in comparison with the uniform flux case ?

The paper is organized around these two questions, and rather traditional aspects are left out. In section 2, we summarize some of the known results for the problem at hand and give a rather consistent set of notation and definitions. Section 3 is devoted to the treatment of two simple examples : the strip problem and the checkerboard lattice. There, we work out the algebraic formulation and the quantization procedure needed for semiclassical calculations. In section 4, the stripped flux lattice is considered in general and the fully general problem is worked out in section 5. Some applications of our results are given in section 6 : the increase of the energy levels by the magnetic distorsion, the stabilization of the Fermi sea by the gaps. Our concluding remarks are summarized in section 7.

2. Periodic magnetic fields.

The most important work on this class of problems is probably due to the soviet school. To our knowledge, the spectrum of 2D electrons in a periodic magnetic field $B(x, y)$ has been investigated first by Novikov *et al.* [6]. Attention has been focused on the lowest Landau level as well as its evolution under the action of a crystal potential considered as a perturbation. To be self-contained, a brief summary of this work and its extension is described in section 2.1. The general problem of a periodic crystal potential and a periodic magnetic field is defined in section 2.2. In this respect, the same notation as in references [3, 4] will be used.

2.1 FREE PARTICLES IN A PERIODIC MAGNETIC FIELD. — The problem considered in reference [6] is relative to a periodic magnetic field $B(x, y)$, doubly periodic and directed along the z-axis :

$$B(x + T_1, y) = B(x, y + T_2) = B(x, y) ;$$

$$B = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} ;$$

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} = 0 . \tag{2.1}$$

Here (A_x, A_y) is the vector potential, which can be chosen as :

$$A_x = -\frac{\partial F}{\partial y} ; \quad A_y = \frac{\partial F}{\partial x} \tag{2.2}$$

F being a solution of

$$(\partial_x^2 + \partial_y^2) F(x, y) = B(x, y) . \tag{2.3}$$

One of the main results of reference [6] is relative to the degeneracy of the lowest Landau level (LLL). In fact, it turns out that the addition of any doubly periodic increment to the homogeneous field leaves the LLL fully degenerate in spite of the loss of symmetry. Furthermore, the magnetic flux ϕ through the magnetic unit cell $T_1 T_2$ is the principal topological characteristic. In particular, for integral or rational flux, the wave function of the ground state can be obtained in terms of elliptic functions. Qualitative arguments have been used [6] to describe the magnetic band structure resulting from a weak potential perturbation, periodic with the same lattice.

These surprising results deserve some comments. First it is useful to recall that in the presence of a magnetic field, wave functions exhibit nodal points rather than nodal lines, the latter being a generic occurrence in systems with time reversal symmetry. The fact that the LLL wave functions are completely determined by the locations of their nodes is due to constraints of analyticity and periodicity [7]. Due to the double-periodicity of $B(x, y)$, the problem is equivalent to an electron on a torus and the number of nodes of the wave function is now finite and given by the (integral) number of quantum flux ϕ.

This description provides a totally general and gauge-independent characterization of any state of the LLL. In particular this gives an opportunity to study the boundary condition sensitivity in a rather unusual context. In fact if one defines a random potential over the torus, the sensitivity of the wave function to changes in boundary conditions can be probed by considering the « brainding » of the zeros. Such an idea has been exploited in reference [8] to probe the sensitivity of LLL wave functions in disorder-broadened bands to changes in boundary conditions. In particular a marked difference in behavior has been found between localized and extended states.

2.2 2D LATTICES AND PERIODIC MAGNETIC FIELD. — In this paper we investigate the tight-binding model, in a periodic magnetic field. More precisely, we are interested in the eigenvalue problem :

$$\varepsilon \psi(\mathbf{r}) = \sum_{r' \neq r} t \exp(i \gamma_{rr'}) \psi(\mathbf{r}') \tag{2.4}$$

describing the hopping of electrons between nearest-neighbouring sites ($\mathbf{r}$) on a lattice under a magnetic field. Here $\psi(\mathbf{r})$ denotes the wave function amplitude at site $\mathbf{r}$ and t is the hopping matrix element.

Without loss of generality, the energy scale t is fixed at $t = 1$. In equation (2.4), the phase factor $\gamma_{\mathbf{r}\mathbf{r}'}$ is

$$\gamma_{\mathbf{r}\mathbf{r}'} = \frac{2\pi}{\phi_0} \int_r^{r'} \mathbf{ds} \cdot \mathbf{A}(\mathbf{s}) \tag{2.5}$$

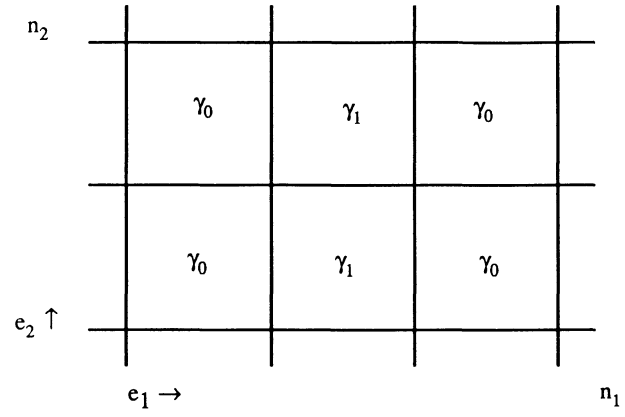
and describes the net effect of the vector potential $\mathbf{A}(\mathbf{s})$ on the hopping matrix elements.

For the sake of simplicity, we will consider the square lattice case. Extensions to other lattices do not offer a major difficulty.

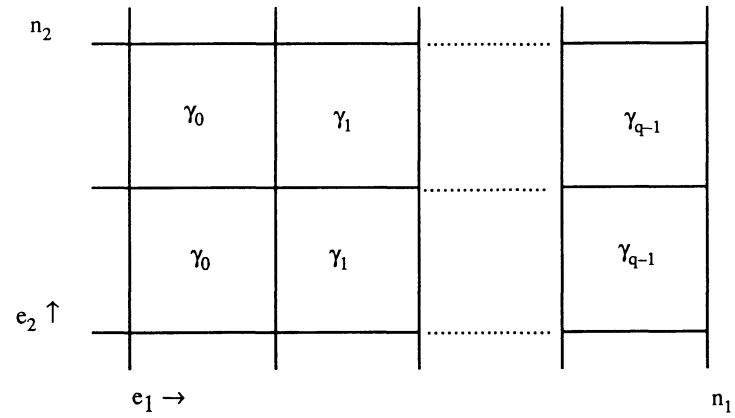
The periodic magnetic field is specified by a magnetic unit cell $q_1 \times q_2$, ($q_1 \geq 1$, $q_2 \geq 1$) corresponding to a pattern of $q_1 q_2$ magnetic fluxes $\phi_{\sigma_1 \sigma_2}$ with $1 \leq \sigma_1 \leq q_1$, $1 \leq \sigma_2 \leq q_2$.

As will be shown below, this is all the information we need in order to solve equation (2.4).

Using the notation of our previous papers, we have $\gamma_{\sigma_1 \sigma_2} = \frac{2\pi}{\phi_0} \phi_{\sigma_1 \sigma_2}$ for the reduced fluxes. Examples of periodic patterns are given in figures 1, 2. For example, $q_1 = q_2 = 1$, describes the uniform field case ; $q_2 = 1$ and arbitrary q_1 corresponds to the stripped lattice, etc.



(a)



(b)

Fig. 1.— Notation used in the text for the magnetic fluxes : (a) simple stripped case with $q = 2$, (b) generalized stripped case.

The main purpose of the next sections is to provide an algebraic formalism to handle the present problem. Therefore well known properties such as : invariance under $\phi_{\sigma_1 \sigma_2} \to \phi_{\sigma_1 \sigma_2} + \phi_0$; $\phi_{\sigma_1 \sigma_2} \to -\phi_{\sigma_1 \sigma_2}$, etc., will be left out. Also, special properties of the spectrum (opening and closure of the gaps, nesting, hierarchical structure, gap labelling, etc.) will not be addressed. The reason is simply that many of these properties, if they survive in the periodic flux case, are natural consequences of the algebraic approach. Of course the same remark holds for properties specific to the rational flux, commensurate effects, etc.

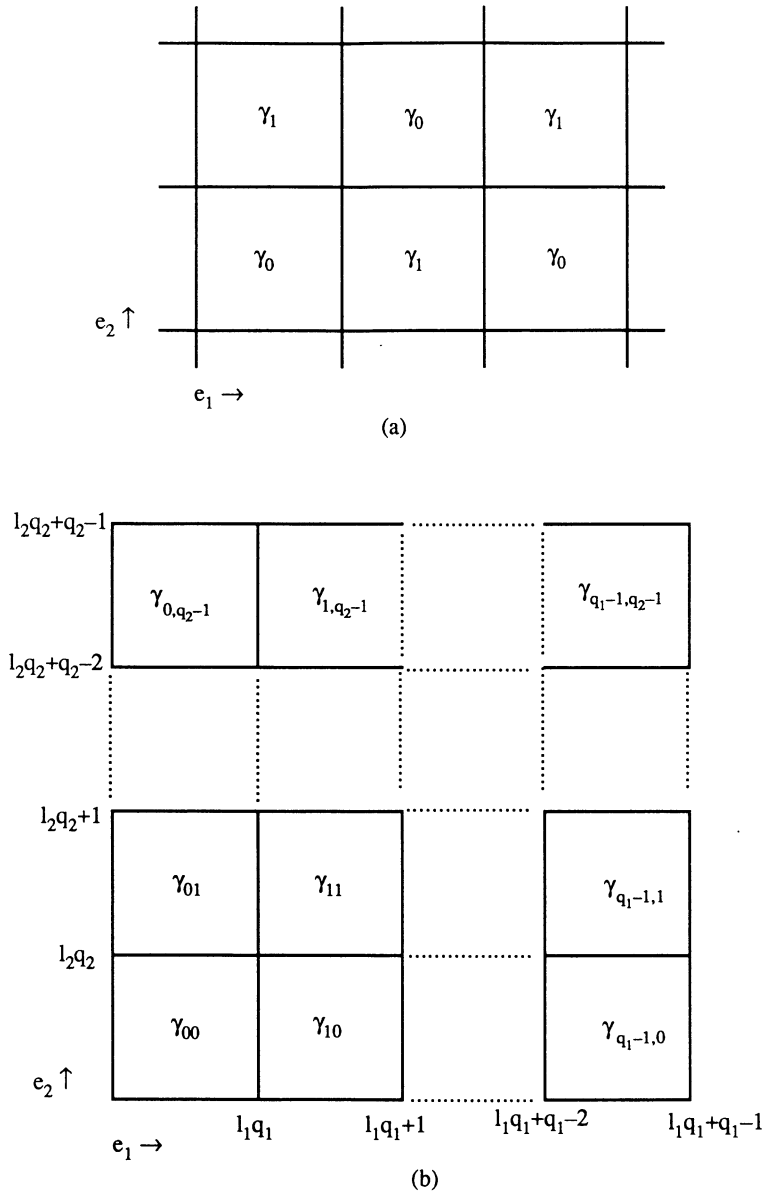


Fig. 2.— (a) simple checkerboard lattice, (b) generalized case for arbitrary q_1 and q_2.

It is important to notice that the problem worked here is somewhat different from the free-lattice case. In this respect the tight-binding model must be viewed as the infinite coupling limit of a lattice potential. Accordingly we limit our calculations below to the weak field limit, where much can already be learnt. Specific extensions to other limits (e.a. rational flux, etc.) can be investigated using the same machinary that we have developed in the uniform magnetic case [3, 4].

Before going to examples let us specify some of our notations.

We consider the lattice of figure 2b.

The wave function will be written as a vector value wave function in the following way :

$$\Psi_{r_1 r_2}(\ell_1, \ell_2) = \psi(q_1 \ell_1 + r_1, q_2 \ell_2 + r_2) \quad 0 \leqslant r_i \leqslant q_i - 1 \quad \ell_1 \in \mathbb{Z} . \tag{2.6}$$

Each component $\Psi_{r_1 r_2}$ can be interpreted as a wave function on a renormalized lattice whose elementary plaquettes are given by the unit cell of periods with size q_1 and q_2.

We can now rewrite the Schrödinger equation (2.4) as a matrix equation for the components of Ψ. This will be given in many details in the next sections.

However we remark that the Schrödinger equation (2.4) is easily expressed in term of the magnetic translations U_1 and U_2 [9]

$$\begin{aligned}
[U_1 \psi](q_1 \ell_1 + r_1, q_2 \ell_2 + r_2) &= \mathrm{e}^{iA_{r_1 r_2}(\ell_2)} \psi(q_1 \ell_1 + r_1 - 1, q_2 \ell_2 + r_2) \\
[U_2 \psi](q_1 \ell_1 + r_1, q_2 \ell_2 + r_2) &= \mathrm{e}^{iA_{r_1 r_2}(\ell_1)} \psi(q_1 \ell_1 + r_1, q_2 \ell_2 + r_2 - 1) .
\end{aligned} \tag{2.7}$$

Indeed equation (2.4) for a square lattice can be rewritten as :

$$(U_1 + U_1^* + U_2 + U_2^*)\, \psi = E\psi . \tag{2.8}$$

For other lattices, similar expressions are found involving only polynomials with respect to U_1 and U_2.

Let us give the expression of these two operators in terms of the vector Ψ. To do so we introduce the ordinary translation operators T_1 and T_2 acting on wave functions as follows :

$$\begin{aligned}
(T_1 \psi)(n_1, n_2) &= \psi(n_1 - 1, n_2) \\
(T_2 \psi)(n_1, n_2) &= \psi(n_1, n_2 - 1)
\end{aligned} \tag{2.9}$$

then we get :

$$\begin{aligned}
[U_1 \Psi]_{\mathbf{r}}(\mathbf{l}) &= \mathrm{e}^{iA_{\mathbf{r}}(\ell_2)} \Psi_{\mathbf{r} - \mathbf{e}_1}(\mathbf{l}) && r_1 = 1, 2, \ldots, q_1 - 1 \\
[U_1 \Psi]_{0, r_2}(\mathbf{l}) &= \mathrm{e}^{iA_{0, r_2}(\ell_2)} T_1 \Psi_{q_1 - 1, r_2}(\mathbf{l}) && r_1 = 0 \\
[U_2 \Psi]_{\mathbf{r}}(\mathbf{l}) &= \mathrm{e}^{iA_{\mathbf{r}}(\ell_1)} \Psi_{\mathbf{r} - \mathbf{e}_2}(\mathbf{l}) && r_2 = 1, 2, \ldots, q_2 - 1 \\
[U_2 \Psi]_{r_1, 0}(\mathbf{l}) &= \mathrm{e}^{iA_{r_2, 0}(\ell_2)} T_2 \Psi_{r_1, q_2 - 1}(\mathbf{l}) && r_2 = 0 .
\end{aligned} \tag{2.10}$$

In the coming sections we will compute various gauge transformations by simplifying the expression of the Schrödinger operator suitable for semiclassical expansion.

We also use finite Fourier transform :

$$\langle \gamma^{\ell} \rangle = \frac{1}{q_1 q_2} \sum_{r_1 = 1}^{q_1} \sum_{r_2 = 1}^{q_2} \gamma^{\ell}_{r_1 r_2}$$

$$\tilde{\gamma}(s_1, s_2) = \frac{1}{q_1 q_2} \sum_{r_1 = 1}^{q_1} \sum_{r_2 = 1}^{q_2} \gamma^{\ell}_{r_1 r_2} \exp\left[\frac{2i\pi}{q_1} r_1 s_1 + \frac{2i\pi}{q_2} r_2 s_2 \right] . \tag{2.11}$$

3. Two examples.

Before considering the general situation with arbitrary magnetic patterns, we will discuss first two specific examples. The first one is the simplest stripped flux lattice where two values of γ are allowed : γ_0 and γ_1. The next section will be devoted to extend this case to arbitrary

$q \times 1$ patterns. The second example is the so-called checkerboard or staggered flux lattice. This is actually a special case of 2×2 patterns, which is generalized in section 5.

3.1 STRIPPED FLUX LATTICE.

3.1.1 *Algebraic formulation.* — The notation used is depicted in figure 1a.

Using formulae (2.10) we find immediately :

$$U_1 = \begin{pmatrix} 0 & e^{i\hat{n}_2 \gamma_0} T_1 \\ e^{i\hat{n}_2 \gamma_1} & 0 \end{pmatrix} \qquad U_2 = \begin{pmatrix} T_2 & 0 \\ 0 & T_2 \end{pmatrix} \tag{3.1}$$

where $\hat{n}_i$ $i = 1, 2$ are the lattice position operators.

By the following gauge transformation

$$\mathcal{R} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\hat{n}_2 \gamma_1} \end{pmatrix} \tag{3.2}$$

$$U_1' = \mathcal{R} U_1 \mathcal{R}^{-1} = \begin{pmatrix} 0 & e^{i\hat{n}_2(\gamma_0 + \gamma_1)} T_1 \\ 1 & 0 \end{pmatrix},$$

$$U_2' = \mathcal{R} U_2 \mathcal{R}^{-1} = \begin{pmatrix} T_2 & 0 \\ 0 & e^{-i\gamma_1} T_2 \end{pmatrix} \tag{3.3}$$

which allows us to see that, as far as the algebra is concerned, all operators of interest are expressed as matrices with elements given by polynomials with respect to U and V where :

$$U = e^{i\hat{n}_2(\gamma_0 + \gamma_1)} T_1$$
$$V = T_2 . \tag{3.4}$$

They are a pair of unitary operators satisfying

$$UV = e^{i2\gamma} VU \qquad 2\gamma = \gamma_0 + \gamma_1 . \tag{3.5}$$

As shown in the previous paper [2] we can realize U and V by means of two self-adjoint operators K_1 and K_2 with :

$$[K_1, K_2] = i , \quad U = e^{-2i\gamma^{1/2} K_1} , \quad V = e^{i\gamma^{1/2} K_2} . \tag{3.6}$$

Let us introduce the new translation operator :

$$u_r = e^{-i\gamma_r / \gamma^{1/2} K_1} \qquad r = 0,1 \tag{3.7}$$

thanks to the new non commutative gauge transformation given by :

$$\mathcal{R}' = \begin{pmatrix} 1 & 0 \\ 0 & u_1 \end{pmatrix} \tag{3.8}$$

the magnetic translation operators are transformed into :

$$U_1'' = \mathcal{R}' U_1' \mathcal{R}'^{-1} = \begin{pmatrix} 0 & u_0 \\ u_1 & 0 \end{pmatrix}$$

$$U_2'' = \mathcal{R}' U_2' \mathcal{R}'^{-1} = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} . \tag{3.9}$$

3.1.2 *Quantization and semi-classical calculation.* — Using the above notation, the Hamiltonian $\mathcal{H}$ can be expressed as :

$$-\mathcal{H} = 4 - (U_1'' + U_1''^* + U_2'' + U_2''^*) = 4 - H \tag{3.10}$$

and in matrix representation H is :

$$H = \begin{pmatrix} V + V^* & u_1 + u_0^* \\ u_0 + u_1^* & V + V^* \end{pmatrix} . \tag{3.11}$$

Classically, $\gamma_0 = \gamma_1 = 0$, the energy spectrum of H is easily calculated as : $2\ (\cos k_2 \pm \cos k_1)$. The quantized version of H is given by : $V = \mathrm{e}^{i\gamma^{1/2} K_2}$, $U_r = \mathrm{e}^{-i\gamma_r K_1/\gamma^{1/2}}$; $r = 1, 2$. Here we use Weyl's quantization scheme, with two operators K_1 and K_2 satisfying $[K_1, K_2] = i$.

Within this picture, H can be decomposed into a sum of two terms :

$$H = 2 \cos\ (\gamma^{1/2} K_2)\ \mathbb{1} + H_1 \tag{3.12}$$

where $\mathbb{1}$ denotes the 2×2 identity matrix and H_1 is only function of K_1. It turns out that H_1 is a Jacobi-like matrix (tri-diagonal) and a discrete Fourier transform is helpful to perform practical calculations. In what follows, we outline these calculations in some detail. For a given set $\{\psi(r)\}$, $0 \leq r \leq q - 1$, we define the discrete Fourier transform by :

$$\begin{aligned} \hat{\psi}(s) &= \frac{1}{q^{1/2}} \sum_{r=0}^{q-1} \omega^{rs} \psi(r) , \quad \omega = \exp\ (2i\pi/q) \\ \psi(r) &= \frac{1}{q^{1/2}} \sum_{t=0}^{q-1} \omega^{-rt} \hat{\psi}(t) . \end{aligned} \tag{3.13}$$

Under this unitary transformation H_1 becomes :

$$\hat{H}_1 = \frac{1}{2} \begin{pmatrix} -A - A^* & -A + A^* \\ -A + A^* & A + A^* \end{pmatrix} \tag{3.14}$$

where $A = u_1 + u_0^* = \exp\left(\frac{-i\gamma_1}{\gamma^{1/2}} K_1\right) + \exp\left(\frac{i\gamma_0}{\gamma^{1/2}} K_1\right)$, A^* being the hermitian conjugate of A.

With this formulation $\hat{H} = \hat{H}_1 + 2 \cos\ (\gamma^{1/2} K_2)$. $\mathbb{1}$ is easy to manipulate and particularly for semiclassical calculations. In what follows, we limite our discussion to the lower edge of the energy spectrum. More precisely we shall calculate the flux dependence of the Landau energy levels of $\hat{H}$ close to $E = 0$. For this, we use the same procedure as for the uniform field case [3]. Our starting point is Schur's formula which can be written as :

$$\hat{H}_{\text{eff}} = P\hat{H}P + P\hat{H}Q \frac{1}{E - Q\hat{H}Q} Q\hat{H}P , \tag{3.15}$$

here P and Q are the eigenprojectors,

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

over the subspaces spanned by $\psi(0)$ and $\psi(1)$ respectively. The resulting effective hamiltonian $\hat{H}_{\text{eff}}$ is simply a 1×1 matrix and depends explicitly on the unknown energy E.

The semiclassical result is obtained by expanding $\hat{H}_{\text{eff}}$ at low γ_0 and γ_1. Simple calculations lead to :

$$\hat{H}_{\text{eff}} = \gamma (K_1^2 + K_2^2) + O(\gamma^2) \tag{3.16}$$

for first order in γ and this is nothing else than the simple harmonic oscillator. The quantization of $\hat{H}_{\text{eff}}$ gives : $E_n = \gamma (2n + 1)$. The second order result is obtained through a simple iteration of equation (3.15), with $E = E_n$ as an input value. The final result reads as :

$$\begin{aligned}\hat{H}_{\text{eff}} = {} & \gamma (K_1^2 + K_2^2) + \frac{1}{2} (K_1^4 + K_2^4) \times \\ & \times \left[\frac{\gamma^2}{12} + \frac{\gamma_0^4 + \gamma_1^4}{24\,\gamma^2} + \frac{1}{12\,\gamma^2} (-\gamma_0^4 - \gamma_1^4 + \gamma_0\,\gamma_1^3 + \gamma_1\,\gamma_0^3) + \frac{\gamma_0^2 + \gamma_1^2}{32\,\gamma^2} (\gamma_0 - \gamma_1)^2 \right] \\ & - \frac{1}{16} (K_1 K_2^2 K_1)(\gamma_0 - \gamma_1)^2 + \frac{E_0}{16} K_2^2 (\gamma_0 - \gamma_1)^2 .\end{aligned} \tag{3.17}$$

Using the known results :

$$\begin{aligned}\langle n | K_r^4 | n \rangle &= \frac{3}{8} [1 + (2n + 1)^2] \\ \langle n | K_1 K_2^2 K_1 | n \rangle &= \frac{1}{8} [5 + (2n + 1)^2] \\ \langle n | K_r^2 | n \rangle &= \frac{1}{2} (2n + 1) ,\end{aligned}$$

we obtain the energies :

$$E_n = \gamma (2n + 1) - \frac{\gamma^2}{16} [1 + (2n + 1)^2] + \frac{1}{16} (\gamma_0 - \gamma_1)^2 + O(\gamma_r^4) . \tag{3.18}$$

The physical meaning of this simple result will be discussed in section 6.1. Notice that E_n is invariant under the permutation $\gamma_0 \leftrightarrow \gamma_1$ and reduces to the known result for $\gamma_0 = \gamma_1$.

3.2 CHECKERBOARD FLUX LATTICE. — The notation is depicted in figure 2a. Proceeding in a similar fashion, the magnetic translation operators U_1 and U_2 (after gauge transformations) can be written as :

$$U_1'' = \begin{pmatrix} 0 & U \\ U & 0 \end{pmatrix}, \qquad U_2'' = \begin{pmatrix} 0 & e^{-i\gamma_0} V \\ e^{-i\gamma_1} V & 0 \end{pmatrix} \tag{3.19}$$

and the corresponding Hamiltonian is given by :

$$-\mathcal{H} = 4 - (U_1'' + U_1''^* + U_2'' + U_2''^*) = 4 - H . \tag{3.20}$$

Equation (3.20) is actually a starting point for performing semiclassical calculations as will be shown below.

We use the quantization rules : $U = \exp(i(\gamma)^{1/2} K_1)$, $V = \exp(i(\gamma)^{1/2} K_2)$ in the Hamiltonian expression :

$$H = \begin{pmatrix} 0 & U + U^* + e^{-i\gamma_0} V + e^{i\gamma_1} V^* \\ U + U^* + e^{i\gamma_0} V^* + e^{-i\gamma_1} V & 0 \end{pmatrix} . \tag{3.21}$$

At low fields, we expand as usual and use Schur's formula, with the following spectral projectors : $P = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $Q = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. The effective Hamiltonian H_{eff} is then obtained as :

$$H_{\text{eff}} = \gamma (K_1^2 + K_2^2) - \frac{\gamma^2}{12}(K_1^4 + K_2^4) + \frac{\gamma^3}{360}(K_1^6 + K_2^6) + \frac{\gamma'^2}{8} - \frac{\gamma\gamma'^2}{8} K_2^2 - \\ - \frac{\gamma\gamma'^2}{8}(K_1^2 - 7K_2^2 + \tilde{E}) \qquad (3.22)$$

with

$$\tilde{E} = (2n+1), \qquad 2\gamma' = \gamma_0 - \gamma_1 .$$

This leads finally to the energies ($n \geq 0$) :

$$E_n = \gamma(2n+1) - \frac{\gamma^2}{16}[1 + (2n+1)^2] + \frac{\gamma^3}{192}[n^3 + (n+1)^3] + \frac{\gamma'^2}{8} - \frac{\gamma\gamma'^2}{32}(2n+1). \qquad (3.23)$$

up to order $O(\gamma_r^3)$.

4. General strip case.

In this section, we generalize the strip case with q values γ_r of the flux (see Fig. 1b for the notation).

4.1 FORMULATION. — As before the analysis sketched in section 2 we get after gauge transformations the magnetic translation operators U_1 and U_2 in the form :

$$U_1'' = \begin{pmatrix} 0 & 0 & & u_q \\ u_1 & & & \\ 0 & u_2 & & \\ 0 & & u_{q-1} & 0 \end{pmatrix}, \qquad U_2'' = \begin{pmatrix} V & & & \\ & V & & \\ & & V & \\ & & & \ddots \\ & & & & V \end{pmatrix} \qquad (4.1)$$

with

$$u_r = \mathrm{e}^{-i\gamma_r/\gamma^{1/2}K_1}, \quad V = \mathrm{e}^{i\gamma^{1/2}} \quad \text{and} \quad \gamma = \frac{1}{q}\sum_{r=1}^{q} \gamma_r \; (\gamma_{r+q} = \gamma_r) . \qquad (4.2)$$

As for $q = 2$ the Hamiltonian can be written as :

$$\mathcal{H} = 2\cos\gamma^{1/2}K_2 \, \mathbb{1} + H_1 \qquad (4.3)$$

$$H_1 = \begin{pmatrix} 0 & u_1^* & & u_q \\ u_1 & 0 & & \\ & u_2 & & \\ & & & u_{q-1}^* \\ u_q^* & & u_{q-1} & 0 \end{pmatrix} . \qquad (4.4)$$

The operator H_1 is tri-diagonal, so it is useful to use a discrete Fourier transform starting from the action of H_1 on the wave function components $\psi(r)$:

$$(H_1 \psi)(r) = u_{r+1}^* \psi(r+1) + u_r \psi(r-1), \quad u_{r+q} = u_r \; (r = 0, \ldots, q-1) \qquad (4.5)$$

we define :

$$\hat{\psi}(s) = \frac{1}{q^{1/2}} \sum_{s=0}^{q-1} \omega^{rs} \psi(r) ,$$

$$\psi(r) = \frac{1}{q^{1/2}} \sum_{t=0}^{q-1} \omega^{-rt} \hat{\psi}(t) , \quad \omega \equiv \exp(2i\pi/q) . \tag{4.6}$$

This gives in particular

$$(\hat{H}_1 \psi)(s) = \sum_t \hat{H}_1 \hat{\psi}(t) \tag{4.7}$$

where we have defined :

$$\hat{H}_1(s, t) = \omega^{(t-s)/2} \frac{1}{q} \sum_{r=0}^{q-1} \omega^{r(s-t)} (u_r \omega^{(s+t)/2} + u_r^* \omega^{-(s+t)/2}) . \tag{4.8}$$

This allow us to have simple matrix elements for $\hat{H}$:

$$\langle s|\hat{H}|t\rangle = 2\cos\gamma^{1/2} K_2 . \delta_{st} + \frac{\omega^{(t-s)/2}}{q} \sum_{r=0}^{q-1} \omega^{r(s-t)} (u_r \omega^{(s+t)/2} + u_r^* \omega^{-(s+t)/2}) . \tag{4.9}$$

4.2 QUANTIZATION AND SEMI-CLASSICAL EXPANSION. — As quantization rules we use $V = \exp(i\gamma^{1/2} K_2)$, $u_r = \exp\left(i \frac{\gamma_r}{\gamma^{1/2}} K_1\right)$; $[K_1, K_2] = i$. For instance, we have

$$\hat{H}_1(0, 0) = \frac{1}{q} \sum_{r=0}^{q-1} (u_r + u_r^*) = \frac{1}{q} \sum_{r=0}^{q-1} 2\cos\left(\frac{\gamma_r}{\gamma^{1/2}} K_1\right) . \tag{4.10}$$

Now the expansion close to $\gamma_r = 0$ can be performed as before. Up to second order (in γ_r), the relevant matrix elements are obtained as :

$$\langle 0|\hat{H}|0\rangle = 2 - \gamma K_2^2 + \gamma^2 \frac{K_2^4}{12} + 2 - \frac{K_1^2}{\gamma} \left(\frac{1}{q} \sum_r \gamma_r^2\right) + \frac{K_1^4}{12\gamma^2} \frac{1}{q} \sum \gamma_r^4 + O(\gamma_r^4) , \tag{4.11}$$

$$\langle s|\hat{H}|0\rangle = -\frac{K_1}{\gamma^{1/2}} \omega^{-5/2} \frac{1}{q} \sum_r \gamma_r \omega^{rs} 2\sin\frac{\pi s}{q} - \frac{K_1^2}{\gamma} \omega^{-5/2} \frac{1}{q} \sum_r \gamma_r^2 \omega^{rs} \cos\frac{\pi s}{q}$$
$$+ \frac{K_1^3}{\gamma^{3/2}} \frac{\omega^{-5/2}}{3!} \frac{1}{q} \sum_r \gamma_r^3 \omega^{rs} 2\sin\frac{\pi s}{q} + O(\gamma_r^4) \quad (s \neq 0) \tag{4.12}$$

and

$$\langle s|\hat{H}|t\rangle = \left(2 - \gamma K_2^2 + 2\cos\frac{2\pi s}{q}\right) \delta_{st} - 2\sin\frac{\pi(s-t)}{q} \omega^{(t-s)/2} \frac{1}{q} \sum_r \gamma_r \omega^{r(s-t)} \frac{K_1}{\gamma^{1/2}}$$
$$- \frac{K_1^2}{\gamma} \omega^{(t-s)/2} \cos\frac{\pi(s+t)}{q} \frac{1}{q} \sum_r \gamma_r^2 \omega^{r(s-t)} + O(\gamma^{3/2}) \qquad (s \neq 0, t \neq 0) . \tag{4.13}$$

Using now Schur's formula, one obtains the expansion of the effective Hamiltonian in powers of $\{\gamma_r\}$. It is convenient to use the following notation :

$$\tilde{\gamma}^{\ell}(s) = \frac{1}{q} \sum_{r=1}^{q} \gamma_r^{\ell} e^{2i\pi rs/q}, \quad \text{where } \ell \text{ is an integer .} \tag{4.14}$$

To first order in γ_r one obtains :

$$H_{\text{eff}}(E) = \left[4 - \gamma K_2^2 - \frac{K_1^2}{\gamma}\left(\frac{1}{q}\sum_r \gamma_r^2\right)\right] + \frac{K_1^2}{\gamma}\sum_s |\tilde{\gamma}(s)|^2 \frac{4\sin^2 \pi s/q}{E - 2(1 + \cos 2\pi s/q)} \tag{4.15}$$

This expression can be reduced to a more simple form :

$$H_{\text{eff}} = 4 - \gamma (K_1^2 + K_2^2) + O(\gamma^{3/2}) \tag{4.16}$$

and this leads to the energy levels :

$$E_n = 4 - \gamma (2n + 1) + O(\gamma^{3/2}) . \tag{4.17}$$

The next order calculations are more involving. After a tedious algebra, one ends up with :

$$E_n = 4 - \gamma (2n + 1) + \frac{\gamma^2}{16}[1 + (2n + 1)^2] + \frac{3}{8}[1 + (2n + 1)^2]\, S - \sum_{s=1}^{q-1} \frac{|\tilde{\gamma}(s)|^2}{4\sin^2 \pi s/q} . \tag{4.18}$$

Here S is a rather complicated expression, given by a sum of 15 different terms each involving complicated sums over the moments of the $\{\gamma_r\}$ distribution. The calculations are too lenghty and tedious to be reproduced here. Careful examination shows that S vanishes for every pattern $\{\gamma_r\}$. Thus, the net result can simply be written as :

$$E_n = 4 - \gamma (2n + 1) + \frac{\gamma^2}{16}[1 + (2n + 1)^2] - \delta \tag{4.19}$$

where δ is a positive quantity given by :

$$\delta = \sum_{s=1}^{q-1} \frac{|\tilde{\gamma}(s)|^2}{2 - 2\cos(2\pi s/q)} . \tag{4.20}$$

The meaning of δ is very simple. $\tilde{\gamma}(s)$ is somehow a « magnetic form factor », describing the distorsion of the magnetic flux pattern, and vanishes in the uniform flux case $\gamma_r = \gamma$ ($r = 0, 1, ..., q - 1$). The denominator in the expression of δ is the spacing energy between the ground state ($s = 0$) and the excited energy levels ($s \geq 1$) of a discrete 1D tight-binding chain of length q. This form of δ allows for the prediction of its expression in the general case, worked out in the next section.

5. The general case.

This section will be devoted to the calculation of various gauge transformations necessary to simplify the matrix form of the magnetic translations U_1 and U_2.

We choose here a Landau gauge where $A_2 = 0$. Then the potential vector can be written as follows :

$$A_{r_1 r_2}(\ell_2) = \frac{2\pi}{\phi_0} \int_{n'}^{n} \mathrm{ds}\, \mathbf{A}(\mathbf{s}) \tag{5.1}$$

where $n = (\ell_1 q_1 + r_1, \ell_2 q_2 + r_2)$ and $n' = (\ell_1 q_1 + r_1 - 1, \ell_2 q_2 + r_2)$

$$A_{\mathbf{r}_1 \mathbf{r}_2}(\ell_2) = \sum_{s_2 = 0}^{r_2} \gamma_{r_1, s_2} + \ell_2 \sum_{s_2 = 0}^{q_2 - 1} \gamma_{r_1, s_2} . \tag{5.2}$$

The expression for U_1 and U_2 becomes (2.10) :

$$[U_1 \Psi]_{\mathbf{r}}(\mathbf{l}) = \mathrm{e}^{iA_{\mathbf{r}}(\ell_2)} \Psi_{\mathbf{r} - \mathbf{e}_1}(\mathbf{l}) ,$$

$$r_1 = 1, 2, ..., q_1 - 1$$

$$[U_1 \Psi]_{0, r_2}(\mathbf{l}) = \mathrm{e}^{iA_{0, r_2}(\ell_2)} T_1 \Psi_{q_1 - 1, r_2}(\mathbf{l}) ,$$

$$r_1 = 0$$

$$[U_2 \Psi]_{\mathbf{r}}(\mathbf{l}) = \mathrm{e}^{iA_{\mathbf{r}}(\ell_1)} \Psi_{\mathbf{r} - \mathbf{e}_2}(\mathbf{l}) ,$$

$$r_2 = 1, 2, ..., q_2 - 1$$

$$[U_2 \Psi]_{r_1, 0}(\mathbf{l}) = \mathrm{e}^{iA_{r_2, 0}(\ell_2)} T_2 \Psi_{r_1, q_2 - 1}(\mathbf{l}) ,$$

$$r_2 = 0 .$$

We first consider the following gauge transformation $\mathcal{R}$ given by the $q_1 q_2 \times q_1 q_2$ matrix :

$$\mathcal{R} = ((\mathcal{R}_{\mathbf{r}_1 \mathbf{r}_2}(\hat{n}_1, \hat{n}_2) \, \delta_{r_1, r_1'} \, \delta_{r_2, r_2'})) . \tag{5.3}$$

Here $0 \leqslant r_i \leqslant q_i - 1$ $(i = 1, 2)$ and $\hat{n}_i$ denote the lattice position operators. Moreover :

$$\mathcal{R}_{\mathbf{r}_1 \mathbf{r}_2}(\ell_1, \ell_2) = \exp\left[- i \sum_{s_1 = 0}^{r_1} \left\{ \gamma_{s_1, 0} + \ell_2 \sum_{s_2 = 0}^{q_2 - 1} \gamma_{s_1, s_2} \right\} \right] . \tag{5.4}$$

This matrix is computed in such a way that the new magnetic translations $U_i' = \mathcal{R} U_i \, \mathcal{R}^{-1}$ be given by :

$$\begin{aligned} (U_1)_{\mathbf{r}\mathbf{r}'} &= \delta_{r_2, r_2'} \, \delta_{r_1, r_1' + 1} \left\{ (1 - \delta_{r_1, 0}) + \delta_{r_1, 0} \, \Lambda_{r_2} \, U \right\} \\ (U_2)_{\mathbf{r}\mathbf{r}'} &= \delta_{r_1, r_1'} \, \delta_{r_2, r_2' + 1} \left\{ (1 - \delta_{r_2, 0}) + \delta_{r_2, 0} \, \Lambda^{r_1} \, V \right\} \end{aligned} \tag{5.5}$$

with

$$\gamma = \frac{1}{q_1 q_2} \sum_{r_i = 0}^{q_i} \gamma_{r_1, r_2} , \qquad \Lambda_{r_2} = \exp\left(i \sum_{s_1 = 0}^{q_1 - 1} \sum_{s_2 = 0}^{r_2} \gamma_{s_1, s_2} \right) .$$

$$\Lambda^{r_1} = \exp\left(- i \sum_{s_1 = 0}^{r_1} \sum_{s_2 = 0}^{q_2 - 1} \gamma_{s_1, s_2} \right) .$$

This allows us to see that every operator of interest can be actually written as a $q_1 q_2$ matrix with coefficients depending polynomially upon the unitary operators U and V which satisfy :

$$UV = \mathrm{e}^{iq_1 q_2 \gamma} \, VU . \tag{5.6}$$

So, as before we will represent U and V in terms of operators K_1 and K_2 as follows :

$$U = e^{-iq_1 \gamma^{1/2} K_1} \qquad V = e^{iq_2 \gamma^{1/2} K^2}. \tag{5.7}$$

Introducing the operators :

$$u_{r_1 r_2} = e^{-i\alpha(r_1) K_1}, \qquad v_{r_1 r_2} = e^{i\beta(r_2) K_2} \tag{5.8}$$

$$\alpha(r_1) = \frac{1}{q_2 \gamma^{1/2}} \sum_{s_2=0}^{q_2-1} \gamma_{r_1, s_2}, \qquad \beta(r_2) = \frac{1}{q_1 \gamma^{1/2}} \sum_{s_1=0}^{q_1-1} \gamma_{s_1, r_2}$$

and the following non commutative gauge transformation $\mathcal{R}'$:

$$\mathcal{R}' = ((\mathcal{R}'_{r_1 r_2} \delta_{r_1, r'_1} \delta_{r_2, r'_2})) \tag{5.9}$$

with $$\mathcal{R}'_{r_1 r_2} = \exp(i\omega^{(1)}_{r_1 r_2} K_2 - \omega^{(2)}_{r_1 r_2} K_1)$$

and $$\omega^{(1)}_{r_1 r_2} = \sum_{s_1=0}^{q_1-1} \sum_{s_2=0}^{r_2} \gamma_{s_1, s_2}, \qquad \omega^{(2)}_{r_1 r_2} = \sum_{s_1=0}^{r_1} \sum_{s_2=0}^{q_2-1} \gamma_{s_1, s_2}.$$

We get :

$$\begin{aligned} U''_1 &= \mathcal{R}' U'_1 \mathcal{R}'^{-1} = ((u_{r_1 r_2} e^{i\psi_1(r_1, r_2)} \delta_{r_1, r'_1 + 1} \delta_{r_2, r'_2})) \\ U''_2 &= \mathcal{R}' U'_2 \mathcal{R}'^{-1} = ((v_{r_1 r_2} e^{i\psi_2(r_1, r_2)} \delta_{r_1, r'_1} \delta_{r_2, r'_2 + 1})) \end{aligned} \tag{5.10}$$

with :

$$\begin{aligned} \psi_1(r_1, r_2) &= -\frac{\alpha(r_1)}{2} \sum_{t=0}^{r_2} \beta(t) && 1 \leqslant r_1 \leqslant q_1 - 1 \\ \psi_1(0, r_2) &= \frac{\alpha - \alpha(0)}{2} \sum_{t=0}^{r_2} \beta(t) \\ \psi_2(r_1, r_2) &= \frac{\beta(r_2)}{2} \sum_{s=0}^{r_1} \alpha(s) && 1 \leqslant r_2 \leqslant q_2 - 1 \\ \psi_2(r_1, 0) &= \frac{\beta(0) - \beta}{2} \sum_{s=0}^{r_1} \alpha(s), && \alpha = \beta = \gamma^{1/2}. \end{aligned} \tag{5.11}$$

Using this transformation we can perform the semiclassical expansion in much the same way as in sections 3 and 4. The calculation is rather cumbersome and we postpone it to a forth coming paper [10]. Nevertheless we conjecture that the Landau levels for the Hamiltonian :

$$H = U''_1 + U''^*_1 + U''_2 + U''^*_2 \tag{5.12}$$

are given by the following formula :

$$E_n = 4 - \langle \gamma \rangle (2n + 1) + \frac{\langle \gamma \rangle^2}{16} [1 + (2n + 1)^2] - \delta \tag{5.13}$$

where

$$\delta = \sum_{s_1=0}^{q_1-1} \sum_{s_2=0}^{q_2-1} \frac{|\tilde{\gamma}(s_1, s_2)|^2}{4 - 2\left(\cos \frac{2\pi s_1}{q_1} + \cos \frac{2\pi s_2}{q_2}\right)} \tag{5.14}$$

and

$$\langle \gamma^{\ell} \rangle = \frac{1}{q_1 q_2} \sum_{r_1=1}^{q_1} \sum_{r_2=1}^{q_2} \gamma^{\ell}_{r_1 r_2}$$

$$\tilde{\gamma}(s_1, s_2) = \frac{1}{q_1 q_2} \sum_{r_1=1}^{q_1} \sum_{r_2=1}^{q_2} \gamma^{\ell}_{r_1 r_2} \exp\left[\frac{2i\pi}{q_1} r_1 s_1 + \frac{2i\pi}{q_2} r_2 s_2 \right].$$

6. Discussions and conclusions.

6.1 INCREASE OF THE MAGNETIC ENERGY. — Let us first discuss the physical content of the general result equation (5.13) of the previous sections. Up to second order in the magnetic flux, the Landau level energy E_n contains two terms. The first one is the usual result [3, 4] for the uniform magnetic field case with ϕ as magnetic flux. This contribution is expected on general grounds. Indeed classical trajectories, which correspond to energies near the zero field band edge ($E = \pm 4$), have length scales much larger than that of the magnetic unit cell. Accordingly, the quantization is controlled by the average flux ϕ, rather than the detailed structure $\phi_{\alpha\beta}$ inside the magnetic unit cell. Similarly the degeneracy of E_n is also preserved for the same reasons. The second contribution δ to E_n is independent of the integer n, and corresponds to the shift of the zero field band edge when $\phi = 0$. The new term δ is governed by the magnetic « structure factor » $\tilde{\gamma}$ as defined above and describes a narrowing of the zero field band width because δ is always positive. This also means that inhomogeneities in the flux pattern increase the Landau level energy E_n, the increment being independent of n. The reason for this general trend can be outlined as follows. Consider the general case of the one-electron Hamiltonian :

$$\mathcal{H} = \frac{1}{2m} \left(\mathbf{p} + \frac{e}{c} \mathbf{A} \right)^2 + V(\mathbf{x}) \tag{6.1}$$

and compare the following two cases : the first one $\mathcal{H}_0$ describes a uniform magnetic flux ϕ and the second, $\mathcal{H}$, corresponds to a non-uniform flux but having the same averaged ϕ. The Hamiltonian $\mathcal{H}$ can be expanded formally by writing :

$$\mathcal{H} = \mathcal{H}_0 + \frac{e}{c} \int d\mathbf{x} \cdot \delta\mathbf{A} \cdot \mathbf{j} + \frac{e^2}{2mc^2} \int d\mathbf{x} \cdot [\delta\mathbf{A}(x)]^2 \cdot \rho(x) . \tag{6.2}$$

Here $j(x)$ and $\rho(x)$ are the electron current and density operators, and $\delta\mathbf{A}(x)$ is the change in vector potential upon going from $\mathcal{H}_0$ to $\mathcal{H}$. For an infinite system $\delta\mathbf{A}(x)$ increases without limit as a function of $\mathbf{x}$. Finite systems are therefore needed for practical calculations with equation (6.2). However, for our purpose, it is sufficient to consider equation (6.2) as a formal expansion. On a lattice, $\mathcal{H}$ assumes the following form :

$$\mathcal{H} = - \sum_{\langle ij \rangle} (e^{iA_{ij}} C_i^+ C_j + \text{h.c.}) \tag{6.3}$$

where we have used the standard notation. The local current density from node j to node i is given by : 2 Im $(e^{iA_{ij}} \langle C_i^+ C_j \rangle)$. Therefore in a uniform magnetic field, the current density vanishes because $e^{iA_{ij}} \langle C_i^+ C_j \rangle$ is nothing else than the energy per bond and then a translation invariant. This remark implies that in equation (6.2), the linear terms in δA give a vanishing contribution to $\mathcal{H}$. The only remaining terms are quadratic in δA and therefore inhomogeneities of the magnetic flux increase the magnetic energy.

6.2 STABILIZATION OF THE FERMI SEA. — The general expression equation (5.13), for the Landau levels, leads to some interesting consequences regarding the stabilization of the Fermi sea by the gaps. This problem has already been discussed by us in the case of a uniform field. Here we consider the same question but for a non uniform magnetic flux. Actually two cases seem to appear : $\phi \neq 0$ and $\phi = 0$.

Case $\phi \neq 0$: For non vanishing $\gamma = 2\pi \frac{\phi}{\phi_0}$, the Landau levels structure allows for an extension of our previous result to the present case. More precisely consider $N = xN_s$ fermions on a square lattice made of N_s sites, and let us introduce η such that $\delta = \gamma^2 \eta^2$ is the global shift in equation (5.13) :

$$E_n = -4 + (2n+1)\gamma - \frac{1}{16}\gamma^2[1 + (1+2n)^2] + \delta . \tag{6.4}$$

The energy per site can then be written as :

$$\frac{E}{N_s} = \sum_{n=0}^{m-1} \frac{\gamma}{2\pi} E_n \tag{6.5}$$

when the m first Landau levels, each having a degeneracy $\frac{\gamma}{2\pi}$, are completely filled. Under these conditions we have : $m\gamma = 2\pi x$, and then

$$\frac{E}{N_s} = -4x + 2\pi x^2 - \frac{\pi^2}{3}x^3 - \frac{\pi^2}{6}\frac{x^3}{m^2}(1 - 24\eta^2) + O(x^4) . \tag{6.6}$$

As a consequence, $m = 1$ corresponds to the minimum of $\frac{E}{N_s}$ as long as $(1 - 24\eta^2) > 0$, which is satisfied for $\eta \ll 1$, i.e. small fluctuations in the flux pattern.

This result remains true if the semiclassical expansion is extended to next orders. To see this, we consider the case of the staggered lattice with two values of the flux : γ_0 and γ_1.

In terms of : $\gamma = \frac{1}{2}(\gamma_0 + \gamma_1)$, $\gamma' = \frac{1}{2}(\gamma_0 - \gamma_1)$ and $\xi = \frac{\gamma'}{\gamma}$ the energy E_n can be written (see Eq. (3.23)) :

$$E_n = -4 + (2n+1)\gamma - \frac{1}{16}\gamma^2[1 + (1+2n)^2] + \frac{\gamma^3}{192}[n^3 + (n+1)^3] + \\ + \frac{\gamma^2\xi^2}{8} - \frac{\gamma^2\xi^2}{32}(2n+1) \tag{6.7}$$

and for m field levels :

$$\frac{E}{N_s} = -4x + 2\pi x^2\left(1 - \frac{\xi^2}{32}\right) - \frac{\pi^2}{3}x^3 + \frac{\pi^3}{48}x^4 - \frac{\pi^2}{6}\frac{x^3}{m^2}\left(1 - \frac{\pi}{8}x - 3\xi^2\right) . \tag{6.8}$$

Therefore $m = 1$ is stabilized as long as $\left(1 - \frac{\pi}{8}x - 3\xi^2\right)$ is positive. Therefore providing some additional conditions on ξ ($\xi \ll 1$), the situation is identical to that of the uniform flux : cusps, metastable states... etc.).

In the opposite limit where $\xi \gg 1$, $m = \infty$ is stabilized and this corresponds, for fixed x, to $\gamma = 0$ i.e. the vanishing flux limit where the discrete level structure disappears.

Case $\phi = 0$: This is exemplified in the case $\gamma_1 = -\gamma_0$ of the staggered square lattice. With the previous notation, this corresponds to $\gamma = 0$, $\xi = \infty$, and one obtains a band spectrum (the corresponding density of states is given in the Appendix). For this particular value of γ, equation (6.6) reduces to :

$$E_n = -4 + \frac{\gamma'^2}{8} + \cdots \tag{6.9}$$

which is simply the new lower band edge. At low fermion densities x, it is sufficient to consider the limiting density of states $\rho(E)$, close to $E_0 = -4 + \frac{\gamma'^2}{8}$. In this limit a simple calculation gives : $\left(a = \cos\frac{\gamma'}{2}\right)$

$$\frac{E}{N_s} = -4\left(\frac{1+a}{2}\right)^{1/2} x + 2\pi\left(\frac{1+a}{2}\right)^{1/2} x^2 - \frac{\pi^2}{3}\left(\frac{2}{1+a}\right)^{1/2} x^3 + O(x^4)\,. \tag{6.10}$$

In particular for $a = 1$ we recover the known result in zero field. However, as a function of a, $\frac{E}{N_s}$ increases for $\gamma' \neq 0$, i.e. at $a < 1$.

This example shows again that E increases with the field distorsions, even in the case when the Fermi sea is stabilized at zero field.

7. Conclusion.

The content of this paper has been summarized in the introduction. Let us conclude with two remarks.

i) In this paper we have introduced a new algebraic formalism which is the appropriate one for solving the periodic flux problem. Evidently, the enlarged algebraic structure so introduced, calls for further studies. In fact, for the sake of simplicity, we have limited our attention to the semiclassical region. However, it is not difficult to consider other problems where the methods and tools elaborated in our previous papers can be used. Similarly, for purely mathematical purposes, a deep investigation of these new algebra is called for.

ii) As a direct consequence of our preliminary results, we were able to extend the range of validity of the mechanism of Fermi sea stabilization. This sheds a new light on this concept, which we believe is a result of very general importance. Furthermore, its extension to other physical problems is certainly a valuable task.

Appendix.

Staggered flux lattice.

In the special case of a staggered lattice (Fig. 1b), with $\gamma_1 = \gamma'$, $\gamma_2 = -\gamma'$, the band structure can be explicitly calculated. The eigenvalues reduce to two coupled sets of linear equations :

$$\begin{aligned}
\varepsilon\psi_A(x,y) &= -\left[\psi_B(x-1,y) + \psi_B(x+1,y) + e^{-i\gamma'/2}\psi_B(x,y-1) + e^{-i\gamma'/2}\psi_B(x,y+1)\right] \\
\varepsilon\psi_B(x,y) &= -\left[\psi_A(x-1,y) + \psi_A(x+1,y) + e^{i\gamma'/2}\psi_A(x,y-1) + e^{i\gamma'/2}\psi_A(x,y+1)\right].
\end{aligned} \tag{A1}$$

Here we have adopted an alternate Landau gauge in order to have a simple expression for the coupling between the two sublattices A and B. Periodic solutions

$$\psi_A(\mathbf{x}) = \psi_A e^{i\mathbf{kx}}, \quad \psi_B(\mathbf{x}) = \psi_B e^{i\mathbf{kx}}, \quad \mathbf{x} = (x, y), \quad \mathbf{k} = (k_x, k_y)$$

lead to a 2×2 determinant. The solution of the secular equation is $\left(a = \cos \frac{\gamma'}{2}\right)$

$$\varepsilon^2 = 4(\cos^2 k_x + \cos^2 k_y + 2\, a \cos k_x \cos k_y) = \varepsilon^2(k_x, k_y) \tag{A2}$$

where $\gamma' = 0$ (i.e. $a = 1$) and $\gamma' = \pi$ (i.e. $a = 0$) reproduce two known limiting cases. The density of states $\rho(E)$ corresponding to the above dispersion relation is given by $(E < 0)$:

$$\rho(E) = \sum_{k_x k_y} \delta\{E - \varepsilon(k_x, k_y)\}, \quad \rho(-E) = \rho(E) \tag{A3}$$

where the sum is taken over the first Brillouin zone. Making the change of variables $u = k_x + k_y$, $v = k_x - k_y$, one obtains :

$$\rho(E) = E \frac{1}{4\pi^2} \int_0^\pi du\, \mathrm{Re} \left[(\cos u + a)^2 - \left(\frac{E^2}{4} - 1 - a \cos u\right)^2\right]. \tag{A4}$$

The calculation of $\rho(E)$ in the general case is rather cumbersome. Let us first consider the case $k_x \sim 0$, $k_y \sim 0$, i.e. close to the lower band edge $-4\left(\frac{1+a}{2}\right)^{1/2}$. In this limit one has :

$$\frac{\varepsilon^2}{8(1+a)} = 1 - \frac{1}{2}(k_x^2 + k_y^2) + \frac{4+a}{24(1+a)}(k_x^4 + k_y^4) + \frac{a}{4+a} k_x^2 k_y^2 + O(k_{x;y}^6).$$

This allows for a simple calculation of $\rho(E)$ to first order in $E + E_0$, $E_0 = 4\left(\frac{1+a}{2}\right)^{1/2}$. $\rho(E)$ assumes the following form :

$$\rho(E) = \frac{1}{\pi E_0} + \frac{1}{8\pi(1+a)^2}(E + E_0) + O[(E + E_0)^2]. \tag{A5}$$

In the general case, $\rho(E) = \rho(-E)$ can be calculated explicitly starting from equation (A4). Letting $x_1 = 1 - \frac{E^2}{4(1-a)}$, $x_2 = -1 + \frac{E^2}{4(1+a)}$, one obtains three regions :

* For $8(1-a) \leqslant E^2 \leqslant 8(1+a)$,

$$\rho(E) = \frac{E}{4\pi^2}(1-a^2)^{-1/2} \cdot 2[2(x_2 - x_1)]^{-1/2} K\left[\left(\frac{(1-x_2)(1-x_1)}{2(x_2-x_1)}\right)^{1/2}\right]. \tag{A6}$$

* For $4(1-a^2) \leqslant E^2 \leqslant 8(1-a)$,

$$\rho(E) = \frac{E}{4\pi^2}(1-a^2)^{-1/2} \cdot 4[(1-x_1)(1+x_2)]^{-1/2} K\left[\left(\frac{(1-x_2)(1+x_1)}{(1-x_1)(1+x_2)}\right)^{1/2}\right] \tag{A7}$$

and finally,

* *For* $0 \leq E^2 \leq 4(1-a^2)$,

$$\rho(E) = \frac{E}{4\pi^2}(1-a^2)^{-1/2} \cdot 4[(1-x_2)(1+x_1)]^{-1/2} K\left[\left(\frac{(1-x_1)(1+x_2)}{(1-x_2)(1+x_1)}\right)^{1/2}\right] \quad \text{(A8)}$$

Here $K(k) = \int_0^{\pi/2} d\varphi (1-k^2\sin^2\varphi)^{-1/2}$ and $K'(k) = K[(1-k^2)^{1/2}]$ are the usual elliptic integrals. One notices the following properties of $\rho(E)$ for $a \neq 1$:

i) $\rho(E) = 0$ at $E = 0$; close to $E = 0$, $\rho(E)$ has a cusp and takes the form :

$$\rho(E) = |E| \frac{1}{4\pi(1-a^2)^{1/2}},$$

which becomes singular at $a \lesssim 1$.

ii) There is a logarithmic divergence of $\rho(E)$ at $E = \pm 2(1-a^2)^{1/2}$.

iii) For $E = [8(1-a)]^{1/2}$, $\rho(E)$ has a jump, corresponding to the complete filling of one of the two Fermi « pockets », present at $a \neq 1$. Indeed the Fermi surface is made of two disconnected pieces, and this as long as $a \neq 1$.

As a final remark notice that for $a = 1$, $x_1 = -\infty$ and one recovers the zero field result :

$$\rho(E) = \frac{1}{2\pi^2} K'\left(\frac{E}{4}\right).$$

Similarly, for $a = 0$, one obtains the known expression :

$$\rho(E) = \frac{1}{2\pi^2} E K'\left(\frac{E^2}{4} - 1\right).$$

References

[1] BELLISSARD J., Operators algebras and applications, vol. II, Eds. E. V. Evans and M. Takesaki (Cambridge University Press, Cambridge) 1988.

[2] BELLISSARD J., The Harald Bohr Centenary, Eds. C. Berg and F. Fluegege (Royal Danish Acad. Science, Copenhagen) 1989.

[3] RAMMAL R. and BELLISSARD J., *J. Phys. France* **51** (1990) n° 17.

[4] RAMMAL R. and BELLISSARD J., *J. Phys. France* **51** (1990) n° 17.

[5] BELLISSARD J., Bad Schandau conference on localization, Eds. W. Weller and P. Ziesche, Teubner, Leipzig (1988).

[6] DUBROVIN B. A. and Novikov S. P., *Sov. Phys. JETP* **52** (1980) 511 ;
DUBROVIN B. A. and NOVIKOV S. P., *Sov. Math. Dokl.* **22** (1980) 240 ;
NOVIKOV S. P., *Sov. Math. Dokl.* **23** (1981) 298.

[7] HALDANE F. D. M. and REZAYI E. H., *Phys. Rev. B* **31** (1985) 2529.

[8] AROVAS D. P., BHATT R. N., HALDANE F. D. M., LITTLEWOOD P. B. and RAMMAL R., *Phys. Rev. Lett.* **60** (1988) 619 and unpublished report.

[9] ZAK J., *Phys. Rev. A* **134** (1964) 1602 and 1607.

[10] BARELLI A., BELLISSARD J. and RAMMAL R., in preparation (1990).