# A Kinetic Model of Quantum Jumps

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**Abstract**

A new class of models describing the dissipative dynamics of an open quantum system $S$ by means of random time evolutions of pure states in its Hilbert space $\mathcal{H}$ is considered. The random evolutions are linear and defined by Poisson processes. At the random Poissonian times, the wavefunction experiences discontinuous changes (quantum jumps). These changes are implemented by some non-unitary linear operators satisfying a locality condition. If the Hilbert space $\mathcal{H}$ of $S$ is infinite dimensional, the models involve an infinite number of independent Poisson processes and the total frequency of jumps may be infinite. We show that the random evolutions in $\mathcal{H}$ are then given by some almost-surely defined unbounded random evolution operators obtained by a limit procedure. The average evolution of the observables of $S$ is given by a Quantum Dynamical Semigroup, its generator having the Lindblad form [1]. The relevance of the models in the field of electronic transport in Anderson insulators is emphasised.

Key words: Open Quantum Systems; Quantum Kinetic Models.

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# 1 Introduction

Kinetic models with random collision events are widely used to study transport properties of systems of classical particles [2]. They generally lead, if the motion between collisions is balistic (respectively, anomalous diffusive) and the time delay between consecutive collisions has a finite mean value, to a conductivity given by the Drude formula [3] (resp., by the anomalous Drude formula [4]). It is natural to ask whether one can construct quantum kinetic models which can describe electronic transport in solids exhibiting different conductivity behaviors than that given by the Drude formula, such as, for instance, disordered solids in the strong localization regime (Anderson insulators) [5]. We study in this paper a quantum kinetic model describing non-interacting electrons in Anderson insulators coupled to external particles like phonons. The model is built in such a way as to give the classical kinetic theory back, such as Boltzmann's equation, when quantum effects can be neglected. The central question addressed in the work, which constitutes a preliminary step towards a kinetic theory of hopping transport, concerns the infinite volume limit of the model. Since the model is related to random wavefunctions models studied in quantum optics [6, 7] and quantum measurement theory [8, 9, 10, 11, 12], and may apply as well to other physical systems interacting with their environment, we shall present our results in the general framework of open quantum systems theory.

The dissipative dynamics of an open quantum system $S$ can be described in two different ways. The first and most popular approach consists in coupling $S$ with a reservoir $R$ [13]. The density matrix $\rho_{\rm tot}$ of the total system $S + R$ is assumed to follow a Liouville-von Neumann equation, i.e., one assumes that $S + R$ is closed. A state of $S$ is specified by the reduced density matrix $\rho$, defined as the partial trace of $\rho_{\rm tot}$ over the reservoir's Hilbert space. $\rho$ does not describe a single system but a statistical ensemble. By tracing out the degrees of freedom of $R$ in the Liouville-von Neumann equation, one obtains an integro-differential equation for $\rho$ (Nakajima-Zwanzig equation [14]). Using a suitable Markov approximation to eliminate memory effects, this equation is then transformed into a simpler first-order linear differential equation, called the *master equation* [13]. One can justify rigorously the Markov approximation in the so-called van Hoove limit, i.e., the weak coupling limit with an appropriate time rescaling [15, 16, 17] (see also [18, 19] for other limits). The reduced dynamics does not conserve pure states. It has been shown by Lindblad [1] that the Markovian master equation is of the form:

$$\frac{d\rho}{dt} = (-\mathcal{L}_H + \mathcal{C}_*)\rho = -i[H,\rho] + \frac{1}{2}\sum_{\ell}\left([L_\ell\,\rho, L_\ell^*] + [L_\ell, \rho L_\ell^*]\right) \ . \tag{1}$$

$H$ is the Hamiltonian of $S$ (including the energy shifts due to the coupling with the reservoir), and $L_\ell$ are some operators acting on the Hilbert space $\mathcal{H}$ of $S$, called the Lindblad operators in the sequel. An alternative approach for the same problem is based on stochastic evolutions of pure states. The state of $S$ is specified by a random wavefunction (RW) in $\mathcal{H}$, evolving according to a linear or nonlinear *stochastic Schrödinger equation*. Different stochastic evolutions have been proposed in the last two decades in various fields of physics and mathematics, especially quantum optics [6, 7], quantum measurement theory [8, 9, 10, 11, 12, 20, 21, 22, 23, 24], the theory of open quantum systems [16] and electronic transport in solids [25, 26, 27]. Consistency with the master equation approach requires that the pure state evolution gives the density matrix evolution back after averaging over the dynamical noise. Apart from being intuitively appealing, the RW models provide quite efficient tools for solving master equations numerically. Actually, one is led to integrate $N$ coupled differential equations for the wavefunction, where $N$ is the dimension of $\mathcal{H}$, for a large enough number of realizations of the dynamical noise. For large $N$, this is generally much more efficient than integrating the $N \times N$ coupled master equations for the density matrix. However, the RW models are more than simple mathematical or numerical tools: they describe the real evolution of the system $S$ under continuous monitoring by means of measurements (photons counting, homodyne or heterodyne detections) [28]. The randomness of their dynamics is a consequence of our ignorance of the result of a measurement in quantum mechanics. At the end of the eighties, experiments on the fluorescence of single ions in magnetic traps have shown records of 'quantum jumps' between an excited atomic state and a lower state, occurring at random times [28]. These sudden jumps - which were already assumed to exist by Einstein in his paper on the A and B atomic coefficients [29] - correspond to the absorption or emission of a photon by the ion at the corresponding transition. Such direct observations have motivated the study of the RW models in quantum optics.

The physical situation which motivates our work is electronic transport in strongly disordered solids [30]. It is well known [5, 31] that the spectrum is pure-point and the electronic eigenfunctions $|i\rangle$ are exponentially localized in such solids (Anderson localization). The electrical conductivity thus vanishes at zero temperature. At non zero temperature $T > 0$, transport occurs via phonon-assisted hopping of electrons from one localized eigenstate into another. At small $T$, a phenomenological argument due to Mott [32] shows that the hopping conductivity $\sigma$ is given by $\sigma = \sigma_0 \exp(-(T_0/T)^\gamma)$, where the exponent $\gamma = 1/(d+1)$ depends on the dimension $d$ ($d = 1, 2, 3$) only and $T_0$ is a constant which depends on the localization length and the density of states at the Fermi energy. Efros and Shklovskii have shown that the presence of a wide Coulomb gap in the density of states modifies $\gamma$, which then equals $1/2$ for all dimensions [30]. The regime of validity of Mott's formula is called the *variable range hopping* regime. Variable range hopping transport occurs for instance in lightly doped compensated semiconductors at low temperature, in amorphous solids [30], in two-dimensional electron gases in zero or strong magnetic field [33, 34], and in the quasicrystal $i$-AlPdRe [35]. The electrons in the disordered potential created by the ions, impurities or defects are coupled to low energy acoustic phonons. Since phonons do not carry current, the study of transport requires the knowledge of the electron dynamics only. The system $S$ of all electrons is thus an open quantum system. If we ignore electron-electron interactions, it can be shown [36] that, at low enough temperature, its dissipative dynamics is correctly described by the master equation (1) with $\ell = (i, j)$, $i \neq j$, and:

$$L_{i\to j} = \sqrt{\Gamma_{i\to j}}\, |j\rangle\langle i| \ ,$$

where $\Gamma_{i\to j}$ is the transition rate from the eigenfunction $|i\rangle$ to the eigenfunction $|j\rangle$. Starting from the electron-phonon interaction Hamiltonian, $\Gamma_{i\to j}$ can be calculated perturbatively by means of Fermi golden rule [37]. It decreases exponentially with the distance $|i - j|$ between the localization centers of $|i\rangle$ and $|j\rangle$, and depends strongly at low temperature on their energies $E_i$ and $E_j$. The widely used relaxation time approximation, which amounts to replace all the $\Gamma_{i\to j}$'s by a single damping constant, is thus completely unjustified in hopping transport.

In this paper, the dissipative dynamics of an electron in an Anderson insulator under electron-phonon or another coupling is described by means of electronic quantum jumps between the localized eigenfunctions $|i\rangle$, occurring at random times. Between jumps, the electronic wavefunction evolves according to Schrödinger equation with an effective non self-adjoint Hamiltonian describing both the disordered potential and some complex energies (inverse lifetime of the eigenfunctions). The rate of occurrence of quantum jumps are given by the above transition rates $\Gamma_{i\to j}$. Unlike in the models studied in [6, 7, 8], the stochastic evolution for the wavefunction is linear, which makes the mathematical analysis easier. The price we pay for this convenience is the non conservation of the norm of the random wavefunction. A fundamental question addressed by mathematical physicists in the theory of solids concerns the study of the spectrum and of the time evolution of the relevant electronic observables at the thermodynamic limit. Letting the volume of a strongly disordered solid tend to infinity, an infinite number of localized eigenfunctions $|i\rangle$ with energies close to the Fermi energy $E_F$ comes into play. Moreover, the double sum $\sum_{i,j} \Gamma_{i\to j}$ diverges, which means that an infinite number of jumps occur in any finite time interval in our kinetic model. Our main result shows that, provided the discontinuous changes of the wavefunction at the jumps are sufficiently 'local', the stochastic dynamics of the wavefunction is also well-defined in this case, and is given by an almost surely unbounded random evolution operator, obtained by a limit procedure. We also prove that the average evolution for the density matrix is given by the Lindblad master equation (1).

Our paper is organized as follows. Section 2 is devoted to the description of the model. Section 3 gives our main results on the stochastic evolution of wavefunctions for infinite dimensional Hilbert spaces (infinite volume limit). Section 4 concerns the averaged evolution; we obtain in particular the Lindblad generator of the average dynamics. It is shown in section 5 that the RW can be found by solving a stochastic time-dependent Schrödinger equation with a kicked Hamiltonian. The model is compared our model with other known RW models in section 6. The last section before the conclusion contains the technical proofs of the two main results of sections 3 and 4.

# 2 The Model

## 2.1 The Stochastic Scheme

Let us consider a quantum system, with separable Hilbert space $\mathcal{H}$ and Hamiltonian $H$, coupled to its environment. We shall assume that $H = V + T$ is the sum of a (possibly unbounded) self-adjoint operator $V$, with dense domain $\mathcal{D}(H)$ and pure point spectrum, and of a bounded self-adjoint operator $T$. Let $\{|i\rangle; i \in \Lambda_\infty\}$ be an orthonormal basis of $\mathcal{H}$ formed by the eigenfunctions of $V$, where $\Lambda_\infty$ is an infinite subset of $\mathbb{Z}^d$. If the system is a doped semiconductor, we can think of $\Lambda_\infty$ as the impurity sites in the host crystal $\mathbb{Z}^3$; then $H$ acts on the Hilbert space $\mathcal{H} = \ell^2(\Lambda_\infty)$ and $\{|i\rangle; i \in \Lambda_\infty\}$ is the canonical basis (see below). For each pair $(i,j) \in \Lambda_\infty^{\times 2}$, we denote $|i-j|$ the Euclidean distance between $i \in \mathbb{Z}^d$ and $j \in \mathbb{Z}^d$. *Instantaneous jumps* take place at some random times

$$\ldots \leq t^{-n}_{i\to j} \leq \ldots \leq t^{-1}_{i\to j} \leq 0 \leq t^{1}_{i\to j} \leq \ldots \leq t^{n}_{i\to j} \leq \ldots \quad , \quad (i,j) \in \Lambda_\infty^{\times 2} \; .$$

These jumps are labelled by pairs $(i,j) \in \Lambda_\infty^{\times 2}$ of indices and by an integer $n$, which counts the number of jumps $(i,j)$ that occurred since the initial time $t=0$. Let us set $t^0_{i\to j} = 0$ and denote by $s^n_{i\to j} = t^n_{i\to j} - t^{n-1}_{i\to j}$ the time delays between two consecutive jumps $(i,j)$ (or, if $n=0$ or 1, the time delay between the initial time $t=0$ and the jump $(i,j)$ immediately preceding or following it). The positive numbers $s^n_{i\to j}$, for any $n \in \mathbb{Z}$ and $i,j \in \Lambda_\infty$, are assumed to be mutually independent random variables distributed according to the exponential law $p(ds) = \Gamma_{i\to j} e^{-s\Gamma_{i\to j}} ds$, where $\Gamma_{i\to j} \geq 0$ depends on $(i,j)$ but not on $n$. In other words, for any fixed $(i,j)$, the jump times $t^n_{i\to j}$, $n \in \mathbb{Z}^\star$ are given by a Poisson process with parameter $\Gamma_{i\to j}$. The *transition rates* $\Gamma_{i\to j}$ are considered here as phenomenological parameters. In concrete situations, they can be computed by using the Fermi golden rule. They contain all the quantitative physical information on the interaction of the system with its environment (e.g., the coupling constant). If the environment is a thermal bath, they depend on its temperature.

Each jump modifies in a discontinuous way the wavefunction of the system. These discontinuous changes are implemented by some bounded operators $W_{i\to j}$ (called *jump operators* in the sequel). More precisely, if the system's wavefunction is $|\psi\rangle \in \mathcal{H}$ just before a jump $(i,j)$, it becomes $W_{i\to j}|\psi\rangle$ just after it:

$$\text{jump } (i,j): \qquad |\psi\rangle \to W_{i\to j}|\psi\rangle \; . \tag{2}$$

The jump operators describe the qualitative effects on the system of its coupling with its environment (e.g., how it is affected by the absorption or the emission of external particles like phonons, photons, ... ). They do not depend on the damping rates or on the temperature of the bath.

Between two consecutive jumps, the system evolves according to the Schrödinger equation with the Hamiltonian $H + K$, where $K$ is a bounded operator describing some complex renormalizations of the energies due to the coupling with the environment (*damping operator*). We will restrict ourself in this work to systems with a norm-preserving average dynamics, namely, such that $\mathbb{E}\ \|\psi(t)\|^2 = 1$, where $\mathbb{E}$ is the average over all times $t^n_{i\to j}$. As we shall see below, in order that $\|\psi(t)\|^2$ be conserved in average, the damping operator $K$ must be given, *up to a self-adjoint operator*, by:

$$K = \frac{1}{2\mathrm{i}} \sum_{i,j\in\Lambda_\infty} \Gamma_{i\to j} \big(W^*_{i\to j} + 1\big)\big(W_{i\to j} - 1\big)\ . \tag{3}$$

This means that $K$ is *not* self-adjoint.

The wavefunction at time $t$, $t_p \le t < t_{p+1}$, is thus formally given by:

$$|\psi(t)\rangle = e^{-i(t-t_p)(H+K)} W_{i_p\to j_p} e^{-i(t_p-t_{p-1})(H+K)} \ldots W_{i_1\to j_1} e^{-it_1(H+K)} |\psi\rangle\ , \tag{4}$$

where $|\psi\rangle$ is the wavefunction at time $t = 0$, $0 \le t_1 \le \ldots \le t_p \le t_{p+1}$ are the times of occurrence of *any* jump, and $(i_p, j_p) \in \Lambda_\infty^{\times 2}$ is the (random) pair of indices corresponding to the actual jump that takes place at time $t_p$. As it will be clear below, the formula (4) is meaningful if:

$$\Gamma \equiv \sum_{i,j\in\Lambda_\infty} \Gamma_{i\to j} < \infty\ . \tag{5}$$

It will be seen in the next section how to define the random wavefunction when $\Gamma = \infty$.

From a mathematical point of view, it is convenient to represent each sequence of random variables $(t^n_{i\to j})_{n\in\mathbb{Z}^\star}$ by a counting process $(N_{i\to j}(t))_{t\in\mathbb{R}}$ [38]. Here $t^n_{i\to j}$, $n \in \mathbb{Z}^\star$, are the left discontinuities of the counting function $N_{i\to j}(t)$. For any compact interval $I \subset \mathbb{R}$, $N_{i\to j}(I)$ is the (random) number of jumps $(i,j)$ occurring at times $t \in I$, namely:

$$N_{i\to j}(I) = \sum_{n=-\infty, n\neq 0}^{\infty} \chi(t^n_{i\to j} \in I) \ , \tag{6}$$

where $\chi$ is the characteristic function ($\chi(\mathcal{P}) = 1$ if the property $\mathcal{P}$ is true, 0 otherwise). We set $N_{i\to j}(t) = N_{i\to j}([0,t])$ if $t \geq 0$ and $N_{i\to j}(t) = -N_{i\to j}([t,0])$ if $t < 0$. The stochastic scheme described above is thus specified by the infinite set of independent Poisson processes $(N_{i\to j}(t))_{t\in\mathbb{R}}$ with parameters $\Gamma_{i\to j}$, for all $i,j \in \Lambda_\infty$. If (5) holds, the staircase function:

$$N(t) = \sum_{i,j\in\Lambda_\infty} N_{i\to j}(t) \ , \tag{7}$$

which counts the total number of jumps between 0 and $t$, define a Poisson process of parameter $\Gamma$. The left discontinuities of this function are the jump times $t_p$ above. As follows from the independence of the Poisson processes $(N_{i\to j}(t))_{t\in\mathbb{R}}$, the probability that the $p$-th jump is a jump $(i,j)$ is:

$$\mathbb{P}\big((i_p, j_p) = (i,j)\big) = \frac{\Gamma_{i\to j}}{\Gamma} \ . \tag{8}$$

## 2.2 Examples

In order to be more concrete, let us give two examples corresponding to physical systems for which the above scheme applies.

### 2.2.1 'Classical' Model

Consider a system initially in an eigenstate of the Hamiltonian $H = V$. We assume that $T = 0$, so that $|i\rangle$ are eigenfunctions of $H$. Let $\Gamma_{i\to i} = 0$ and:

$$\begin{aligned} W_{i\to j} &= 1 + \big(|j\rangle - |i\rangle\big)\langle i| \ , \ i \neq j \in \Lambda_\infty \\ K &= 0 \ . \end{aligned} \tag{9}$$

The jump operators $W_{i\to j}$ simply transform $|i\rangle$ into $|j\rangle$ and leave unchanged all other eigenfunctions $|k\rangle, k\neq i$. Let the system evolve from $t=0$ until time $t$ according to the quantum jumps scheme described above. If we restrict ourself to initial wavefunctions $|\psi(0)\rangle = |i\rangle$, $i\in\Lambda_\infty$, the wavefunction $|\psi(t)\rangle$ at time $t\geq 0$ is equal, up to a phase factor, to $|i(t)\rangle$, $i(t)\in\Lambda_\infty$; in particular, its norm $\|\psi(t)\|$ is conserved. Actually, $|\psi(t)\rangle$ remains unchanged up to a phase as long as there is no jump. When a jump occurs, it may (or may not) jump from an eigenfunction $|i\rangle$ into another eigenfunction $|j\rangle$. The stochastic dynamics is thus completely specified by a set of random numbers $p_i(t), i\in\Lambda_\infty$, with $p_i(t)=1$ if $|\psi(t)\rangle$ is proportional to $|i\rangle$ and 0 otherwise. Define the average population $\pi_i(t)$ in the eigenfunction $|i\rangle$ as the average of $p_i(t)$, $\pi_i(t)=\mathbb{E}\ p_i(t)$. In order to determine the equation satisfied by the $\pi_i$'s, let us compute the amount of change $d\pi_i = d\pi_i^+ - d\pi_i^-$ of $\pi_i(t)$ between times $t$ and $t+dt$. The gain $d\pi_i^+$ is equal to the sum over $j\in\Lambda_\infty$ of the probability $\Gamma_{j\to i}\,dt$ that a jump $(j,i)$ occurs between $t$ and $t+dt$, multiplied by the probability $\pi_j(t)$ that this jump modifies the wavefunction. Similarly, the loss $d\pi_i^-$ is equal to the sum over $j\in\Lambda_\infty$ of the probability $\Gamma_{i\to j}\,dt$ that a jump $(i,j)$ occurs between $t$ and $t+dt$, multiplied by $\pi_i(t)$. Thus $\pi_i(t)$ satisfies the Boltzmann equation:

$$\frac{d\pi_i}{dt} = \sum_{j\in\Lambda_\infty, j\neq i} \left(\Gamma_{j\to i}\,\pi_j(t) - \Gamma_{i\to j}\,\pi_i(t)\right) . \tag{10}$$

The above considerations clearly still hold if the system is initially in an impure state given by a diagonal density matrix, $\rho(0)=\sum_i \pi_i(0)|i\rangle\langle i|$.

This example shows that the stochastic model above can be seen as a quantum generalization of a classical kinetic model. In the quantum case, the classical 'random collisions' between particles of the system and of its environment are replaced by 'random quantum jumps'.

### 2.2.2 The Anderson Model

Let us consider a crystal the atoms of which are located at the vertices of a Bravais lattice in dimension $d$. Using labelling of the lattice sites by integers, we can identify it with $\mathbb{Z}^d$. Some random sites are actually occupied by impurities instead of atoms of the original species. These sites form an infinite (random) set $L \subset \mathbb{Z}^d$. At low enough temperature, conducting electrons are almost all in the impurity band, i.e., they are in linear combinations of impurity orbitals. Neglecting the other electrons and assuming only one orbital par impurity, the one-electron Hilbert space $\mathcal{H}$ is identified with $\ell^2(L)$. The electronic Hamiltonian can be chosen as the Anderson Hamiltonian:

$$H = \sum_{x \in L} \epsilon_x |x\rangle\langle x| + \sum_{x,y \in L} t_{xy} |x\rangle\langle y| \; .$$

$|x\rangle$, $x \in L$, are the canonical basis vectors, describing an electronic state in the impurity orbital at site $x$. $\epsilon_x$ are independent identically distributed random variables and $t_{xy}$ are hopping terms. The randomness of the site energies $\epsilon_x$, describing disorder in the solid, must be distinguished from the dynamical randomness above, which describes dissipation. If the system is submitted to a uniform electric field $\mathcal{E}(t)$, a term $q\,\mathcal{E}(t)X$ must be added to $H$, $q$ being the charge of the carriers and $X$ the position operator. Then, $H$ becomes unbounded. A first choice for the basis vectors $|i\rangle$ is to take them equal to the canonical basis vectors $|x\rangle$. Then $\Lambda_\infty = L$ and $H$ is non-diagonal, i.e. $T \neq 0$. As is well-known, for zero field and strong enough disorder (i.e., for large enough $\langle \Delta\epsilon_x^2 \rangle / t_{xy}$), the eigenfunctions of $H$ with energies close to the Fermi energy are exponentially localized [5, 31]. Another interesting choice for the basis vectors $|i\rangle$ is to take them equal to these localized eigenfunctions, so that $T = 0$. This choice is motivated by an adiabatic approximation, which implies that the generator of the average evolution commutes with the Liouvillian $\mathcal{L}_H = i[H, .]$ and that the jump operators are simple functions of the eigenvectors of $H$ [36]. One can argue [36] that this approximation is valid at low enough temperature in the variable range hopping regime. The set $\Lambda_\infty$ can be considered as the set of the localization centers of $|i\rangle$, i.e., as the lattice points $i \in L$ where the amplitude of $|i\rangle$ is maximum.

We now describe how the electrons in the impurity band are kicked by phonons. We take $\Gamma_{i\to i} = 0$ and:

$$W_{i\to j} = 1 + |j\rangle\langle i| \tag{11}$$

for any $i, j \in \Lambda_\infty$. $K$ is chosen according to (3), i.e.

$$K = -\frac{\mathrm{i}}{2} \sum_{i\in\Lambda_\infty} \Gamma_i |i\rangle\langle i| - \mathrm{i} \sum_{i\neq j\in\Lambda_\infty} \Gamma_{i\to j} |j\rangle\langle i| \ . \tag{12}$$

The effect of $K$ is thus to add an imaginary part $-\mathrm{i}\,\Gamma_i/2$ to the energies $E_i = \langle i|H|i\rangle$ and to introduce a hopping term $-\mathrm{i}\,\Gamma_{i\to j}$. Here $\Gamma_i = \sum_j \Gamma_{i\to j}$ is the inverse life-time of the $|i\rangle$ and describes how instable is this localized wavefunction due to the coupling with phonons.

# 3 Stochastic Evolution of Wavefunctions

## 3.1 Main Hypothesis on $\Gamma_{i\to j}$ and $W_{i\to j}$

If the double sum $\Gamma$ in (5) is infinite, the total number $N(I)$ of jumps in any compact interval $I \subset \mathbb{R}$ may be infinite with probability one. For example, it is shown in the appendix (lemma 4) that, if there exists $\beta > 0$ such that $\sum_j \Gamma_{i\to j} \geq \beta$ for infinitely many indices $i \in \Lambda_\infty$, then the sup over $i$ of $\sum_j N_{i\to j}(I)$ and thus $N(I)$ are infinite with probability one. The random times $t_p$ and the random indices $(i_p, j_p)$ in the formula (4) giving $|\psi(t)\rangle$ are therefore not well-defined. The idea for computing the random evolution in $\mathcal{H}$ when $\Gamma = \infty$ runs as follows: (1) restrict $\Lambda_\infty$ to a finite 'box' $\Lambda \subset \Lambda_\infty$ and determine the wavefunction at time $t$ using (4); (2) let the size of the box increase to reach the limit $\Lambda \uparrow \Lambda_\infty$. The main result of this section states that the limit exists provided suitable conditions on the $\Gamma_{i\to j}$ and $W_{i\to j}$ are made.

Let us denote $L = \mathcal{F}(\Lambda_\infty)$ the set of all *finite* subsets of $\Lambda_\infty$. $L$ is ordered by inclusion. We define the limit $\Lambda \uparrow \Lambda_\infty$ by means of the increasing sequence $(\Lambda_m)_{m\in\mathbb{N}}$ in $L$, with:

$$\Lambda_m = \left\{ i \in \Lambda_\infty ; |i| \leq m \right\} \ , \ m \in \mathbb{N} \ . \tag{13}$$

More precisely, a net $(|\psi_\Lambda\rangle)_{\Lambda\in L}$ in $\mathcal{H}$ is said to converge to $|\psi\rangle$ if the sequence $(|\psi_{\Lambda_m}\rangle)_{m\in\mathbb{N}}$ in $\mathcal{H}$ converges to $|\psi\rangle$. For any $\Lambda \in L$, the projector on the finite subspace span$\{|i\rangle, i \in \Lambda\}$ is denoted by $P_\Lambda = \sum_{i\in\Lambda} |i\rangle\langle i|$.

We will assume in what follows that $\Gamma_{i\to j}$, $W_{i\to j}$, $K$ and $T$ satisfy the following requirements.

**Assumption A**

*There is $r_1 > 0$ such that:*

$$\sup_{i\in\Lambda_\infty} \sum_{j\in\Lambda_\infty} \Gamma_{i\to j}\, e^{r_1|i-j|} < \infty \quad , \quad \sup_{i\in\Lambda_\infty} \sum_{j\in\Lambda_\infty} \Gamma_{j\to i}\, e^{r_1|i-j|} < \infty \ .$$

**Assumption B**

*For any $i, j, k, l \in \Lambda_\infty$,*

$$\left|\langle l|(W_{i\to j} - 1)|k\rangle\right| \leq (f_{il} + f_{jl})(f_{ik} + f_{jk}) \ ,$$

*where $f_{ij} = f_{ji} > 0$ is such that there is $r_2 > 0$, $\sup_{i\in\Lambda_\infty} \sum_{j\in\Lambda_\infty} f_{ij}\, e^{r_2|i-j|} < \infty$.*

**Assumption C**

*There is $r_3 > 0$ such that:*

$$\sup_{i\in\Lambda_\infty} \sum_{j\in\Lambda_\infty} |\langle i|(T+K)|j\rangle|\, e^{r_3|i-j|} < \infty \quad , \quad \sup_{i\in\Lambda_\infty} \sum_{j\in\Lambda_\infty} |\langle i|(T+K^*)|j\rangle|\, e^{r_3|i-j|} < \infty \ .$$

Assumption A is satisfied if the rates $\Gamma_{i\to j}$ tend to 0 exponentially as $|i-j| \to \infty$. Assumption B means that: (1) $W_{i\to j}$ affects appreciably the $l$-component of $|\psi\rangle$, i.e. $\langle l|W_{i\to j}|\psi\rangle$

is appreciably different from $\langle l|\psi\rangle$, only if $l$ is 'exponentially close' to $i$ or $j$; (2) $W_{i\to j}|k\rangle$ differs appreciably from $|k\rangle$ only if $k$ is 'exponentially close' to $i$ or $j$. For instance, $W_{i\to j} = 1 + |j\rangle\langle i|$ satisfies B with $f_{ij} = \delta_{ij}$ (Kronecker delta).

If $H$ is not bounded, some care about domains must be taken. Let $\mathcal{B}(\mathcal{H})$ be the $C^*$-algebra of bounded operators on $\mathcal{H}$. Consider the subalgebra of $\mathcal{B}(\mathcal{H})$:

$$D(\mathcal{L}_H) = \big\{ A \in \mathcal{B}(\mathcal{H}); A\,\mathcal{D}(H) \subset \mathcal{D}(H) \text{ and } \big[H, A\big] : \mathcal{D}(H) \to \mathcal{H} \text{ is bounded} \big\} . \tag{14}$$

This subalgebra is the ultraweakly dense domain of the Liouvillian $\mathcal{L}_H$ (see [44], Proposition 3.2.55):

$$\mathcal{L}_H(A) = \mathrm{i}\big[H, A\big] \ , \ A \in D(\mathcal{L}_H) \ . \tag{15}$$

**Assumption D**

*$T, K$ and $W_{i\to j}$ are elements of $D(\mathcal{L}_H)$ for any $i, j \in \Lambda_\infty$.*

## 3.2 Notions of Quantum Trajectory and of Random Propagator

Let $(\Xi, \Sigma, \mathbb{P})$ be the probability space for the infinite set of the Poisson processes $(N_{i\to j}(t))_{t\in\mathbb{R}}$ for all $i, j \in \Lambda_\infty$. The expectation (mean) value on this space is denoted by $\mathbb{E}$. For any $s \in \mathbb{R}$, we define a bimeasurable bijection $T(s)$ from $\Xi$ into itself, by translating simultaneously all the counting functions $N_{i\to j}(t)$:

$$N_{i\to j}^{T(s)\,\xi}(t) = \begin{cases} N_{i\to j}^{\xi}(t-s) + N_{i\to j}^{\xi}([-s, 0[) & \text{if } s \geq 0 \\[2ex] N_{i\to j}^{\xi}(t-s) - N_{i\to j}^{\xi}([0, -s[) & \text{if } s < 0. \end{cases}$$

It follows from the stationarity of Poisson processes that the probability measure $\mathbb{P}$ is invariant under the group of translations $\{T(s); s \in \mathbb{R}\}$.

The state of the system is specified by a random wavefunction (RW), i.e. by a measurable function $\Psi : \xi \in \Xi \mapsto |\Psi\rangle^\xi \in \mathcal{H}$ with a $\mathbb{P}$-integrable square norm $\||\Psi\rangle^\xi\|^2$. Two RW that differ on a set of $\xi$'s of probability zero are said to be equivalent. The space $\mathcal{K} = L^2(\Xi, \mathcal{H}, \mathbb{P})$ of the equivalence classes of RW is isomorphic to the tensor product:

$$\mathcal{K} = L^2(\Xi, \mathbb{P}) \otimes \mathcal{H} \ .$$

In other words, the coupling of the system with its environment forces us to enlarge the system's Hilbert space, by taking its tensor product with $L^2(\Xi, \mathbb{P})$. If $\Psi_1, \Psi_2 \in \mathcal{K}$, then $\xi \mapsto {}^{\xi}\langle \Psi_1 | \Psi_2 \rangle^{\xi}$ and $\xi \mapsto |\Psi_1\rangle^{\xi}\,{}^{\xi}\langle \Psi_2|$ are integrable and weakly integrable, respectively. The time translations act on $\mathcal{K}$ through a group of unitaries $\{\mathcal{S}(s); s \in \mathbb{R}\}$, with:

$$|\mathcal{S}(s)\,\Psi\rangle^{\xi} = |\Psi\rangle^{T(s)\,\xi} \quad \text{for almost every } \xi \in \Xi \ . \tag{16}$$

The dynamics of the system is given by a map $t \in [0, \infty[ \mapsto \Psi(t) \in \mathcal{K}$. Using the same terminology as in [7], we call the map $t \in [0, \infty[ \mapsto |\Psi(t)\rangle^{\xi} \in \mathcal{H}$ for a fixed outcome $\xi \in \Xi$ a *quantum trajectory*. Note that quantum trajectories are discontinuous in the quantum jumps schemes.

Let us assume that quantum trajectories do not interact, i.e. that for any $0 \leq t_0 \leq t$, $|\Psi(t)\rangle^{\xi}$ depends only on $|\Psi(t_0)\rangle^{\xi}$ for the same $\xi$. This property is satisfied in the scheme described in section 2.1 and, more generally, in all schemes for which the RW is the solution of a stochastic Schrödinger equation with classical noise. In the scheme of section 2.1, $|\Psi(t)\rangle^{\xi}$ depends moreover *linearly* on $|\Psi(t_0)\rangle^{\xi}$. The quantum trajectories are thus completely determined by a family $\{\mathcal{U}(t, t_0); t \geq t_0 \geq 0\}$ of linear evolution operators $\mathcal{U}(t, t_0)$ on $\mathcal{K}$, satisfying:

$$|\mathcal{U}(t, t_0)\,\Psi\rangle^{\xi} = U^{\xi}(t, t_0)\,|\Psi\rangle^{\xi} \ , \tag{17}$$

where $U^{\xi}(t, t_0)$ are some operators on $\mathcal{H}$. The quantum trajectory $t \mapsto |\Psi(t)\rangle^{\xi}$ such that $|\Psi(t_0)\rangle^{\xi} = |\psi\rangle \in \mathcal{H}$ is given in terms of the evolution $\mathcal{U}(t, t_0)$ by:

$$|\Psi(t)\rangle^{\xi} = |\mathcal{U}(t, t_0)(1 \otimes |\psi\rangle)\rangle^{\xi} = U^{\xi}(t, t_0)|\psi\rangle \ .$$

Here 1 is the constant function with value one on $\Xi$. Since the jump operators are not assumed to be unitary, the operators $U^\xi(t,t_0)$ are not unitary, and it can be expected in some cases that they be unbounded on a set of non-zero probability (we shall see below that this is indeed happens). Hence some care about domains must be taken.

**Definition 1** *Let* $\mathcal{U} = \{\mathcal{U}(t,t_0); t \geq t_0 \geq 0\}$ *be a family of linear operators* $\mathcal{U}(t,t_0)$ *on* $\mathcal{K}$ *satisfying* (17) *for any* $\Psi$ *in their domains* $\mathcal{D}(\mathcal{U}(t,t_0))$. *Let* $\mathcal{F} \subset \mathcal{K}$ *be the subspace of all finite linear combinations of the tensor products* $f \otimes |\psi\rangle$ *with* $f \in L^\infty(\Xi, \mathbb{P})$ *and* $|\psi\rangle \in \mathcal{H}$. $\mathcal{U}$ *is called a* Random Propagator *if for any* $0 \leq t_0 \leq \tau \leq t$, $|\psi\rangle, |\varphi\rangle \in \mathcal{H}$ *and* $\Psi \in \mathcal{F}$, *one has:*

**(RP1)** $|\psi\rangle \in \bigcap_{s \geq 0} \mathcal{D}(U^\xi(t_0+s,t_0))$ *for a.e.* $\xi \in \Xi$;

**(RP2)** $\mathcal{U}(t,t) = 1$;

**(RP3)** $\mathcal{U}(t,\tau)\big(1 \otimes P_\Lambda\big)\mathcal{U}(\tau,t_0)\,\Psi \to \mathcal{U}(t,t_0)\,\Psi$ *as* $\Lambda \uparrow \Lambda_\infty$;

**(RP4)** $\mathcal{U}(t+s,t_0+s)\,\Psi = \mathcal{S}(-s)\,\mathcal{U}(t,t_0)\,\mathcal{S}(s)\,\Psi$ *for any* $s \geq -t_0$;

**(RP5)** $U^\xi(t,\tau)|\psi\rangle$ *and* $U^\xi(\tau,t_0)|\varphi\rangle$ *are stochastically independent* $\mathcal{H}$*-valued random variables;*

**(RP6)** $\big\|\mathcal{U}(t,t_0)\big(1 \otimes |\psi\rangle\big)\big\|_{\mathcal{K}} = \big\|\psi\big\|_{\mathcal{H}}$;

**(RP7)** $t \in \mathbb{R}_+ \mapsto \mathcal{U}(t,0)\,\Psi \in \mathcal{K}$ *is continuous.*

The condition (17) and axioms **(RP1)** and **(RP6)** imply that $\mathcal{F} \subset \mathcal{D}(\mathcal{U}(t,t_0))$ for any $t \geq t_0$. Note that $\mathcal{F} \subset \mathcal{D}(\mathcal{U}(t,t_0))$ does not implies **(RP1)**, which states the existence of a *t-independent* subset $\Xi_\psi \subset \Xi$ of probability one on which $U^\xi(t,t_0)|\psi\rangle$ exists for all $t$'s.

The axioms **(RP1)** to **(RP3)** imply that for any $|\psi\rangle \in \mathcal{H}$ and $t_0 \geq 0$, there is for a.e. $\xi \in \Xi$ a unique quantum trajectory $t \mapsto |\Psi(t)\rangle^\xi = U^\xi(t,t_0)|\psi\rangle$ such that $|\Psi(t_0)\rangle^\xi = |\psi\rangle$. Moreover, $\Psi(t) : \xi \mapsto |\psi(t)\rangle^\xi$ belongs to $\mathcal{K}$ for any $t \geq t_0$.

The axioms **(RP4)** and **(RP5)** mean that quantum trajectories define a stationary process with independent increments in the Hilbert space $\mathcal{H}$ (recall that $\mathbb{P}$ is invariant under $T(s)$). Since $\left\|\mathcal{U}(t,t_0)(1\otimes|\psi\rangle)\right\|_{\mathcal{K}}^2 = \mathbb{E}\,\|U(t,t_0)|\psi\rangle\|^2$, the condition **(RP6)** means that $\|\psi(t)\|^2$ is conserved on average.

In practice, the operators $\mathcal{U}(t,t_0)$ are defined by giving, on a set of probability one, the random operators $U^\xi(t,t_0)$ on $\mathcal{H}$ which appear in (17). By axiom **(RP4)**, these operators satisfy for any $s\geq 0$ and a.e. $\xi$:

$$U^\xi(t)|\psi\rangle \equiv U^\xi(t,0)|\psi\rangle = U^{T(s)\,\xi}(t+s,s)|\psi\rangle \ . \tag{18}$$

To study quantum trajectories, one must define another topology on $\mathcal{K}$ than the norm topology considered in definition 1. The net $(\Psi_\Lambda)_{\Lambda\in L}$ in $\mathcal{K}$ is said to *converge pointwise* to $\Psi\in\mathcal{K}$ if $\||\Psi_\Lambda\rangle^\xi - |\Psi\rangle^\xi\|\to 0$ as $\Lambda\uparrow\Lambda_\infty$ for any $\xi$ in a subset $\Xi_\Psi\subset\Xi$ of probability one. For any $\Lambda\in L$, let $\tau\mapsto\Psi_\Lambda(\tau)$ be a map from $\mathbb{R}_+$ to $\mathcal{K}$ and let $0\leq t\leq\infty$. One says that $(\Psi_\Lambda(\tau))_{\Lambda\in L}$ converges pointwise to $\Psi(\tau)$ *uniformly with respect to* $\tau$ on $[0,t[$ if:

$$\lim_{\Lambda\uparrow\Lambda_\infty}\ \sup_{0\leq\tau\leq t}\||\Psi_\Lambda(\tau)\rangle^\xi - |\Psi(\tau)\rangle^\xi\| = 0$$

for a.e. $\xi\in\Xi$. As follows from the dominated convergence theorem, pointwise convergent $\|.\|_{\mathcal{K}}$-bounded nets are norm convergent.

**Proposition 1** *Let $\{\mathcal{U}_\Lambda;\Lambda\in L\}$ be a family of random propagators satisfying the following conditions for any $|\psi\rangle\in\mathcal{H}$ and $0<t<\infty$:*

*(i)* *$(\mathcal{U}_\Lambda(\tau,t_0)\,1\otimes|\psi\rangle)_{\Lambda\in L}$ converges pointwise uniformly with respect to $(\tau,t_0)$ on $[0,t]^2$;*

*(ii)* *there exists $g\in L^2(\Xi,\mathbb{P})$ such that $\|U_\Lambda^\xi(\tau,t_0)|\psi\rangle\|\leq g(\xi)$ for any $\Lambda\in L$, $0\leq t_0\leq\tau\leq t$ and a.e. $\xi\in\Xi$;*

*(iii) for any* $0 \leq \tau \leq t$, $\sup_{\Lambda \in L} \big\| \big( \mathcal{U}_\Lambda(t,\tau)\, 1 \otimes P_{\Lambda'}\, \mathcal{U}_\Lambda(\tau) - \mathcal{U}_\Lambda(t) \big) 1 \otimes |\psi\rangle \big\|_{\mathcal{K}} \to 0$ *as* $\Lambda' \uparrow \Lambda_\infty$.

*Let* $\mathcal{U}(t,t_0)$ *be the operators on* $\mathcal{K}$ *satisfying* (17) *such that, for any* $|\psi\rangle \in \mathcal{H}$,

$$|\mathcal{U}(t,t_0)\,(1 \otimes |\psi\rangle)\rangle^{\xi} = U^{\xi}(t,t_0)|\psi\rangle = \lim_{\Lambda \uparrow \Lambda_\infty} U^{\xi}_{\Lambda}(t,t_0)|\psi\rangle \quad \textit{for a.e. } \xi \in \Xi \ .$$

*Then* $\{\mathcal{U}(t,t_0); t \geq t_0 \geq 0\}$ *is a random propagator. Moreover, the same statement holds if axiom* **(RP6)** *is replaced by the following condition:*

**(RP6')** *there is* $c > 0$ *such that* $\big\| \mathcal{U}(t,t_0)\, 1 \otimes |\psi\rangle \big\|_{\mathcal{K}} \leq c\, \|\psi\|$ *for any* $|\psi\rangle \in \mathcal{H}$.

*Proof.* By assumptions *(i-ii)* and the dominated convergence theorem, $\mathcal{U}(t,t_0)\, 1 \otimes |\psi\rangle$ is in $\mathcal{K}$ and $\big\| \mathcal{U}_\Lambda(\tau,t_0)\, 1 \otimes |\psi\rangle \big\|_{\mathcal{K}} \to \big\| \mathcal{U}(\tau,t_0)\, 1 \otimes |\psi\rangle \big\|_{\mathcal{K}}$. The pointwise uniform convergence of $\mathcal{U}_\Lambda(\tau,t_0)\, 1 \otimes |\psi\rangle$ and the $\sigma$-additivity of the measure $\mathbb{P}$ imply that the random operators $U^{\xi}(\tau,t_0)$ satisfy **(RP1)**. Moreover, $\mathcal{U}(t,t_0)$ satisfies **(RP6)** or **(RP6')** if this true for all $\mathcal{U}_\Lambda(t,t_0)$, $\Lambda \in L$. Let $\Psi \in \mathcal{F}$ and $t_0 \leq \tau$, $t_0 \leq t$. By the dominated convergence theorem again, $\big\| (\mathcal{U}_\Lambda(\tau,t_0) - \mathcal{U}_\Lambda(t,t_0))\Psi \big\|_{\mathcal{K}}$ converges to $\big\| (\mathcal{U}(\tau,t_0) - \mathcal{U}(t,t_0))\Psi \big\|_{\mathcal{K}}$ uniformly with respect to $\tau$ on $[t_0,t]$. Since the first norm tends to zero as $\tau \to t$ for any $\Lambda \in L$, this implies **(RP7)**. The same argument shows that:

$$\begin{aligned} &\lim_{\Lambda' \uparrow \Lambda_\infty} \big\| \big( \mathcal{U}(t,\tau)\, 1 \otimes P_{\Lambda'}\, \mathcal{U}(\tau) - \mathcal{U}(t) \big) 1 \otimes |\psi\rangle \,\big\|_{\mathcal{K}} \\ &= \lim_{\Lambda \uparrow \Lambda_\infty} \lim_{\Lambda' \uparrow \Lambda_\infty} \big\| \big( \mathcal{U}_\Lambda(t,\tau)\, 1 \otimes P_{\Lambda'}\, \mathcal{U}_\Lambda(\tau) - \mathcal{U}_\Lambda(t) \big) 1 \otimes |\psi\rangle \,\big\|_{\mathcal{K}} = 0 \ . \end{aligned}$$

Assumption *(iii)* has been used to exchange the limits with respect to $\Lambda'$ and $\Lambda$. This proves **(RP3)**. Now $\mathcal{U}$ clearly satisfies **(RP2)** and **(RP5)**. For any $|\psi\rangle \in \mathcal{H}$ and $s \geq 0$, denote $\Xi_{\psi,s}$ the subset of $\Xi$ such that $\mathbb{P}(T(s)\,\Xi_{\psi,s}) = 1$ and $U_{\Lambda}^{T(s)\,\xi}(t,t_0)|\psi\rangle$ converges to $U^{T(s)\,\xi}(t,t_0)|\psi\rangle$ for any $\xi \in \Xi_{\psi,s}$. By translation invariance, one has $\mathbb{P}(\Xi_{\psi,s}) = \mathbb{P}(T(s)\,\Xi_{\psi,s}) = 1$. As a result, $\mathcal{S}(-s)\,\mathcal{U}_\Lambda(t,t_0)\, 1 \otimes |\psi\rangle \to \mathcal{S}(-s)\,\mathcal{U}(t,t_0)\, 1 \otimes |\psi\rangle$ pointwise. By unicity of the limit and (17), **(RP4)** is true for any $\Psi = f \otimes |\psi\rangle$ with $f \in L^{\infty}(\Xi,\mathbb{P})$, $|\psi\rangle \in \mathcal{H}$. $\square$

### 3.3 Existence of the Random Propagator when $\Gamma = \infty$

Let $\Lambda \subset \Lambda_\infty$ be is finite box of $\Lambda_\infty$. Set:

$$N_\Lambda(t) = \sum_{i,j\in\Lambda} N_{i\to j}(t) \ .$$

$(N_\Lambda(t))_{t\in\mathbb{R}}$ is a Poisson process of parameter $\Gamma_\Lambda = \sum_{i,j\in\Lambda}\Gamma_{i\to j}$. For any fixed $(t_0,\xi)\in\mathbb{R}\times\Xi$, the left discontinuities of the counting function $N_\Lambda(t)$ are denoted by:

$$\ldots \le t_\Lambda^{-p} \le \ldots \le t_\Lambda^{-1} \le t_0 < t_\Lambda^{1} \ldots \le t_\Lambda^{p-1} \le t_\Lambda^{p} \le \ldots$$

We set $t_\Lambda^0 = t_0$ and denote by $s_\Lambda^p = t_\Lambda^p - t_\Lambda^{p-1}$ the time delay between the $(p-1)$-th and the $p$-th jumps (or, if $p=0$ or $1$, between the time $t_0$ and the jump immediately preceding or following it). For each $p\in\mathbb{Z}^\star$, the random couple of indices $(i_\Lambda^p, j_\Lambda^p)\in\Lambda^{\times 2}$ is defined by demanding that the $p$-th jump is a jump $(i_\Lambda^p, j_\Lambda^p)$, i.e.

$$t^n_{i_\Lambda^p\to j_\Lambda^p} = t_\Lambda^p \quad \text{for some } n\in\mathbb{Z}^\star \ . \tag{19}$$

Since the processes $(N_{i\to j}(t))_{t\in\mathbb{R}}$, $i,j\in\Lambda$, are independent, the random variables $s_\Lambda^1,\ldots,s_\Lambda^p$ and $(i_\Lambda^1, j_\Lambda^1),\ldots,(i_\Lambda^p, j_\Lambda^p)$ are mutually independent.

Let us project the infinite dimensional Hilbert space $\mathcal{H}$ of the system onto the finite dimensional Hilbert space $P_\Lambda\mathcal{H} = \mathrm{span}\{|i\rangle; i\in\Lambda\}$, and forget all jumps $(i,j)\notin\Lambda$. Assume that the RW is a.s. equal to $|\psi\rangle\in\mathcal{H}$ at time $t_0$: $|\Psi_\Lambda(t_0)\rangle^\xi = |\psi\rangle$ for a.e. $\xi$. Then, the RW at time $t$ in the stochastic scheme of section 2.1 is given by $|\Psi_\Lambda(t)\rangle^\xi = U_\Lambda^\xi(t,t_0)|\psi\rangle$, with:

$$U_\Lambda(t,t_0) = \begin{cases} U_0(t-t_0) & \text{if } t_0\le t < t_\Lambda^1 \\ U_0(t-t_\Lambda^p)\, W_{i_\Lambda^p\to j_\Lambda^p}\, U_0(s_\Lambda^p)\ldots W_{i_\Lambda^1\to j_\Lambda^1}\, U_0(s_\Lambda^1) & \text{if } t_0 < t_\Lambda^p \le t < t_\Lambda^{p+1} \end{cases} \tag{20}$$

and:

$$U_0(s) \equiv e^{-\mathrm{i}s(H+K)} \ , \ s\ge 0 \ . \tag{21}$$

**Proposition 2** *Let assumption D hold. The operators on $\mathcal{K}$ associated to the $U_\Lambda(t,t_0)$'s by (17) are denoted by $\mathfrak{U}_\Lambda(t,t_0)$. Then $\mathfrak{U}_\Lambda(t,t_0)$ satisfy the axioms* **(RP1-RP5)** *of definition 1 and, for any $t_0 \le t$, $U_\Lambda(t,t_0) \in \mathcal{D}(\mathcal{L}_H)$ almost surely.*

*Proof.* It follows from the self-adjointness of $H$ and from a Dyson expansion of $U_0(s)$ in powers of $K$ that $\|U_0(s)\| \le e^{s\|K\|}$ for any $s \ge 0$. The Dyson series exists since $K \in \mathcal{D}(\mathcal{L}_H)$ (see proposition 5 below). This yields:

$$\big\|U_\Lambda(t,t_0)\big\| \le w_\Lambda^{N_\Lambda([t,t_0])}\, e^{(t-t_0)\|K\|} \ , \tag{22}$$

with $w_\Lambda = \max_{i,j\in\Lambda} \|W_{i\to j}\|$. Since $N_\Lambda([t,t_0]) < \infty$ a.s., $U_\Lambda(t,t_0)$ is a.s. bounded and a.s. in the subalgebra $\mathcal{D}(\mathcal{L}_H)$. The fact that $\mathfrak{U}_\Lambda(t,t_0)(1\otimes|\psi\rangle)$ belongs to $\mathcal{K}$ and **(RP3)** are consequences of (22) and of the dominated convergence theorem. **(RP4)** follows from the transformation rules of $t_\Lambda^p$ and $(i_\Lambda^p, j_\Lambda^p)$ under the time translations: if $\xi \longrightarrow T(s)\,\xi$ and $t_0 \longrightarrow t_0 + s$, then $t_\Lambda^p \longrightarrow t_\Lambda^p + s$ and $(i_\Lambda^p, j_\Lambda^p) \longrightarrow (i_\Lambda^p, j_\Lambda^p)$, which imply:

$$U_\Lambda^\xi(t)|\psi\rangle \equiv U_\Lambda^\xi(t,0)|\psi\rangle = U_\Lambda^{T(s)\,\xi}(t+s,s)|\psi\rangle$$

for any $s \ge 0$. $\square$

The main result of this section is the following theorem. It is proven in section 7.

**Theorem 1** *Let assumptions A to D hold and $|\psi\rangle \in \mathcal{H}$. Then $(\mathfrak{U}_\Lambda(t,t_0)\,1\otimes|\psi\rangle)_{\Lambda\in L}$ converges pointwise as $\Lambda \uparrow \Lambda_\infty$ uniformly with respect to $(t,t_0)$ on compacts of $\mathbb{R}_+^2$. Let us denote its limit by $\mathfrak{U}(t,t_0)\,1\otimes|\psi\rangle : \xi \mapsto U^\xi(t,t_0)\,|\psi\rangle$. For any $r \ge 0$ and $\eta > 0$ such that $r \le r_1\eta/(d+\eta)$, $r \le r_2$ and $r \le r_3$, there is a random constant $c_t > 0$ such that the following property holds for a.e. $\xi \in \Xi$:*

$$\big|\langle i|U^\xi(t,t_0)|j\rangle\big| \le c_t^\xi\, \min\big\{|i|,|j|\big\}^\eta e^{-r|i-j|} \ , \ i,j \in \Lambda_\infty \ . \tag{23}$$

*Suppose moreover that* $|\psi\rangle \in \mathcal{D}(H)$ *and that there is* $0 < r_0 < r_1/8$, $r_0 \leq r_2$, $r_0 \leq r_3$ *such that the domain* $\mathcal{D}(H)$ *of* $H$ *is invariant under the operator:*

$$\Delta_{r_0} = \sum_{i,j \in \Lambda_\infty} e^{-r_0|i-j|} \, |i\rangle\langle j| \; . \tag{24}$$

*Then* $U(t,t_0)|\psi\rangle \in \mathcal{D}(H)$ *almost surely.*

**Remark 1** The subset of $\Xi$ of probability 1 on which $U_\Lambda^\xi(t,t_0)|\psi\rangle \to U^\xi(t,t_0)|\psi\rangle$ for all $t$'s depends on $\psi$.

**Remark 2** The random evolution operator $U(t)$ may be unbounded with probability one. Actually, let $H = K = 0$ and $W_{i\to j} = 1 + |j\rangle\langle i|$. It follows from (20) that:

$$\langle j|U(t)|i\rangle \geq \lim_{\Lambda \uparrow \Lambda_\infty} \langle j| \Big( 1 + \sum_{p=1}^{N_\Lambda(t)} |j_\Lambda^p\rangle\langle i_\Lambda^p| \Big) |i\rangle = \delta_{ij} + N_{i\to j}(t) \; .$$

By the lemma 4 in the appendix, $\big\| U(t) \big\|^2 \geq \sup_{i \in \Lambda_\infty} \big\{ 1 + 2N_{i\to i}(t) + \sum_{j\in\Lambda_\infty} N_{i\to j}^2(t) \big\}$ is a.s. infinite.

**Remark 3** Under the assumptions of the theorem, $(U_\Lambda(\tau,t_0)^*|\psi\rangle)_{\Lambda\in L}$ converges pointwise uniformly w.r.t. $(\tau,t_0)$ on compacts, for any $|\psi\rangle \in \mathcal{H}$. In fact, one obtains the adjoint operators $U_\Lambda(t,t_0)^*$ of $U_\Lambda(\tau,t_0)$ by inverting $t$ and $t_0$ and replacing $W_{i\to j}$ and $K$ by their adjoints in (20). Since $W_{i\to j}^*$ satisfy assumption B if and only if this is true for $W_{i\to j}$, theorem 1 is also true for the adjoints $U_\Lambda(t,t_0)^*$.

It follows from theorem 1 and from the $\sigma$-additivity of the measure $\mathbb{P}$ that, with probability one, $U_\Lambda(t) \to U(t)$ strongly on the dense subspace $\mathcal{D}$ of finite linear combinations the basis vectors $|i\rangle$, $i \in \Lambda_\infty$. Assume that $\|W_{i\to j}\| \leq 1$ for any $i,j \in \Lambda_\infty$. By (22), the random operators $U_\Lambda(t)$ are uniformly bounded in $\Lambda$. Therefore $U(t)$ is bounded with probability 1. By an $\epsilon/3$ argument, $U_\Lambda(t) \to U(t)$ strongly on $\mathcal{H}$ a.s. (which means: $U_\Lambda^\xi(t)|\psi\rangle \to U^\xi(t)|\psi\rangle$ for any

$|\psi\rangle \in \mathcal{H}$ on a $\psi$-independent set of $\xi$'s of probability one). Similarly, $U_\Lambda(t)^*$ converge strongly on $\mathcal{H}$ and therefore $U_\Lambda(t)^* \to U(t)^*$ strongly a.s.. The statement below is obtained from (20) and an $\epsilon/3$ argument.

**Corollary 1** *Let the assumptions of theorem 1 hold. If $K$ is self-adjoint and $W_{i\to j}$ is unitary for any $i, j \in \Lambda_\infty$, then $U(t, t_0)$ is almost surely unitary.*

# 4 Average Evolution of Observables

The aim of this section is to establish a link between the above quantum jumps approach and the master equation approach of open quantum systems.

## 4.1 Markov Equations and Semigroups

Let us first recall few basic facts on the master equation approach. Consider a quantum system in contact with a thermal bath. If the decay time of correlations of the coupling agent of the bath is much shorter than the typical relaxation time of the system, memory effects can be neglected and the evolution of the density matrix $\rho(t)$ of the system is approximately given by a first-order linear differential equation [13]:

$$\frac{d\rho(t)}{dt} = \mathcal{L}_* \, \rho \ , \tag{25}$$

where $\mathcal{L}_*$ is a linear operator from the Banach space $L^1(\mathcal{H})$ of trace-class operators on $\mathcal{H}$ into itself. More precise conditions under which such a so-called *Markovian master equation* gives a correct description of the dynamics of open quantum systems have been given elsewhere [15, 16, 17, 18, 19]. The solution $\rho(t) = \Phi_*(t)\rho(0)$ of (25) is given by a one-parameter semigroup $\{\Phi_*(t); t \geq 0\}$ of positive linear operators $\Phi_*(t) : L^1(\mathcal{H}) \to L^1(\mathcal{H})$. Actually, it has been shown by Kraus [40] that for physical Markovian master equations, the operators $\Phi_*(t)$ satisfy a condition stronger than positivity, called *complete positivity.* An operator $\mathcal{J}_*$ on $L^1(\mathcal{H})$ is completely positive (CP) if for any positive integer $m$ and any $L^1(\mathcal{H})$-valued $m \times m$ matrix $\rho^{(m)} = (\rho_{\mu\nu})_{\mu,\nu=1,..m}$,

$$\rho^{(m)} \geq 0 \quad \Rightarrow \quad \left(\mathcal{J}_* \rho_{\mu\nu}\right)_{\mu,\nu=1,\ldots,m} \geq 0 \ .$$

The $m \times m$ matrix in the right hand side is the matrix of matrix elements $\mathcal{J}_* \rho_{\mu\nu}$. A similar definition holds for operators on a $C^*$-algebra $\mathcal{A}$ [41]. Complete positivity and positivity are equivalent only if $\mathcal{A}$ is commutative [42].

In this paper, the observables $A$ of the system are assumed to be elements of the von Neumann algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. This case is simpler and has been mostly studied in the literature. Note, however, that the $C^*$-algebra $\mathcal{A}$ describing electrons in solids at the thermodynamic limit is different from $\mathcal{B}(\mathcal{H})$ [43]. For such systems, one has to work with the Banach space $L^1(\mathcal{A}, \mathcal{T})$ and with its dual, the von Neumann algebra $L^\infty(\mathcal{A}, \mathcal{T})$, where $\mathcal{T}$ is the trace per unit volume on $\mathcal{A}$ [43]. Then the Hilbert space $\mathcal{H}$ has to be replaced by the Hilbert space $L^2(\mathcal{A}, \mathcal{T})$ of the G.N.S. representation of $\mathcal{T}$.

As is well-known [44], $\mathcal{B}(\mathcal{H})$ is the dual of $L^1(\mathcal{H})$ under the duality:

$$A : \rho \in L^1(\mathcal{H}) \mapsto \mathrm{tr}(A\,\rho) \ . \tag{26}$$

The evolution of observables $A(t)$ in the Heisenberg picture is given by the dual operators of the $\Phi_*(t)$'s, $\Phi(t) : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ (in fact, the thermal averages must coincide in the two pictures, which implies $\mathrm{tr}(A(t)\rho(0)) = \mathrm{tr}(A(0)\rho(t))$ ). The family $\Phi = \{\Phi(t); t \geq 0\}$ of the dual operators is a semigroup on $\mathcal{B}(\mathcal{H})$. Following the definition of ref. [1, 45], a *Quantum Dynamical Semigroup* (QDS) on $\mathcal{B}(\mathcal{H})$ is an ultraweakly continuous semigroup of CP operators on $\mathcal{B}(\mathcal{H})$ preserving the identity, i.e, it is a semigroup $\Phi = \{\Phi(t); t \geq 0\}$ on $\mathcal{B}(\mathcal{H})$ satisfying, for any

$t \geq 0$ and $A \in \mathcal{B}(\mathcal{H})$:

**(QDS1)** $\Phi(t) : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is ultraweakly continuous (normal);

**(QDS2)** $\Phi(t)A \to A$ ultraweakly as $t \to 0$;

**(QDS3)** $\Phi(t)$ is CP;

**(QDS4)** $\Phi(t)\,1 = 1$ for any $t \geq 0$.

Note that the ultraweak topology is sometimes called '$\sigma$-weak topology' in the literature. **(QDS1)** holds automatically if $\Phi(t)$ is the dual of an operator on $L^1(\mathcal{H})$ for the duality (26) (the ultraweak topology on $\mathcal{B}(\mathcal{H})$ coincide with the $*$-weak topology for this duality [44]). The two first conditions imply the existence of a generator $\mathcal{L}$ of the semigroup, defined on an ultraweakly-dense domain $D(\mathcal{L})$ in $\mathcal{B}(\mathcal{H})$. $\mathcal{L}\,A$ is, by definition, the derivative $d\Phi(t)A/dt$, which exists in the ultraweak topology for any $A \in D(\mathcal{L})$ [44]. The last condition **(QDS4)** is equivalent to the requirement that $\Phi_*(t)$ be trace-preserving. The general form of the (bounded) generators $\mathcal{L}$ of norm continuous QDS has been found by Lindblad [1]; the case of unbounded generator, i.e. of non norm continuous QDS, has been treated by Davies [46]. The result of these authors is that $\mathcal{L}_* = -\mathcal{L}_H + \mathcal{C}_*$ is given by (1). In the particular case of no coupling between the system and the bath, $\mathcal{L} = \mathcal{L}_H$ generates an ultraweakly continuous group $\{e^{t\mathcal{L}_H}; t \in \mathbb{R}\}$ of $*$-automorphisms of $\mathcal{B}(\mathcal{H})$ (given by $e^{t\mathcal{L}_H}A = e^{itH}Ae^{-itH}, A \in \mathcal{B}(\mathcal{H})$). This is not true if the coupling with the bath is turned on. In this sense, the notion of QDS generalizes to open quantum systems the notion of one-parameter group of $*$-automorphism, describing the dynamics of closed systems.

## 4.2 Random Propagators and Quantum Dynamical Semigroups

It is shown in this subsection that the average evolution of observables of a system with quantum trajectories $t \mapsto |\Psi(t)\rangle^\xi = U^\xi(t,t_0)|\psi\rangle$, where $\{\mathfrak{U}(t,t_0); t \geq t_0 \geq 0\}$ is a random propagator, define a quantum dynamical semigroup.

Let $\mathcal{U}$ be a RP, $t \geq t_0 \geq 0$ and $A \in \mathcal{B}(\mathcal{H})$. Define the hermitian form:

$$h_{A,t-t_0}(\varphi,\psi) = \big(\mathcal{U}(t,t_0)\,1\otimes|\varphi\rangle\,,1\otimes A\,\mathcal{U}(t,t_0)\,1\otimes|\psi\rangle\big)_{\mathcal{K}}\ ,\ |\varphi\rangle,|\psi\rangle \in \mathcal{H}\ .$$

The right hand side depends on $t, t_0$ only through their difference because of axiom **(RP4)** and of the stationarity of $\mathbb{P}$ under $T(s)$. By axiom **(RP6)**, $|h_{A,t}(\varphi,\psi)| \leq \|A\|\|\varphi\|\|\psi\|$. The Riesz lemma shows that there exists a unique bounded operator $\Phi(t) : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that, for any $|\varphi\rangle, |\psi\rangle \in \mathcal{H}$, $A \in \mathcal{B}(\mathcal{H})$ and $\tau \geq 0$,

$$\begin{aligned}\langle\varphi|\Phi(t)A|\psi\rangle &= \big(\mathcal{U}(t+\tau,\tau)\,1\otimes|\varphi\rangle\,,1\otimes A\,\mathcal{U}(t+\tau,\tau)\,1\otimes|\psi\rangle\big)_{\mathcal{K}} \\ &= \mathbb{E}\,\langle U(t+\tau,\tau)\varphi|A|U(t+\tau,\tau)\psi\rangle\ . \end{aligned} \tag{27}$$

**Proposition 3** *Let $\mathcal{U}$ be a random propagator. Then the family $\Phi = \{\Phi(t); t \geq 0\}$ of operators on $\mathcal{B}(\mathcal{H})$ defined by* (27) *is a quantum dynamical semigroup on $\mathcal{B}(\mathcal{H})$. For any $t \geq 0$, $\Phi(t)$ is the dual under the duality* (26) *of the CP operator $\Phi_*(t)$ on $L^1(\mathcal{H})$, which acts on $T = \sum_{n=0}^{\infty} t_n|\psi_n\rangle\langle\varphi_n| \in L^1(\mathcal{H})$ as follows:*

$$\Phi_*(t)\,T = \sum_{n=0}^{\infty} t_n\,\mathbb{E}\,|U(t)\psi_n\rangle\langle U(t)\varphi_n|\ . \tag{28}$$

*Proof.* Let $A \in \mathcal{B}(\mathcal{H})_+$ and $\rho = \sum_n \rho_n|\psi_n\rangle\langle\psi_n| \in L^1(\mathcal{H})_+$, where $\{|\psi_n\rangle; n \in \mathbb{N}\}$ is an orthonormal basis of $\mathcal{H}$ and $\rho_n \geq 0$, $\sum_n \rho_n < \infty$. Since $\Phi(t)A$ is bounded, one has $(\Phi(t)A)\,\rho \in L^1(\mathcal{H})$. One obtains from the cyclicity of the trace and the monotone convergence theorem for sequences:

$$\begin{aligned}\mathrm{tr}\big((\Phi(t)A)\rho\big) &= \sum_{n=0}^{\infty}\rho_n\,\mathbb{E}\,\big\|A^{\frac{1}{2}}\,U(t)|\psi_n\rangle\big\|^2 = \sum_{n=0}^{\infty}\rho_n\ \mathrm{tr}\big(A^{\frac{1}{2}}\,\mathbb{E}\,|U(t)\,\psi_n\rangle\langle U(t)\psi_n|\,A^{\frac{1}{2}}\big) \\ &= \mathrm{tr}\big(A\,\Phi_*(t)\,\rho\big)\ .\end{aligned}$$

As any bounded (trace-class) operator can be expressed as a sum of four positive bounded (trace-class) operators, it follows that $\Phi_*(t) : L^1(\mathcal{H}) \to L^1(\mathcal{H})$ and that $\Phi(t)$ is the dual operator of $\Phi_*(t)$. Hence the operators $\Phi(t)$ are ultraweakly continuous. It is straightforward to check on (27) that they are CP. By axiom **(RP6)**, $\langle\psi|\Phi(t)\,1|\psi\rangle = \|\psi\|^2$ for any $|\psi\rangle \in \mathcal{H}$. It follows from the polarization identity that $\Phi(t)\,1 = 1$. By **(RP3)**,

$$\begin{aligned}\langle\varphi|\Phi(t+\tau)A|\psi\rangle &= \lim_{\Lambda\uparrow\Lambda_\infty}\big(\mathcal{U}(t+\tau,\tau)(1\otimes P_\Lambda)\,\mathcal{U}(\tau)\,1\otimes|\varphi\rangle\,,(1\otimes A)\;\mathcal{U}(t+\tau,\tau)\times\\ &\quad (1\otimes P_\Lambda)\,\mathcal{U}(\tau)\,1\otimes|\psi\rangle\big)_{\mathcal{K}}\\ &= \lim_{\Lambda\uparrow\Lambda_\infty}\langle\varphi|\Phi(\tau)\big(P_\Lambda(\Phi(t)A)P_\Lambda\big)|\psi\rangle = \langle\varphi|\Phi(\tau)\Phi(t)A|\psi\rangle\ .\end{aligned}$$

The second line is obtained by using **(RP5)** and the ultraweak continuity of $\Phi(t)$. Hence $\Phi$ is a semigroup. It is clear from axioms **(RP2)** and **(RP7)** that $\Phi(t)A \to A$ weakly as $t \to 0$. But $\Phi(t)A$ is uniformly bounded with respect to $t$. By using the equivalence [44] of the ultraweak and weak topologies on balls of $\mathcal{B}(\mathcal{H})$, this proves **(QDS2)**. $\square$

**Remark 4** Let $\mathcal{U}$ is a family of operators on $\mathcal{K}$ satisfying (17) and all axioms of RP but with axiom **(RP6)** replaced by the condition **(RP6')** of proposition 1. Then, by the same arguments as above, the operators $\Phi(t)$ in (27) satisfy the properties **(QDS1-QDS3)**.

From a physical point of view, the QDS $\Phi$ gives the average dynamics of the system, as defined as follows. The stochastic density matrix at time $t \geq 0$ is given in the Schrödinger picture in terms of the initial density matrix $\rho(0) \in L^1(\mathcal{H})_+$ by the formal expression:

$$\rho^\xi(t) = U^\xi(t)\,\rho(0)\,U^\xi(t)^*\ .$$

The *average dynamics* is then given in the Schrödinger picture by the average density matrices:

$$\mathbb{E}\,\rho^\xi(t) = \Phi_*(t)\rho(0) \in L^1(\mathcal{H})_+\ .$$

In the Heisenberg picture, it is given by the average observables:

$$\mathbb{E}\,A^\xi(t) = \Phi(t)A(0) \in \mathcal{B}(\mathcal{H})\ .$$

These relations provide the link between the RW approach and the standard approach by means of statistical ensembles of systems (see e.g. [6, 9]). For example, let the system be initially in a pure state, i.e. $\rho(0)$ is a projector $P_\psi$ on a normalized wavefunction $|\psi\rangle \in \mathcal{H}$. Then $\rho^\xi(t) = P_{|\Psi\rangle^\xi(t)}$, with $|\Psi(t)\rangle^\xi = U^\xi(t)|\psi\rangle$. For any fixed outcome, the system remains in a pure state at any time. However, the average state $\mathbb{E}\, P_{|\Psi\rangle^\xi(t)}$ at time $t > 0$ is not a pure state.

### 4.3 Generators of the Average Evolution

It is shown in this subsection that if $K$ is chosen according to (3), the operators $\mathcal{U}(t, t_0)$ of theorem 1 define a RP. The QDS associated to $\mathcal{U}$ is determined explicitly by calculating its generator.

Let $\Lambda \in L$ and $U_\Lambda(t)$ be the random evolution operators defined by (20). By the same arguments as above (section 4.2),

$$\langle\varphi|\Phi_\Lambda(t)A|\psi\rangle = \big(\mathcal{U}_\Lambda(t)\, 1 \otimes |\varphi\rangle\, , 1 \otimes A\ \mathcal{U}_\Lambda(t)\, 1 \otimes |\psi\rangle\big)_{\mathcal{K}}\ \ ,\ \ |\psi\rangle, |\varphi\rangle \in \mathcal{H} \tag{29}$$

defines a unique CP bounded operator $\Phi_\Lambda(t) : A \in \mathcal{B}(\mathcal{H}) \mapsto \mathbb{E}\ U_\Lambda(t)^* A\, U_\Lambda(t) \in \mathcal{B}(\mathcal{H})$. Note that even if $\mathcal{U}_\Lambda(t)$ does not satisfy axiom **(RP6)**, the hermitian form in (29) is bounded thanks to (22):

$$|\langle\varphi|\Phi_\Lambda(t)A|\psi\rangle| \leq \mathbb{E}\ w_\Lambda^{2N_\Lambda(t)}\, e^{2t\|K\|}\big\|\psi\big\|\big\|\varphi\big\|\big\|A\big\| = e^{\,t\,\Gamma_\Lambda(w_\Lambda^2-1)+2t\|K\|}\,\big\|\psi\big\|\,\big\|\varphi\big\|\,\big\|A\big\|\ . \tag{30}$$

The main result of this section is the following:

**Theorem 2** *Let the assumptions B and D be satisfied. Assume that:*

$$\sup_{i\in\Lambda_\infty}\sum_{j\in\Lambda_\infty}\Gamma_{i\to j} < \infty \quad , \quad \sup_{i\in\Lambda_\infty}\sum_{j\in\Lambda_\infty}\Gamma_{j\to i} < \infty\ . \tag{31}$$

*Then:*

*(1) for any $t \geq 0$ and $A \in \mathcal{B}(\mathcal{H})$, $(\Phi_\Lambda(t)A)_{\Lambda \in L}$ converges ultrastrongly as $\Lambda \uparrow \Lambda_\infty$ to a bounded operator $\Phi(t)A$. Moreover, the infinite series in* (3) *converges ultrastrongly as $\Lambda \uparrow \Lambda_\infty$ and, if assumption A holds, its sum $K$ satisfies assumption C with $r_3 = \min\{r_1/2, r_2\}$.*

*(2) If $K$ is given by* (3) *(respectively, by* (3) *up to a self-adjoint bounded operator $H'$), then $\Phi = \{\Phi(t); t \geq 0\}$ is a QDS on $\mathcal{B}(\mathcal{H})$ of generator $\mathcal{L} = \mathcal{L}_H + \mathcal{C}$ (resp. $\mathcal{L} = \mathcal{L}_H + \mathcal{C} + \mathcal{L}_{H'}$), where $\mathcal{C}$ is the Lindblad bounded generator defined by the ultrastrongly convergent sums:*

$$\mathcal{C}A = \sum_{i,j \in \Lambda_\infty} \Big( L_{i\to j}^* A L_{i\to j} - \frac{1}{2} \big\{ L_{i\to j}^* L_{i\to j}, A \big\} \Big) \ , \ A \in \mathcal{B}(\mathcal{H}) \ . \tag{32}$$

*The Lindblad operators $L_{i\to j}$ are given by:*

$$L_{i\to j} = \sqrt{\Gamma_{i\to j}} \big( W_{i\to j} - 1 \big) \ . \tag{33}$$

*(3) If $K$ is given by* (3) *up to a self-adjoint operator and assumptions A and C hold, the operators $\mathfrak{U}(t, t_0)$ in theorem* 1 *define a random propagator $\mathfrak{U}$. Moreover, $\Phi$ is the QDS associated to $\mathfrak{U}$, i.e. for any $|\psi\rangle, |\varphi\rangle \in \mathcal{H}$, $A \in \mathcal{B}(\mathcal{H})$ and $t \geq 0$,*

$$\langle \varphi | \Phi(t) A | \psi \rangle \equiv \lim_{\Lambda \uparrow \Lambda_\infty} \mathbb{E} \ \langle U_\Lambda(t) \varphi | A | U_\Lambda(t) \psi \rangle = \mathbb{E} \ \langle U(t) \varphi | A | U(t) \psi \rangle \ . \tag{34}$$

The brackets denote the anticommutator: $\{A, B\} = AB + BA$. The property *(3)* shows that one can invert the mean and the limit $\Lambda \uparrow \Lambda_\infty$. Formula (32) is the expression of the generators of QDS found by Lindblad [1]. The proof of theorem 2 is based upon the explicit calculation of the generators of the semigroups $\Phi_\Lambda$ for $\Lambda \in L$. We perform this calculation in the remaining of this section. The proof of theorem 2 will be completed in section 7.

**Proposition 4** *Let assumptions D hold and $\Lambda \in L$. Denote $K_1$ and $K_2$ the self-adjoint and anti self-adjoint parts of $K$:*

$$K_1 = \frac{1}{2} \big( K + K^* \big) \quad , \quad K_2 = \frac{1}{2\mathrm{i}} \big( K - K^* \big) \ .$$

*Then the operators $\Phi_\Lambda(t)$ define an ultraweakly continuous semigroup $\Phi_\Lambda$ on $\mathcal{B}(\mathcal{H})$ of generator $\mathcal{L} = \mathcal{L}_H + \mathcal{C}_\Lambda$, where the bounded operator $\mathcal{C}_\Lambda$ on $\mathcal{B}(\mathcal{H})$ is given by:*

$$\mathcal{C}_\Lambda\, A = \mathrm{i}\big[K_1, A\big] + \big\{K_2, A\big\} + \sum_{i,j\in\Lambda} \Gamma_{i\to j}\big(W^*_{i\to j}\, A\, W_{i\to j} - A\big)\ . \tag{35}$$

**Corollary 2** *Let assumption D hold. The following conditions are equivalent:*

*(1)* $\{\mathfrak{U}_\Lambda(t); t\geq 0\}$ *is a random propagator;*

*(2)* $\{\Phi_\Lambda(t); t\geq 0\}$ *is a quantum dynamical semigroup;*

*(3) $K$ is given, up to a self-adjoint operator, by* $K_\Lambda = \dfrac{1}{2\mathrm{i}}\displaystyle\sum_{i,j\in\Lambda}\Gamma_{i\to j}\big(W^*_{i\to j}+1\big)\big(W_{i\to j}-1\big)$.

*If one of these condition is satisfied, then the bounded part $\mathcal{C}_\Lambda$ of the generator of $\Phi_\Lambda$ is given, up to a Liouvillian $\mathcal{L}_{H_1} = \mathrm{i}[H_1, .]$ with $H_1$ a self-adjoint bounded operator, by:*

$$\mathcal{C}_\Lambda\, A = \sum_{i,j\in\Lambda}\Big(L^*_{i\to j}\, A\, L_{i\to j} - \frac{1}{2}\big\{L^*_{i\to j}L_{i\to j}, A\big\}\Big)\ , \tag{36}$$

*where $L_{i\to j}$ are given by* (33).

*Proof of the proposition.* We first prove that the evolution operators $\mathfrak{U}_\Lambda(t)$ satisfy the axiom **(RP7)** of definition 1. It has been shown in section 3.3 that $\mathfrak{U}_\Lambda(t)$ satisfy also axioms **(RP1-RP5)**; by the remark 4 above and (30), it will then follows from **(RP7)** that $\Phi_\Lambda$ is an ultraweakly continuous semigroup.

Let $|\psi\rangle\in\mathcal{H}$, $t\geq 0$ and $0\leq\tau\leq 1$. One has $\mathbb{P}(N_\Lambda([t+\tau,t])=0) = e^{-\tau\Gamma_\Lambda}$. It follows from (22), the Cauchy-Schwartz inequality and axiom **(RP4)** that $\|(\mathfrak{U}_\Lambda(t+\tau)-\mathfrak{U}_\Lambda(t))1\otimes|\psi\rangle\|^2_{\mathcal{K}}$ is bounded by:

$$\begin{aligned}&\int_{N^\xi_\Lambda([t+\tau,t])\geq 1} d\mathbb{P}(\xi)\, \big\|\big(U^\xi_\Lambda(t+\tau)-U^\xi_\Lambda(\tau)\big)|\psi\rangle\big\|^2 + \int d\mathbb{P}(\xi)\,\big\|\big(U_0(\tau)-1\big)U^\xi_\Lambda(t)|\psi\rangle\big\|^2\\ &\leq\ 4\,\big(1-e^{-\tau\Gamma_\Lambda}\big)^{\frac12}\big(\mathbb{E}\ w_\Lambda^{4N_\Lambda(t+1)}\,e^{4(t+1)\|K\|}\big)^{\frac12}\big\|\psi\big\|^2 + e^{-\tau\Gamma_\Lambda}\,\mathbb{E}\,\big\|\big(U_0(\tau)-1\big)U^\xi_\Lambda(t)|\psi\rangle\big\|^2\ .\end{aligned}$$

As $\tau \mapsto U_0(\tau) = e^{-i\tau(H+K)}$ is strongly continuous, one concludes from the dominated convergence theorem that $(\mathcal{U}_\Lambda(t+\tau) - \mathcal{U}_\Lambda(t))1 \otimes |\psi\rangle \to 0$ as $\tau \to 0+$. This shows that $\mathcal{U}_\Lambda(t)$ satisfies **(RP7)**.

Hence $\Phi_\Lambda$ is an ultraweakly continuous semigroup. Let $\mathcal{L}_H + \mathcal{C}_\Lambda$ be its generator and $\mathcal{L}_H + \mathcal{C}_0$ the corresponding generator for $\Lambda = \emptyset$:

$$\Phi_0^*(t)A = U_0(t)^* \, A \, U_0(t) = e^{t(\mathcal{L}_H + \mathcal{C}_0)} \, A \; .$$

For any $|\psi\rangle \in \mathcal{D}(H)$, $\mathrm{i}\, d\,U_0(t)|\psi\rangle/dt = (H+K)|\psi\rangle$ (see proposition 5 below). If $A \in \mathcal{D}(\mathcal{L}_H)$, then

$$t^{-1}\big(\Phi_0(t)A - A\big) \to \mathrm{i}\big(H + K^*\big)A - \mathrm{i}A\big(H+K\big)$$

weakly on $\mathcal{D}(H)$ as $t \to 0$. Thus:

$$\mathcal{C}_0 \, A = \mathrm{i}\big[K_1, A\big] + \big\{K_2, A\big\} \; , \; A \in \mathcal{B}(\mathcal{H}) \; . \tag{37}$$

Let $0 \le t \le \infty$ and $\epsilon > 2\|K\|$. As $\|\Phi_0(t)\| \le e^{2t\|K\|}$, proposition 3.1.6. of [44] shows that:

$$\big(e^{-t\epsilon}\, \Phi_0(t) - 1\big)\big(\epsilon - \mathcal{L}_H - \mathcal{C}_0\big)^{-1} A = -\int_0^t ds\, e^{-s\epsilon}\, \Phi_0(s)A \; , \; A \in \mathcal{B}(\mathcal{H}) \; . \tag{38}$$

It follows from (20) that for any $p \in \mathbb{N}$,

$$\begin{aligned}\int_{t_\Lambda^p}^{t_\Lambda^{p+1}} dt\, e^{-t\epsilon}\, U_\Lambda(t)^* A\, U_\Lambda(t) \;&=\; e^{-t_\Lambda^p \epsilon}\, \Phi_0(s_\Lambda^1)\, \mathcal{W}_{i_\Lambda^1 \to j_\Lambda^1} \ldots \Phi_0(s_\Lambda^p)\, \mathcal{W}_{i_\Lambda^p \to j_\Lambda^p}\big(1 - e^{-s_\Lambda^{p+1}\epsilon}\Phi_0(s_\Lambda^{p+1})\big) \times \\ &\quad \big(\epsilon - \mathcal{L}_H - \mathcal{C}_0\big)^{-1} A \; ,\end{aligned} \tag{39}$$

with $\mathcal{W}_{i\to j} A = W_{i\to j}^* A\, W_{i\to j}$ for any $A \in \mathcal{B}(\mathcal{H})$. Note that, by hypothesis, $\mathcal{W}_{i\to j}$ and $\mathcal{C}_0$ leave $\mathcal{D}(\mathcal{L}_H)$ invariant. All the operators in the right hand side of (39) are stochastically independents. $s_\Lambda^q$ is distributed according to an exponential law with parameter $\Gamma_\Lambda$, so that, thanks to (38),

$$\mathbb{E}\; e^{-s_\Lambda^q \epsilon}\, \Phi_0(s_\Lambda^q)\, A = \Gamma_\Lambda\big(\epsilon + \Gamma_\Lambda - \mathcal{L}_H - \mathcal{C}_0\big)^{-1} A \; , \; A \in \mathcal{B}(\mathcal{H}) \, , \; q = 1, \ldots, p \; .$$

The formula (8) yields:

$$\mathcal{W}_\Lambda \, A = \mathbb{E}\ \mathcal{W}_{i_\Lambda^q \to j_\Lambda^q} A = \frac{1}{\Gamma_\Lambda} \sum_{i,j \in \Lambda} \Gamma_{i\to j} W_{i\to j}^* A \, W_{i\to j} \ , \ A \in \mathcal{B}(\mathcal{H}), q = 1, \dots, p \ .$$

The mean value of the left hand side of (39) is:

$$\Phi_\Lambda^{(p)}(\epsilon)\, A = \Big(\Gamma_\Lambda \big(\epsilon + \Gamma_\Lambda - \mathcal{L}_H - \mathcal{C}_0\big)^{-1} \mathcal{W}_\Lambda\Big)^p \big(\epsilon + \Gamma_\Lambda - \mathcal{L}_H - \mathcal{C}_0\big)^{-1} A \ .$$

By (22) and (38), the norm of $(\epsilon + \Gamma_\Lambda - \mathcal{L}_H - \mathcal{C}_0)^{-1} \mathcal{W}_\Lambda$ is bounded by $w_\Lambda^2/(\epsilon + \Gamma_\Lambda - 2\|K\|)$. Therefore the series $\sum_{p=0}^\infty \Phi_\Lambda^{(p)}(\epsilon)$ converges in norm if $\epsilon > \epsilon_\Lambda = 2\|K\| + (w_\Lambda^2 - 1)\Gamma_\Lambda$. Let $A$ be positive and bounded and $|\psi\rangle \in \mathcal{H}$. The Laplace transform of $\langle \psi | \Phi_\Lambda(t) A | \psi\rangle$ is for $\epsilon > \epsilon_\Lambda$:

$$\begin{aligned}
\int_0^\infty dt\, e^{-t\epsilon} \, \langle \psi | \Phi_\Lambda(t) A | \psi \rangle \;&=\; \langle \psi | \big(\epsilon - \mathcal{L}_H - \mathcal{C}_\Lambda\big)^{-1} A | \psi \rangle \\
&=\; \mathbb{E} \sum_{p=0}^\infty \int_{t_\Lambda^p}^{t_\Lambda^{p+1}} dt\, e^{-t\epsilon} \langle \psi | U_\Lambda(t)^* A U_\Lambda(t) | \psi \rangle = \sum_{p=0}^\infty \langle \psi | \Phi_\Lambda^{(p)}(\epsilon)\, A\, |\psi\rangle \\
&=\; \langle \psi | \big(\epsilon - \mathcal{L}_H - \mathcal{C}_0 - \Gamma_\Lambda(\mathcal{W}_\Lambda - 1)\big)^{-1} A | \psi \rangle \ . \qquad (40)
\end{aligned}$$

The identity (38) has been used in the first line, together with Fubini's theorem and the monotone convergence theorem in the second. By the polarization identity and the linearity in $A$ in (40), $\mathcal{C}_\Lambda = \mathcal{C}_0 + \Gamma_\Lambda(\mathcal{W}_\Lambda - 1)$. □

*Proof of the corollary.* The axiom **(RP6)** of RP is equivalent, by the polarization identity, to the axiom **(QDS4)** of QDS (see the proof of proposition 3). Since it has been shown that $\mathfrak{U}_\Lambda(t)$ satisfy all axioms of RP but **(RP6)** and that $\Phi_\Lambda(t)$ satisfy all axioms of QDS but **(QDS4)**, this proves that *(1)* $\Leftrightarrow$ *(2)*. Now **(QDS4)** is equivalent to $\mathcal{C}_\Lambda 1 = 0$. By (35), this shows that *(2)* $\Leftrightarrow$ *(3)*. □

It follows from proposition 4 that $\Phi_\Lambda$ are bounded perturbations of the group of $*$-automorphisms generated by the derivation $\mathcal{L}_H$. By applying the theorem 3.1.33. of [44], one obtains:

**Corollary 3** *The operators $\Phi_\Lambda(t)A$ can be expressed by means of the following norm convergent Dyson's expansions:*

$$\Phi_\Lambda(t)A = e^{t\mathcal{L}_H} A + \sum_{q=1}^{\infty} \int_0^t d\tau_q \int_0^{\tau_q} d\tau_{q-1} \ldots \int_0^{\tau_2} d\tau_1 \, e^{\tau_1 \mathcal{L}_H} \, \mathcal{C}_\Lambda \, e^{(\tau_2-\tau_1)\mathcal{L}_H} \ldots \mathcal{C}_\Lambda \, e^{(t-\tau_q)\mathcal{L}_H} \, A \ . \quad (41)$$

Let us come back to the Anderson model for electrons in disordered solids of section 2.2. As follows from (11) and (33), the Lindblad operators are $L_{i\to j} = \sqrt{\Gamma_{i\to j}}|j\rangle\langle i|$, $i, j \in \Lambda_\infty$. The corresponding Lindblad equation for the density matrix reads:

$$\frac{d\rho}{dt} + \mathcal{L}_H \, \rho = \mathcal{C}_* \, \rho = \sum_{i,j\in\Lambda_\infty, i\neq j} \Gamma_{i\to j} \Big( \langle i|\rho|i\rangle \, P_j - \frac{1}{2}\{P_i, \rho\} \Big) \ , \quad (42)$$

where $P_i = |i\rangle\langle i|$ is the projector on $|i\rangle$. Equation (42) is the optical master equation [47]. It describes the mean evolution of the system due to phonon absorption/emission processes; it does not take into account the elastic electron-phonon scattering. Note that the action of the collision operator on the right hand side on the diagonal part of the density matrix $\rho$ coincide with the action of the collision operator of the Boltzmann equation (10).

**Remark 5** The classical jumps model of section 2.2 does not define a QDS. Actually, such a model does not describe correctly the evolution of quantum observables $A$ which do not commute with $H = V$.

# 5 Stochastic Hamiltonians

## 5.1 The Stochastic Schrödinger Equation

Let us assume that the jump operators can be expressed as exponentials, i.e. that there exists some bounded operators $V_{i\to j}$ such that:

$$W_{i\to j} = e^{-\mathrm{i}V_{i\to j}} \ , \ i, j \in \Lambda_\infty \ . \quad (43)$$

This is the case for instance in the model for electrons in strongly disordered solids of section 2.2: the jump operators (11) are of the form (43) with $V_{i\to j} = \mathrm{i}\,|j\rangle\langle i|$. It is shown in this section that the stochastic evolution of section 2 can be found by solving formally the stochastic Schrödinger equation:

$$\mathrm{i}\,\frac{d\,|\Psi\rangle}{dt} = \Big(H + K + \sum_{i,j\in\Lambda_\infty} V_{i\to j} \sum_{n=-\infty}^{\infty} \delta\big(t - t^n_{i\to j}\big)\Big)\,|\Psi(t)\rangle\ . \tag{44}$$

The last term in the time-dependent stochastic Hamiltonian is a random kicked Hamiltonian (noise term). The second term, which is time independent and non random, can be interpreted as a 'damping term', by analogy with the Langevin equation describing Brownian motion.

## 5.2 Solution of the Schrödinger Equation

Let us restrict as before $\Lambda_\infty$ to a finite box $\Lambda \subset \Lambda_\infty$ and compute the solution of the corresponding stochastic Schrödinger equation,

$$i\,\frac{d|\Psi_\Lambda\rangle}{dt} = \Big(H + K + \sum_{p=-\infty, p\neq 0}^{\infty} V_{i^p_\Lambda \to j^p_\Lambda}\,\delta\big(t - t^p_\Lambda\big)\Big)\,|\Psi(t)\rangle\ . \tag{45}$$

As usual (see e.g. [48]), this solution is found in two steps: (1) replace the Dirac distribution $\delta$ by a smooth function $g_\varepsilon : \mathbb{R} \to \mathbb{R}_+$ of compact support $\operatorname{supp} g_\varepsilon \subset [-\varepsilon, \varepsilon]$ and integral unity:

$$\int_{-\infty}^{\infty} dt\, g_\varepsilon(t) = 1\ ,\ \varepsilon > 0\ ; \tag{46}$$

(2) find the limit as $\varepsilon \to 0+$ of the corresponding solution $|\Psi_{\Lambda,\varepsilon}(t)\rangle$.

Let us substitute $g_\varepsilon$ to $\delta$ in (45). The time-dependent Hamiltonian must be replaced by the operator:

$$V_{\Lambda,\varepsilon}(t) = < V_\Lambda, g_\varepsilon(t - .) > = \sum_{p=-\infty, p\neq 0}^{\infty} V_{i^p_\Lambda \to j^p_\Lambda}\, g_\varepsilon\big(t - t^p_\Lambda\big)\ .$$

The resulting Schrödinger equation is:

$$\mathrm{i}\,\frac{d\,|\Psi_{\Lambda,\varepsilon}(t)\rangle}{dt} = \big(H + K + V_{\Lambda,\varepsilon}(t)\big)|\Psi_{\Lambda,\varepsilon}(t)\rangle \tag{47}$$

The purpose of the following statements is to show that the limit as $\varepsilon \to 0+$ of its solution $|\psi_{\Lambda,\varepsilon}(t)\rangle$ such that $|\psi_{\Lambda,\varepsilon}(t_0)\rangle = |\psi\rangle \in \mathcal{D}(H)$ is given by:

$$\lim_{\varepsilon \to 0+} |\Psi_{\Lambda,\varepsilon}(t)\rangle = U_\Lambda(t,t_0)|\psi\rangle \ , \tag{48}$$

where $U_\Lambda(t,t_0)$ is the random evolution operator (20). It will follows from theorem 1 that the (formal) solution of the Schrödinger equation (44) with the same initial condition, obtained by letting the size of the box $\Lambda$ tend to infinity, is:

$$|\Psi(t)\rangle^\xi = \lim_{\Lambda \uparrow \Lambda_\infty} \lim_{\varepsilon \to 0+} |\Psi_{\Lambda,\varepsilon}(t)\rangle^\xi = U^\xi(t,t_0)|\psi\rangle \ , \tag{49}$$

which exists for a.e. $\xi$. Note the importance of the order of the limits: we have first taken $\varepsilon \to 0+$ and then $\Lambda \uparrow \Lambda_\infty$.

The next proposition is a well-known result on time-dependent Schrödinger equation. Its proof can be found in standard textbooks (see e.g. [39], problems of section X.12).

**Proposition 5** *Let $H$ be a self-adjoint operator with dense domain $\mathcal{D}(H)$. Let $\mathcal{D}(\mathcal{L}_H)$ be the subset of $\mathcal{B}(\mathcal{H})$ be defined by (14). Let $t \in \mathbb{R} \mapsto V(t) \in \mathcal{D}(\mathcal{L}_H)$ be a norm-continuous application such that $t \in \mathbb{R} \mapsto \mathcal{L}_H V(t)$ is also norm-continuous. Then, for any $|\psi\rangle \in \mathcal{D}(H)$ and $t_0 \in \mathbb{R}$, the time-dependent Schrödinger equation:*

$$\mathrm{i}\,\frac{d\,|\psi(t)\rangle}{dt} = \big(H + V(t)\big)|\psi(t)\rangle$$

*with the initial condition $|\psi(t_0)\rangle = |\psi\rangle$ has a unique solution $|\psi(t)\rangle = U(t,t_0)|\psi\rangle$. The operators $U(t,t_0)$ belong to $\mathcal{D}(\mathcal{L}_H)$ and are given by the norm-convergent Dyson series:*

$$U(t,t_0) = \sum_{q=0}^{\infty} (-\mathrm{i})^q \int_{t_0}^{t} d\tau_q \ldots \int_{t_0}^{\tau_2} d\tau_1\, e^{-\mathrm{i}H(t-\tau_q)}\, V(\tau_q) \ldots e^{-\mathrm{i}H(\tau_2-\tau_1)}\, V(\tau_1)\, e^{-\mathrm{i}H(\tau_1-t_0)} \ . \tag{50}$$

*The norm-convergence of this series is uniform with respect to $(t,t_0)$ on compacts of $\mathbb{R}^2$.*

The last statement implies in particular that $(t, t_0) \in \mathbb{R}^2 \mapsto U(t, t_0)$ is strongly continuous (by hypothesis on $V(t)$ and strong continuity of $t \mapsto e^{-\mathrm{i}Ht}$, all terms in the series are strongly continuous).

**Corollary 4** *Let assumption D hold. Let $\Lambda \in L$, $\varepsilon > 0$, $t_0 \geq 0$ and $|\psi\rangle \in \mathcal{D}(H)$. Then the stochastic Schrödinger equation* (47) *with the initial condition $|\Psi_{\Lambda,\varepsilon}(t_0)\rangle^\xi = |\psi\rangle$ has a unique solution $|\Psi_{\Lambda,\varepsilon}(t)\rangle^\xi = U^\xi_{\Lambda,\varepsilon}(t, t_0)|\psi\rangle$ for all $\xi$ in a $(\psi, \varepsilon)$-independent subset of $\Xi$ of probability* 1.

*Proof.* The functions $h_{\varepsilon,i,j}(\tau) = \sum_n g_\varepsilon(\tau - t^n_{i\to j})$ are a.s. continuous. Actually, given $t > 0$, one can find a set $\Xi_{t,i,j} \subset \Xi$ of probability one such that, for any $\xi \in \Xi_{t,i,j}$, $-t \leq \tau \leq t$ and $0 < \varepsilon \leq 1$, the number of terms different from zero in the series defining $h_{\varepsilon,i,j}(t)$ is less than $N_{i\to j}([-t-1, t+1])$, which is a.s. finite. Hence $K + V_{\Lambda,\epsilon}(t) \in D(\mathcal{L}_H)$ a.s. for any $t \geq 0$ and the applications $t \mapsto V_{\Lambda,\epsilon}(t)$ and $t \mapsto \mathcal{L}_H V_{\Lambda,\epsilon}(t)$ are a.s. norm-continuous. □

The next result shows that the evolutions operators $U_{\Lambda,\varepsilon}(t, t_0)$ converge strongly as $\varepsilon \to 0+$ and that their limit is independent of the choice of the functions $g_\varepsilon$.

**Proposition 6** *Let assumption D hold. Let $\Lambda \in L$ and $\{g_\varepsilon; \varepsilon > 0\}$ be a family of smooth functions $\mathbb{R} \to \mathbb{R}_+$ of supports in $[-\varepsilon, \varepsilon]$ satisfying* (46)*. Then $U_{\Lambda,\varepsilon}(t, t_0)$ converges strongly a.s. for any $t, t_0 \geq 0$. If $t_0 \leq t$, its strong limit is a.s. given by* (20).

This proposition and theorem 1 prove the result announced above:

**Corollary 5** *Let assumptions A to D hold and $|\psi\rangle \in \mathcal{D}(H)$. Then the solution $\Psi(t)$ of the stochastic Schrödinger equation* (44) *with the initial condition $|\Psi(t)\rangle^\xi = |\psi\rangle$ is given by* (49)*, where $U^\xi(t, t_0)$ is the random evolution operator defined in theorem 1.*

*Proof.* The operators $U_{\Lambda,\varepsilon}(t, t_0)$ are uniformly bounded in $\varepsilon$ on $]0, 1]$. In fact, since supp $g_\varepsilon$ is a subset of $[-1,1]$ for such $\varepsilon$'s, it follows from (46) that the $q$-th term of the Dyson series (50) is bounded by:

$$\frac{1}{q!}\Big(\|K\| + \sum_{i,j\in\Lambda} \|V_{i\to j}\| N_{i\to j}\big([t_0-1,t+1]\big)\Big)^q .$$

This also shows that the Dyson expansion (50) for $U_{\Lambda,\varepsilon}(t,t_0)$ converges in norm uniformly with respect to $\varepsilon$ on $]0,1]$. Moreover, the uniqueness of the solution $|\psi_{\Lambda,\varepsilon}(t)\rangle$ implies:

$$U_{\Lambda,\varepsilon}(t,\tau)\,U_{\Lambda,\varepsilon}(\tau,t_0) = U_{\Lambda,\varepsilon}(t,t_0)\ ,\ t_0\le\tau\le t\ .$$

It is thus sufficient to show that $U_{\Lambda,\varepsilon}(t,\tau)$ converges strongly for any $t,\tau\in]\,t_\Lambda^{p-1},t_\Lambda^{p+1}[$, since one may then use this relation to extend this result for any $t,t_0\in\mathbb{R}$.

For $t,\tau\in\,]\,t_\Lambda^{p},t_\Lambda^{p+1}[$, the convergence is evident as $U_\Lambda(t,\tau)=U_0(t,\tau)$ for any $\varepsilon$ smaller than $d(\{\tau,t\},\{t_\Lambda^p,t_\Lambda^{p+1}\})/2$. Assume $t_\Lambda^{p-1}<\tau\le t_\Lambda^p\le t<t_\Lambda^{p+1}$. Let $\varepsilon>0$ be such that $\varepsilon\le(t_\Lambda^{p+1}-t)/2$ and $\varepsilon\le(\tau-t_\Lambda^{p-1})/2$. Using (50), one gets:

$$\begin{aligned} U_{\Lambda,\varepsilon}(t,\tau) \;=\; & \sum_{q=0}^{\infty}\frac{(-\mathrm{i})^q}{q!}\int_{\mathbb{R}^q}\prod_{r=1}^{q}\Big(g_\varepsilon(\tau_r-t_\Lambda^p)\,d\tau_r\Big)\sum_\sigma \chi\big(t\ge\tau_{\sigma(q)}\ge\ldots\ge\tau_{\sigma(1)}\ge\tau\big)\times \\ & U_0(t-\tau_{\sigma(q)})\,V_{i_\Lambda^p\to j_\Lambda^p}\ldots U_0(\tau_{\sigma(2)}-\tau_{\sigma(1)})\,V_{i_\Lambda^p\to j_\Lambda^p}\,U_0(\tau_{\sigma(1)}-\tau)\ . \end{aligned} \tag{51}$$

The sum inside the integral runs over all permutations $\sigma$ of $\{1,2,...,q\}$. It is a strongly continuous operator-valued function of $(\tau_1,...\tau_q)$ with compact support. Therefore each term in the sum over $q$ converges strongly to:

$$\frac{(-\mathrm{i})^q}{q!}\,U_0(t-t_\Lambda^p)\,\big(V_{i_\Lambda^p\to j_\Lambda^p}\big)^q\,U_0(t_\Lambda^p-\tau)\ .$$

as $\varepsilon\to0+$. Since the series (51) converges uniformly with respect to $\varepsilon$, $U_{\Lambda,\varepsilon}(t,\tau)\to U_\Lambda(t,\tau)$ a.s. as $\varepsilon\to0+$. Similar arguments show that $U_{\Lambda,\varepsilon}(\tau,t)$ converge strongly a.s.. $\square$

**Remark 6** For the time reversed stochastic evolution, the strong limit of $U_{\Lambda,\varepsilon}(t,t_0)$ is a.s. given, if $t_\Lambda^{p-1}\le t<t_\Lambda^p\le t_0$, by:

$$U_\Lambda(t,t_0)=U_0(t-t_\Lambda^p)\,W^{-1}_{i_\Lambda^p\to j_\Lambda^p}\,U_0(t_\Lambda^p-t_\Lambda^{p+1})\ldots W^{-1}_{i_\Lambda^{-1}\to j_\Lambda^{-1}}\,U_0(t_\Lambda^{-1}-t_0)\ .$$

**Remark 7** Equation (44) may be rewritten as an Ito stochastic differential equation as follows:

$$\mathrm{i}\, d|\Psi\rangle = \Big( (H+K)\, dt + \mathrm{i} \sum_{i,j\in\Lambda_\infty} (W_{i\to j} - 1)\, dN_{i\to j}(t) \Big) |\Psi(t)\rangle \ . \tag{52}$$

This equation has the same solutions, given by the evolution operators (20), as (44). This can be seen by computing the values of the discontinuities of $|\tilde{\Psi}(t)\rangle = U_0(t_0,t)U_\Lambda(t,t_0)|\Psi(t_0)\rangle$ at the jump times $t_\Lambda^p$, and by noting that $|\tilde{\Psi}(t)\rangle$ is constant between jumps. Note that, although it might be tempting to replace $dN_{i\to j}(t)$ by $\sum_n \delta(t - t^n_{i\to j})dt$, the operator multiplying the stochastic differential in (52) is $\mathrm{i}(W_{i\to j} - 1)$, whereas $V_{i\to j}$ multiplies the Dirac distributions in (44). A similar stochastic differential equation has been introduced by Belavkin [12], but $K$ and the jump operators $W_{i\to j}$ are different in this reference than the one considered here (see section 6). One may prove directly proposition 4 on the expression of the generator of the average dynamics by using Ito stochastic calculus and the stochastic Schrödinger equation (52). The proof given above has the advantage that it shows that Laplace transforms of observables can be computed directly, without having to solve a differential equation.

We end this section by stating a result on the strong convergence of the double sum in (44) in the sense of operator-valued distributions.

**Proposition 7** *Let* (31) *be satisfied and assume that for any* $i,j,k,l \in \Lambda_\infty$,

$$\big|\langle l|V_{i\to j}|k\rangle\big| \leq (f'_{il} + f'_{jl})(f'_{ik} + f'_{jk}) \ , \tag{53}$$

*where the matrix elements* $f'_{ij} = f'_{ji} > 0$ *are such that there is* $a > 0$ *and* $\eta > 0$,

$$f'_{ij} \leq a\, |i-j|^{-2d-\eta} \ , \ i,j \in \Lambda_\infty \ .$$

*Let* $g$ *be a real function of class* $C^\infty$ *with compact support and set:*

$$< V_\Lambda, g > = \sum_{i,j\in\Lambda} V_{i\to j} \sum_{n=-\infty}^{\infty} g(t^n_{i\to j}) \ , \ \Lambda \in L \ .$$

*Then there is a non-random dense domain $\mathcal{D} \subset \mathcal{H}$ such that $(< V_\Lambda, g >)_{\Lambda \in L}$ converges strongly on $\mathcal{D}$ as $\Lambda \uparrow \Lambda_\infty$ with probability one. Denote by $< V, g >$ its limit. There are a random constant $c$ and a non-random unbounded positive self-adjoint operator $A$ of domain $\mathcal{D}$ such that the inequality:*

$$\big\| < V, g > |\psi\rangle \big\| \leq c \,\big\| A|\psi\rangle \big\| \ , \ |\psi\rangle \in \mathcal{H} \tag{54}$$

*holds with probability one.*

This proposition is proven in the appendix, where it is also shown that $< V, g >$ can be unbounded with probability one.

**Remark 8** If (53) holds with $\sup_{i \in \Lambda_\infty} \sum_{j \in \Lambda_\infty} f'_{ij} e^{r_2 |i-j|} < \infty$, then assumption B is satisfied. In fact, for any $\nu \in \mathbb{N}^*$,

$$\begin{aligned} \big|\langle l | V^{\nu}_{i \to j} | k \rangle\big| &\leq (f'_{il} + f'_{jl})(f'_{ik} + f'_{jk}) \sum_{l_1, \dots l_{\nu-1} \in \Lambda_\infty} (f'_{il_1} + f'_{jl_1})^2 \cdots (f'_{il_{\nu-1}} + f'_{jl_{\nu-1}})^2 \\ &\leq c^{2\nu-2} (f'_{il} + f'_{jl})(f'_{ik} + f'_{jk}) \end{aligned}$$

with $c = 2 \sup_{i \in \Lambda_\infty} \sum_{l \in \Lambda_\infty} f'_{il}$. Expanding $W^{\pm 1}_{i \to j} = e^{\mp i V_{i \to j}}$ as a power series gives:

$$\big|\langle l \,|(W^{\pm 1}_{i \to j} - 1)|k \rangle\big| \leq \frac{\exp(c^2) - 1}{c^2} \big(f'_{il} + f'_{jl}\big)\big(f'_{ik} + f'_{jk}\big) \ .$$

# 6 Comparison with Other Stochastic Schemes

Other stochastic dynamical schemes with Poisson processes have been introduced by Ghirardi, Rimini and Weber [8], Dalibard, Castin and Mølmer [6], Carmichael [7], and Barchielli and Belavkin [11, 12]. Different schemes using Wiener processes have been studied by Gisin and Percival [9], Ghirardi, Pearle and Rimini [10], van Kampen [20] and by other authors [11, 21, 22, 23, 24]. We outline in this section the main similarities and differences of these models with the model presented above. The reader can find more information and other relevant references in the reviews [7, 28, 49].

## 6.1 Quantum Jump schemes

To our knowledge, the first quantum jump scheme is due to Ghirardi *et al.* [8], who introduced the following model in connection with the problem of the linear superpositions of macroscopically distinguishable states (Schrödinger cat states). The authors consider some jump operators

$$L_{x,n} = (\sqrt{\pi}a)^{-\frac{1}{2}} \exp\Big(-\frac{(x-X_n)^2}{2a^2}\Big)$$

which implement 'spontaneous collapses' in the position space around some point $x \in \mathbb{R}^3$. $X_n$ is the position operator of the $n$-th particle of a composite system of $\mathcal{N}$ particles. The collapse $(x,n)$ localizes the $n$-th particle around $x$ with an accuracy $a > 0$. The operators $L_{x,n}$ are self-adjoint and satisfy:

$$\int_{\mathbb{R}^d} dx\, L_{x,n}^2 = 1 \ , \ n = 1, \dots, \mathcal{N} \ . \tag{55}$$

In the wavefunction formulation of the model [50], the collapses (jumps) modify discontinuously the wavefunction of the system according to the non-linear transformation:

$$\text{collapse } (x,n) : \ |\psi\rangle \to \big\| L_{x,n}|\psi\rangle \big\|^{-1} L_{x,n}|\psi\rangle \ .$$

The probability that a collapse $(x,n)$ (resp. that *any* collapse) occurs between times $t$ and $t+dt$ is equal to $dp_n(x) = \lambda \| L_{x,n}|\psi(t)\rangle\|^2 dt$ (resp. to $dp = \sum_n \int dx\, dp_n(x) = \lambda\mathcal{N}\|\psi(t)\|^2 dt$), where $\lambda$ is a characteristic frequency. Between jumps, the composite system evolves according to Schrödinger equation (with no damping operator; this fact is related to (55)). As shown in [8], for macroscopic systems ($\mathcal{N} \gg 1$), the stochastic collapses kill very rapidly the coherences between states localized a distance greater than $a$. Provided one chooses $\lambda$ small enough, they have little effect on the dynamics of microscopic systems ($\mathcal{N} \simeq 1$) at times accessible in a laboratory experiment.

Dalibard *et al.* [6] and Carmichael [7] have studied a similar but different model in order to describe photon-atom interactions in quantum optics. The original motivation was to reproduce experimental data on the fluorescence of single atoms [28]. Quantum jumps occur as a result of a continuous measurement of photon emission from the atom. The discontinuous change of the wavefunction occurring at a jump is given, as in the collapse model above, by a non-linear transformation. In the notations of section 2.1:

$$\text{jump } (i,j) : |\psi\rangle \to \big\|(W_{i\to j}-1)|\psi\rangle\big\|^{-1} (W_{i\to j}-1)|\psi\rangle \ . \tag{56}$$

The operators $W_{i\to j}-1$ are arbitrary (they do not satisfy a relation similar to (55)), as in the case considered in this paper. A jump $(i,j)$ occurs if a photon of energy equal to the Bohr frequency $\omega_{ij} = \langle i|V|i\rangle - \langle j|V|j\rangle$ is detected. Between jumps, the atom evolves in the following way. Successive measurements on the fluorescence of the atom are performed at short time intervals $\delta t$, with the result of no photon detected. These measurements increase our knowledge on the state of the system; it can be shown that the wavefunction evolves between two measurements according to Schrödinger equation but with a non self-adjoint Hamiltonian $H + K_0$. Perturbation theory gives [6]:

$$K_0 = \frac{1}{2\mathrm{i}} \sum_{i,j} \Gamma_{i\to j} (W_{i\to j}-1)^* (W_{i\to j}-1) \tag{57}$$

(note the difference with the damping operator $K$ defined in (3)). Immediately after a zero-photon measurement, the wavefunction is normalized, $|\psi\rangle \to \|\psi\|^{-1}|\psi\rangle$. If, on the contrary, a photon is detected, the wavefunction is transformed as in (56). The probability of detection of a photon of frequency $\omega_{ij}$ is $\delta p_{i\to j} = \Gamma_{i\to j}\|(W_{i\to j}-1)|\psi(t)\rangle\|^2\,\delta t$. It depends upon the wavefunction $|\psi(t)\rangle$ before the jump, and thus upon $t$. As a consequence, the time delays $s_{i\to j}$ between consecutive jumps $(i,j)$ are not given by simple exponential laws. The quantum jumps scheme of Dalibard *et al.* is therefore more involved than the one given by a set of independent Poisson processes (this conclusion also holds for the collapse model of ref. [8]). Despite this mathematical complexity, its dynamics is very simple to implement numerically [6]. On time scales greater than $\delta t$, the stochastic dynamics is norm-preserving. The same model has been derived by a completely different and more abstract method using quantum stochastic calculus by Barchielli and Belavkin [11].

The main difference between the models of ref. [6, 7, 8, 11] with the model presented in section 2.1 is that the stochastic dynamics is linear and not norm-preserving in the latter, and vice versa in the formers. A linear non norm-preserving model based on Poisson processes has also been discussed by Belavkin [12]. The quantum jumps in this work are given by (56) excepted that no normalization of the wavefunction is made, and the damping operator is different from both damping operators $K$ and $K_0$ defined in (3) and (57).

## 6.2 Quantum Diffusion schemes

As in the schemes of ref. [6, 7, 8, 11], the stochastic scheme investigated by Gisin *et al.* [9], Ghirardi *et al.* [10] and Barchielli *et al.* [11] has a norm-preserving and non linear stochastic quantum evolution. However, this evolution is given by a stochastic Schrödinger equation with Wiener processes (quantum diffusion). Another model based on Wiener processes but with linear stochastic dynamics has been studied in [10, 20]. Its wavefunction satisfies the Ito stochastic Schrödinger equation:

$$\mathrm{i}\, d|\Psi\rangle = \Big( (H + K_0)\, dt + \mathrm{i} \sum_{i,j} \sqrt{\Gamma_{i\to j}} (W_{i\to j} - 1)\, d\xi_{i\to j}(t) \Big) |\Psi(t)\rangle \ , \tag{58}$$

where $K_0$ is given by (57) and $(\xi_{i\to j}(t))_{t\in\mathbb{R}}$ are independent complex Wiener processes. The Ito differentials $d\xi_{i\to j}$ satisfy:

$$d\xi_{i\to j}\, d\bar{\xi}_{k\to m} = \delta_{i,k}\delta_{j,l}\, dt \quad , \quad d\xi_{i\to j}\, d\xi_{k\to m} = 0 \ .$$

The link between this linear non norm-preserving model and the norm-preserving non linear one has been emphasized in [10].

As shown in [6, 7, 10, 9, 20], provided that the appropriate damping operator $K$ is added to the Hamiltonian $H$, all the above models lead to the same trace-preserving average dynamics, given by the Lindblad equation (1); it has been shown in section 4 that the same holds true in our model. More general stochastic dynamical models, which lead to non-Markovian master equations and use correlated noise, have been introduced recently by several authors [21, 22, 23, 24]. In ref. [21, 22], two nice derivations of these models by mean of a path integral and a coherent states method have been proposed.

### 6.3 Comparison with the Model of Section 2.1

The main advantage of the RW model presented in this work compared with the non linear quantum jumps scheme of e.g. Dalibard *et al.* is its simplicity. Because of the use of Poisson processes and of the linearity of the dynamics, the solution of the stochastic Schrödinger equation is known exactly: it is given by formula (4). The mathematical analysis is also more easy, as the operator theory framework can be used to study the stochastic evolution in the Hilbert space. From the point of view of the average dynamics (i.e., for statistical ensembles of systems), our model is equivalent to the model of Dalibard *et al.* and to the other models discussed in the preceding subsections. Actually, as seen in section 4, the density matrix $\rho(t) = \mathbb{E}\ |\psi(t)\rangle\langle\psi(t)|$ is given by the same quantum dynamical semigroup with Lindblad generator as in these models.

For general (non unitary) jump operators $W_{i\to j}$, the norm of the wavefunction for a fixed outcome is not constant (and not continuous) in time (although, as said before, the square norm is conserved *on average*). Indeed, if one insists in describing an open system by a wavefunction, its norm may not be necessarily conserved by the dynamics, since, unlike in a closed system, the dynamics is not unitary. This is related to the fact that the interaction with the environment may provide us with some information on the system.

From the numerical side, the stochastic dynamics in our model could be of interest if the exponentials $e^{-is(H+K)}$ were known on a broad interval of times $s$. This happens, for example, if $H+K$ can be diagonalized analytically. Then, the computation of the wavefunction at time $t$ requires a multiplication of $2p$ matrices, where $p$ is the number of jumps between $t=0$ and $t$ (formula (4)), whereas the non linear quantum jumps and the Wiener schemes involve a time integration between 0 and $t$. However, the range of application of the model is clearly limited to systems for which the RW for a typical trajectory has a norm that do not decrease or grow too rapidly on the considered time interval.

# 7 Proofs

This section is devoted to the proofs of theorem 1 and 2. It is organized as follows. We introduce few definitions and a deterministic estimate in the first subsection. A probabilistic estimate is proved in the second subsection. The proof of theorem 1 on the stochastic evolution is then obtained in the third subsection. The last subsection present the proof of theorem 2 on the average dynamics; it is independent of the two preceding subsections.

## 7.1 Notations - Deterministic Estimate

If $|\psi\rangle$ and $A$ are respectively a vector and an operator on $\mathcal{H}$, we denote by $|\tilde{\psi}\rangle$ and $\tilde{A}$ the vector and operator:

$$|\tilde{\psi}\rangle = \sum_{i\in\Lambda_\infty} \big|\langle i|\psi\rangle\big|\,|i\rangle \quad , \quad \tilde{A} = \sum_{i,j\in\Lambda_\infty} \big|\langle i|A|j\rangle\big|\,|i\rangle\langle j|\ .$$

We define the following order relations on the Hilbert space $\mathcal{H}$ and on the space of operators on $\mathcal{H}$:

$$|\varphi\rangle \prec |\psi\rangle \quad \Leftrightarrow \quad \langle i|\varphi\rangle \leq \langle i|\psi\rangle \;\; \forall\, i \in \Lambda_\infty$$

$$A \prec B \quad \Leftrightarrow \quad \langle i|A|j\rangle \leq \langle i|B|j\rangle \;\; \forall\, i,j \in \Lambda_\infty\ .$$

It is clear that if $\tilde{A} \prec B$ and $\tilde{C} \prec D$, then $0 \prec (\tilde{AC}) \prec BD$ and, for any $|\psi\rangle \in \mathcal{H}$,

$$\big\|A|\psi\rangle\big\| \leq \big\|B|\tilde{\psi}\rangle\big\|\ . \tag{59}$$

For any $r \geq 0$, the $*$-algebraic norms $\|.\|_r$ on subspaces of $\mathcal{B}(\mathcal{H})$ are defined by:

$$\big\|A\big\|_r = \max\Big\{\sup_{i\in\Lambda_\infty}\sum_{j\in\Lambda_\infty}\big|\langle i|A|j\rangle\big|e^{r|i-j|}\ ,\ \sup_{i\in\Lambda_\infty}\sum_{j\in\Lambda_\infty}\big|\langle j|A|i\rangle\big|e^{r|i-j|}\Big\}\ . \tag{60}$$

Let us set:

$$F = \sum_{i,j\in\Lambda_\infty} f_{ij}|i\rangle\langle j| \quad , \quad G = \sum_{i,j\in\Lambda_\infty} \Gamma_{i\to j}(|i\rangle + |j\rangle)(\langle i| + \langle j|) \tag{61}$$

With these notations, assumption B reads:

$$\tilde{W}_{i\to j} \prec 1 + \delta W_{i\to j}\ , \tag{62}$$

with:

$$\delta W_{i\to j} = F\big(|i\rangle + |j\rangle\big)\big(\langle i| + \langle j|\big)F\ . \tag{63}$$

The hypothesis on $f_{ij}$ in assumption B, assumption A and assumption C are respectively equivalent to:

$$\big\|F\big\|_{r_2} < \infty \ \ , \ \ \big\|G\big\|_{r_1} < \infty \ \ , \ \ \big\|T+K\big\|_{r_3} < \infty \ . \tag{64}$$

The more general assumption (31) is equivalent to $\|G\|_0 < \infty$.

**Remark 9** Any positive, increasing and $\|.\|$-bounded sequence $(|\psi_m\rangle)_{m\in\mathbb{N}}$ in $\mathcal{H}$ with respect to $\prec$ (i.e. such that $0 \prec |\psi_m\rangle \prec |\psi_{m+1}\rangle \ \forall \ m \in \mathbb{N}$ and $\sup_m \|\psi_m\| < \infty$) is convergent. For any positive and increasing sequence of operators $(A_m)_{m\in\mathbb{N}}$ with respect to $\prec$ such that there is $A \in \mathcal{B}(\mathcal{H})$, $\langle i|A_m|j\rangle \to \langle i|Aj|\rangle \ \ \forall\, i,j \in \Lambda_\infty$, one has $A_m \to A$ strongly. The first affirmation follows from the estimate:

$$\big\|\psi_m - \psi_n\big\|^2 = \big\|\psi_m\big\|^2 + \big\|\psi_n\big\|^2 - 2\sum_{i\in\Lambda_\infty}\langle i|\psi_m\rangle\langle i|\psi_n\rangle \leq \big\|\psi_m\big\|^2 - \big\|\psi_n\big\|^2 \ , \ m \geq n \ .$$

The sequence $(\|\psi_m\|^2)_{m\in\mathbb{N}}$ being increasing and bounded, it is Cauchy and thus so is $(|\psi_m\rangle)_{m\in\mathbb{N}}$. The second affirmation follows from an $\varepsilon/3$ argument.

Let assumption D be satisfied. By applying proposition 5 with $H \longrightarrow V$, $V(t) \longrightarrow T+K$, the Dyson's expansion:

$$U_0(s) = e^{-\mathrm{i}sV} + \sum_{q=1}^{\infty}(-\mathrm{i})^q \int_{0\leq\tau_1\ldots\leq\tau_q\leq s} d\tau_1\ldots d\tau_q\, e^{-\mathrm{i}(s-\tau_q)V}(T+K)\ \ldots\ e^{-\mathrm{i}(\tau_2-\tau_1)V}(T+K)e^{-\mathrm{i}\tau_1 V}$$

converges. Since $(\widetilde{e^{-\mathrm{i}sV}}) = 1$ for any $s \in \mathbb{R}$, this implies that $\tilde{U}_0(s) \prec e^{s(T\tilde{+}K)}$. Together with (20) and (62), this yields to the following deterministic estimate.

**Lemma 1** *Let the assumptions B and D hold. Set $S = (T \tilde{+} K)$. Then $\tilde{U}_\Lambda^\xi(t) \prec C_\Lambda^\xi(t)$ for any $\xi \in \Xi$, $\Lambda \in L$ and $t \geq 0$, where the random operators $C_\Lambda(t)$ are given by:*

$$C_\Lambda(t) = e^{(t-t_\Lambda^p)S}\left(1 + \delta W_{i_\Lambda^p \to j_\Lambda^p}\right) e^{s_\Lambda^p S}\ \ldots\ \left(1 + \delta W_{i_\Lambda^1 \to j_\Lambda^1}\right) e^{s_\Lambda^1 S} \ , \ 0 \leq t_\Lambda^p \leq t < t_\Lambda^{p+1} \ . \tag{65}$$

*Moreover, $C_\Lambda(t)$ is an increasing function of $\Lambda$ and $t$ for the order relation $\prec$, i.e. for any $\Lambda, \Lambda' \in L$ and $t, t' \geq 0$, $(\Lambda \subset \Lambda'$ or $t \leq t') \Rightarrow C_\Lambda(t) \prec C_{\Lambda'}(t')$.*

## 7.2 Probabilistic Estimate

For any $m \in \mathbb{N}^*$, let $\mathcal{H}^{\otimes m}$ be the $m$-th tensor product of the Hilbert space $\mathcal{H}$. If $A \in \mathcal{B}(\mathcal{H})$, the bounded operators $A^{\otimes m}$ and $A_m$ on $\mathcal{H}^{\otimes m}$ are defined by:

$$A^{\otimes m} = A \otimes \ldots \otimes A \quad , \quad A_m = \sum_{\mu=1}^{m} 1 \otimes \ldots \otimes 1 \otimes A \otimes 1 \otimes \ldots \otimes 1$$

($A$ is the $\mu$-th factor in the $\mu$-th term of the sum). For $r \geq 0$, an algebraic $*$-norm $\|.\|_r$ on subspaces of $\mathcal{B}(\mathcal{H}^{\otimes m})$ is defined in much the same way as in (60):

$$\begin{aligned} \left\|A^{(m)}\right\|_r &= \max\Big\{ \sup_{i_1,..,i_m \in \Lambda_\infty} \sum_{j_1,...,j_m \in \Lambda_\infty} \left|\langle i_1, ..., i_m | A^{(m)} | j_1, ..., j_m\rangle\right| e^{r \sum_{\mu=1}^m |i_\mu - j_\mu|} \, , \\ &\qquad \left(A^{(m)} \leftrightarrow (A^{(m)})^*\right) \Big\} \, . \end{aligned} \tag{66}$$

Here $|i_1, i_2, \ldots, i_m\rangle \in \mathcal{H}^{\otimes m}$ denote the tensor product of the $m$ vectors $|i_1\rangle, \ldots |i_m\rangle$. It is clear that $\|A^{\otimes m}\|_r = \|A\|_r^m$ and $\|A_m\|_r = m\,\|A\|_r$.

**Lemma 2** *Let $m \in \mathbb{N}^\star$ and assumptions A to D hold. Then, for almost all $t$ with respect to the Lebesgue measure, $\mathbb{E}\ C_\Lambda(t)^{\otimes m} \to e^{t(R^{(m)}+S_m)}$ strongly as $\Lambda \uparrow \Lambda_\infty$, where $R^{(m)}$ is the strongly convergent sum:*

$$R^{(m)} = \sum_{i,j \in \Lambda_\infty} \Gamma_{i\to j} \Big( \left(1 + \delta W_{i \to j}\right)^{\otimes m} - 1 \Big) \, . \tag{67}$$

*Moreover if $0 \leq r \leq r_2$, $r \leq r_3$ and $r \leq r_1/m$, then $\sup_{\Lambda \in L} \left\|\mathbb{E}\ C_\Lambda(t)^{\otimes m}\right\|_r$ is finite.*

*Proof.* Let $\epsilon > m\,\|S\|_0$. The Laplace transform of $C_\Lambda(t)^{\otimes m}$ is:

$$\begin{aligned} C_\Lambda^{(m)}(\epsilon) &= \int_0^\infty dt e^{-t\epsilon} C_\Lambda(t)^{\otimes m} = \sum_{p=0}^{\infty} (\epsilon - S_m)^{-1} \left(1 - e^{s_\Lambda^{p+1}(S_m - \epsilon)}\right) \times \\ &\qquad \left(1 + \delta W_{i_\Lambda^p \to j_\Lambda^p}\right)^{\otimes m} e^{s_\Lambda^p (S_m - \epsilon)} \ldots \left(1 + \delta W_{i_\Lambda^1 \to j_\Lambda^1}\right)^{\otimes m} e^{s_\Lambda^1 (S_m - \epsilon)} \, . \end{aligned}$$

A similar calculation than the one performed in the proof of proposition 4 shows:

$$\mathbb{E}\ C_\Lambda^{(m)}(\epsilon) = \big(\Gamma_\Lambda + \epsilon - S_m\big)^{-1} \sum_{p=0}^{\infty} \Big( \big(\Gamma_\Lambda + R_\Lambda^{(m)}\big)\big(\Gamma_\Lambda + \epsilon - S_m\big)^{-1}\Big)^p \tag{68}$$

provided that the series converges, with:

$$R_\Lambda^{(m)} = \Gamma_\Lambda \Big(\mathbb{E}\ \big(1 + \delta W_{i_\Lambda^q \to j_\Lambda^q}\big)^{\otimes m} - 1\Big) = \sum_{i,j\in\Lambda} \Gamma_{i\to j} \Big(\big(1 + \delta W_{i\to j}\big)^{\otimes m} - 1\Big) \ \succ\ 0\ . \tag{69}$$

Let $0 \le r \le r_2$, $r \le r_3$, $r \le r_1/m$. A simple but somehow lengthly estimate gives:

$$\Big\| \sum_{i,j\in\Lambda} \Gamma_{i\to j}\, \delta W_{i\to j}^{\otimes\mu} \Big\|_r \ \le\ 2^{2\mu-1} \big\|F\big\|_r^{2\mu} \big\|G\big\|_{\mu r} \ <\ \infty\ ,\ \mu = 1, \dots, m, \Lambda \in L\ .$$

By expanding the tensor product in (69), it follows that $\exists\ a > 0$ such that $\|R_\Lambda^{(m)}\|_r \le a$ $\forall\ \Lambda \in L$. The matrix elements $\langle i_1, \dots, i_m | R_\Lambda^{(m)} | j_1, \dots, j_m\rangle$ converge when $\Lambda \uparrow \Lambda_\infty$, as non decreasing bounded functions of $\Lambda$. Hence $R^{(m)}$ is bounded and $R_\Lambda^{(m)} \to R^{(m)}$ strongly as $\Lambda \uparrow \Lambda_\infty$ (see remark 9). Let $\epsilon > m\,\|S\|_r + a$. For any $\Lambda \in L$, $\|(\Gamma_\Lambda + R_\Lambda^{(m)})(\Gamma_\Lambda + \epsilon - S_m)^{-1}\|_r < 1$ and the series in (68) converges for the norm $\|.\|_r$. Then:

$$\mathbb{E}\ C_\Lambda^{(m)}(\epsilon) = \big(\epsilon - R_\Lambda^{(m)} - S_m\big)^{-1} \to \big(\epsilon - R^{(m)} - S_m\big)^{-1}$$

strongly. By the monotone convergence and Fubini theorems, the Laplace transform of the limit of $\langle i_1, \dots, i_m | \mathbb{E}\ C_\Lambda(t)^{\otimes m} | j_1, \dots, j_m\rangle$ as $\Lambda \uparrow \Lambda_\infty$ is equal to:

$$\lim_{\Lambda\uparrow\Lambda_\infty} \langle i_1, \dots, i_m | \mathbb{E}\ C_\Lambda^{(m)}(\epsilon) | j_1, \dots, j_m\rangle = \langle i_1, \dots, i_m | \big(\epsilon - R^{(m)} - S_m\big)^{-1} | j_1, \dots, j_m\rangle\ .$$

The first statement follows by inverse Laplace transforms. Since $\|R^{(m)}\|_r = \lim_{\Lambda\uparrow\Lambda_\infty} \|R_\Lambda^{(m)}\|_r \le a$ and $\|S_m\|_r = m\,\|S\|_r$ are finite, hence so is $\|e^{t(R^{(m)}+S_m)}\|_r$. This shows the second statement of the lemma because $C_\Lambda(t)$ increases with $\Lambda$. □

**Lemma 3** *Let assumptions A to D hold. Then, for any* $0 < r \le r_1\eta/(d+\eta)$, $r \le r_2$, $r \le r_3$ *and any* $\eta > 0$, *there is a random constant* $c > 0$ *such that with probability one,*

$$\langle i | C_\Lambda(t) | j\rangle \le c\,\big(\min\{|i|, |j|\}\big)^\eta\, e^{-r|i-j|}\ \ ,\ \ \forall\ i,j \in \Lambda_\infty,\ \forall\ \Lambda \in L\ . \tag{70}$$

*Proof.* One must show that the set of outcomes $\Xi_0$:

$$\Xi_0 = \bigcap_{c>0} \bigcup_{\Lambda \in L} \bigcup_{i,j \in \Lambda_\infty} \Xi_{\Lambda,c,i,j} \quad , \quad \Xi_{\Lambda,c,i,j} = \left\{ \xi \in \Xi; \langle i | C_\Lambda^\xi(t) | j \rangle \geq c\, e^{-r|i-j|} |i|^\eta \right\}$$

has probability zero. Let $m \in \mathbb{N}^*$ be such that $d/\eta < m \leq d/\eta + 1$. Since $C_\Lambda(t)$ is an increasing function of $\Lambda$, $\Xi_{\Lambda,c,i,j} \subset \Xi_{\Lambda',c',i,j}$ if $\Lambda \subset \Lambda'$ or $c \geq c'$. Hence:

$$\mathbb{P}(\Xi_0) \leq \lim_{c \to \infty} \lim_{\Lambda \uparrow \Lambda_\infty} \sum_{i,j \in \Lambda_\infty} \mathbb{P}(\Xi_{\Lambda,c,i,j}) \ . \tag{71}$$

By Tchebychev's inequality,

$$\mathbb{P}(\Xi_{\Lambda,c,i,j}) \leq \mathbb{E}\ \langle i | C_\Lambda(t) | j \rangle^m \frac{e^{mr|i-j|}}{|i|^{m\eta}\, c^m} \ .$$

But $m\eta > d$ and $rm \leq r_1$, thus by lemma 2,

$$\mathbb{P}(\Xi_0) \leq \lim_{c \to \infty} c^{-m} \sup_{\Lambda \in L} \left\| \mathbb{E}\ C_\Lambda(t)^{\otimes m} \right\|_r \sum_{i \in \mathbb{Z}^d} |i|^{-m\eta} = 0 \ . \tag{72}$$

The same argument holds by interchanging $i$ and $j$. $\square$

### 7.3 Proof of Theorem 1

*Proof of theorem 1.* Let $|\psi\rangle \in \mathcal{H}$ and $t \geq 0$. One has to show that the net $(U_\Lambda(.)|\psi\rangle)_{\Lambda \in L}$ is Cauchy with probability one for the sup norm $\|\psi(.)\|_t = \sup_{0 \leq \tau \leq t} \|\psi(\tau)\|$. Let $\Lambda, \Lambda' \in L, \Lambda \subset \Lambda'$ and $0 \leq \tau \leq t$. $U_{\Lambda'}$ can be expressed by a formula similar to (20), but with $U_0$ replaced by $U_\Lambda$ and the $(i_\Lambda^p, j_\Lambda^p)$'s by the pairs $(i_{\Lambda',\Lambda}^p, j_{\Lambda',\Lambda}^p)$ of indices in $\Lambda'$ with at least one index outside $\Lambda$. This gives:

$$U_{\Lambda'}(\tau) - U_\Lambda(\tau) = \sum_{i,j \in \Lambda', i \,\mathrm{or}\, j \notin \Lambda} \ \sum_{n=1}^{N_{i\to j}(\tau)} U_{\Lambda'}(\tau, t_{i\to j}^n +) \big( W_{i\to j} - 1 \big) U_\Lambda(t_{i\to j}^n -) \ . \tag{73}$$

From this equality, assumption B and lemma 1, it follows:

$$\begin{aligned} \left\| \big( U_{\Lambda'}(.) - U_\Lambda(.) \big) |\psi\rangle \right\|_t &\leq \sup_{0 \leq \tau \leq t} \left\| \sum_{i,j \in \Lambda', i \,\mathrm{or}\, j \notin \Lambda} N_{i\to j}(\tau)\, C_{\Lambda'}(\tau)\, \delta W_{i\to j}\, C_\Lambda(\tau) |\tilde{\psi}\rangle \right\| \\ &\leq \left\| \sum_{i,j \in \Lambda', i \,\mathrm{or}\, j \notin \Lambda} |\varphi_{i,j,\Lambda''}\rangle \right\| \end{aligned}$$

with and $\Lambda'' \supset \Lambda' \supset \Lambda$ and:

$$|\varphi_{i,j,\Lambda''}\rangle = N_{i\to j}(t)\, C_{\Lambda''}(t)\, \delta W_{i\to j}\, C_{\Lambda''}(t)|\tilde{\psi}\rangle \ \succ\ 0\ .$$

By the remark 9 above, $(U_\Lambda(.)|\psi\rangle)_{\Lambda\in L}$ is Cauchy provided that:

$$\sup_{\Lambda,\Lambda''\in L} \Big\| \sum_{i,j\in\Lambda} |\varphi_{i,j,\Lambda''}\rangle \Big\|^2 < \infty\ . \tag{74}$$

To prove that (74) is true with probability one, it is enough to show that the mean value of the left hand side exists and is finite. Successive applications of the monotone convergence theorem and Fatou's lemma shows that this mean value is finite if:

$$\sum_{k\in\Lambda_\infty} \sup_{\Lambda,\Lambda''\in L} \mathbb{E}\, \Big( \sum_{i,j\in\Lambda} \langle k|\varphi_{i,j,\Lambda''}\rangle \Big)^2 < \infty\ . \tag{75}$$

It remains to prove (75).

By applying twice the Cauchy-Schwartz inequality, it follows that $\mathbb{E}\, \big(\sum_{i,j\in\Lambda}\langle k|\varphi_{i,j,\Lambda''}\rangle\big)^2$ is bounded by:

$$\sum_{l,l',m,m',h,h'\in\Lambda_\infty} \langle l|B_\Lambda|h\rangle\langle l'|B_\Lambda|h'\rangle \Big(\mathbb{E}\, \langle k,h,k,h'|C_{\Lambda''}(t)^{\otimes^4}|l,m,l',m'\rangle^2\Big)^{\frac{1}{2}} \langle m|\tilde{\psi}\rangle\langle m'|\tilde{\psi}\rangle\ ,$$

with:

$$B_\Lambda = \sum_{i,j\in\Lambda} \big(\mathbb{E}\ N^4_{i\to j}(t)\big)^{\frac{1}{4}}\, \delta W_{i\to j}\ .$$

Now $\mathbb{E}\ N^4_{i\to j}(t)$ is a polynomial of degree 4 in $\Gamma_{i\to j}$ and $\Gamma_{i\to j} \le \|G\|_{r_1} e^{-r_1|i-j|}$. Hence there is a constant $c>0$ such that $\mathbb{E}\ N^4_{i\to j}(t) \le c\, e^{-r_1|i-j|}\ \ \forall\ i,j\in\Lambda_\infty$. This means that for any $0<r<r_1/4$, $r\le r_2$:

$$\big\|B_\Lambda\big\|_r \le c^{\frac{1}{4}}\, \big\|F\big\|_r^2\, \Big\| \sum_{i,j\in\Lambda_\infty} e^{-\frac{r_1|i-j|}{4}} \big(|i\rangle + |j\rangle\big)\big(\langle i| + \langle j|\big) \Big\|_r < \infty\ .$$

Moreover, it follows from lemma 2 that if $0\le r\le r_1/8$ and $r\le r_2$, $r\le r_3$,

$$\mathbb{E}\ \langle k,h,k,h'|C_{\Lambda''}(t)^{\otimes^4}|l,m,l',m'\rangle^2 \le const.\ e^{-2r(|k-l|+|h-m|+|k-l'|+|h'-m'|)}\ .$$

By using the bound $\langle l|B_\Lambda|h\rangle \leq \|B_\Lambda\|_r \, e^{-r|l-h|}$, the following inequality holds:

$$\sup_{\Lambda,\Lambda''\in L} \mathbb{E} \Big( \sum_{i,j\in\Lambda} \langle k|\tilde{V}^q|\varphi_{i,j,\Lambda''}\rangle \Big)^2 \leq const. \, \langle k|V|k\rangle^{2q} \, \langle k|\Delta_r^3|\tilde{\psi}\rangle^2 \ ,$$

where $q = 0$ or $1$ and $\Delta_r$ is given by (24) and is clearly bounded. This estimate, with $q = 0$, proves (75), i.e., that $(U_\Lambda(.)|\psi\rangle)_{\Lambda\in L}$ converges for the sup norm $\|.\|_t$ with probability one. The same estimate for $q = 1$ proves that if $|\psi\rangle \in \mathcal{D}(H)$ and $\Delta_r\mathcal{D}(H) \subset \mathcal{D}(H)$, $(V\,U_\Lambda(t)|\psi\rangle)_{\Lambda\in L}$ converges with probability one (recall that $\mathcal{D}(V) = \mathcal{D}(H)$). By the closeness of $V$, this implies that $U(t)|\psi\rangle$ is a.s. in $\mathcal{D}(H)$. The property (23) follow immediately from lemma 3. $\square$

### 7.4 Proof of Theorem 2

We deduce in this subsection the theorem 2 on the average dynamics from the proposition 4 and its corollary.

*Proof of statements (1) and (2).* We first prove that for any $A \in \mathcal{B}(\mathcal{H})$, the net $(\mathcal{C}_\Lambda A)_{\Lambda\in L}$ converges ultrastrongly and the operators $\mathcal{C}_\Lambda$ in (35) are uniformly bounded in $\Lambda$. Then, by applying corollary 3, it will follow that the same holds for the operators $\Phi_\Lambda(t)$, $t \geq 0$.

Let $\Lambda, \Lambda' \in L$, $\Lambda \subset \Lambda'$ and $A \in \mathcal{B}(\mathcal{H})$ and set:

$$\mathcal{Q}_\Lambda A = \sum_{i,j\in\Lambda} \Gamma_{i\to j}\big(W_{i\to j}^* - 1\big)A\big(W_{i\to j} - 1\big) \quad , \quad Y_\Lambda = \sum_{i,j\in\Lambda} \Gamma_{i\to j}\big(W_{i\to j} - 1\big) \ .$$

Equation (35) can be rewritten as:

$$\mathcal{C}_\Lambda \, A = \mathrm{i}\, K^* \, A - \mathrm{i}\, A\, K + \mathcal{Q}_\Lambda \, A + Y_\Lambda^* \, A + A\, Y_\Lambda \ .$$

One has $G_\Lambda \equiv \sum_{i,j\in\Lambda} \Gamma_{i\to j}(|i\rangle + |j\rangle)(\langle i| + \langle j|) \to G$ strongly as $\Lambda \uparrow \Lambda_\infty$. This is because, by assumption (31), $G$ is bounded and for any $|\psi\rangle \in \mathcal{H}$,

$$\big\|(G_{\Lambda'} - G_\Lambda)|\psi\rangle\big\| \leq \big\|\big((1 - P_\Lambda)G + G(1 - P_\Lambda)\big)|\tilde{\psi}\rangle\big\| + \Big\| \sum_{i\in\Lambda_\infty} \sum_{j\in\Lambda_\infty, j\notin\Lambda} \big(\Gamma_{i\to j} + \Gamma_{j\to i}\big)|i\rangle\langle i|\tilde{\psi}\rangle \Big\| \ .$$

The first term in the right hand side converges to 0; so does the second, because the sum over $i$ converges uniformly with respect to $\Lambda$. Now, by assumption B, $\|F\|_0 < \infty$ and:

$$\big(\mathcal{Q}_{\Lambda'}\, A \,\tilde{-}\, \mathcal{Q}_\Lambda\, A\big) \prec \sum_{i,j\in\Lambda', i \text{ or } j\notin\Lambda} \Gamma_{i\to j}\,\delta W_{i\to j}\,\tilde{A}\,\delta W_{i\to j} \;\prec\; 4\,\big\|F\big\|_0^2\big\|A\big\|\,F(G_{\Lambda'} - G_\Lambda)F\ .$$

The last inequality follows from the estimate:

$$\big(\langle i| + \langle j|\big)F\tilde{A}F\big(|i\rangle + |j\rangle\big) \leq 4\big\|F\big\|_0^2\Big(\sup_{k\in\Lambda_\infty}\sum_{l\in\Lambda_\infty}\big|\langle l|A|k\rangle\big|^2\Big)^{\frac{1}{2}} \leq 4\big\|F\big\|_0^2\,\big\|A\big\|\ .$$

Similarly, $(Y_{\Lambda'} \,\tilde{-}\, Y_\Lambda) \prec F(G_{\Lambda'} - G_\Lambda)F$. Then, by (59), the nets $(Y_\Lambda)_{\Lambda\in L}$, $(Y_\Lambda^*)_{\Lambda\in L}$ and $(\mathcal{Q}_\Lambda\, A)_{\Lambda\in L}$ in $\mathcal{B}(\mathcal{H})$ are Cauchy. Therefore they converge respectively to $Y$, $Y^*$ and $\mathcal{Q}\, A \in \mathcal{B}(\mathcal{H})$. This proves that $(\mathcal{C}_\Lambda\, A)_{\Lambda\in L}$ and the truncated sums in (3),

$$K_\Lambda = \frac{1}{2\mathrm{i}}\sum_{i,j\in\Lambda}\Gamma_{i\to j}\big(W_{i\to j}^* + 1\big)\big(W_{i\to j} - 1\big) = \frac{1}{2\,\mathrm{i}}\,\mathcal{Q}_\Lambda\, 1 - \mathrm{i}\, Y_\Lambda\ ,$$

converge strongly. The same estimates show that $\mathcal{C}_\Lambda$ and $K_\Lambda$ are uniformly bounded in $\Lambda$, e.g.

$$\big\|\mathcal{C}_\Lambda\big\| \leq \big(8\|F\|_0^2 + 4\big)\big\|F\,G\,F\big\| \quad , \quad \tilde{K}_\Lambda \prec \big(2\|F\|_0^2 + 1\big)F\,G\,F\ ,\ \Lambda\in L\ .$$

By corollary 3 and the equivalence between the strong and ultrastrong topologies on balls in $\mathcal{B}(\mathcal{H})$ [44], $\Phi_\Lambda(t)A \to e^{t(\mathcal{L}_H + \mathcal{C})}A$ ultrastrongly, where $\mathcal{C}\, A$ is the ultrastrong limit of $(\mathcal{C}_\Lambda\, A)_{\Lambda\in L}$,

$$\mathcal{C}\, A = -\frac{1}{2}\,(\mathcal{Q}\,1)\,A - Y^*\,A - \frac{1}{2}\,A\,(\mathcal{Q}\,1) - A\,Y + \mathcal{Q}\,A + Y^*\,A + A\,Y = \mathcal{Q}\,A - \frac{1}{2}\big\{\mathcal{Q}\,1, A\big\}\ .$$

*Proof of statement (3).* Let the assumptions A to D hold. A similar proof of that of proposition 4 shows that the operators $\Theta_\Lambda(t)$ on $\mathcal{B}(\mathcal{H})$ such that $\langle\psi|\Theta_\Lambda(t)A|\varphi\rangle = \mathbb{E}\,\langle C_\Lambda(t)\psi|A|C_\Lambda(t)\varphi\rangle$ are equal to $\exp(t\,\mathcal{G}_\Lambda)\,A$, with:

$$\mathcal{G}_\Lambda\, A = S^* A + A\,S + \sum_{i,j\in\Lambda}\Gamma_{i\to j}\Big(\big(1 + \delta W_{i\to j}^*\big)A\big(1 + \delta W_{i\to j}\big) - A\Big)\ .$$

It follows from the same arguments as above that $(\mathcal{G}_\Lambda\, A)_{\Lambda\in L}$ and thus $(\Theta_\Lambda(t)A)_{\Lambda\in L}$ converge ultrastrongly respectively to $\mathcal{G}\, A$ and $\Theta(t)A$. It has been shown in section 7.3 that for any

$|\psi\rangle \in \mathcal{H}$, $(C_\Lambda(t)|\tilde{\psi}\rangle)_{\Lambda\in L}$ converges a.s.. Let $C(t)|\tilde{\psi}\rangle$ be its limit. By lemma 1 and the monotone convergence theorem, $\|C(t)|\tilde{\psi}\rangle\|^2 \in L^1(\Xi, \mathbb{P})$ (its integral is equal to $\langle\tilde{\psi}|\Theta(t)\,1|\tilde{\psi}\rangle$). The operators $\mathcal{U}_\Lambda(t,t_0)$ satisfy all the axioms of RP but **(RP6)** and:

$$\left\|U_\Lambda^\xi(\tau,t_0)|\psi\rangle\right\| \leq \left\|C^\xi(t)|\tilde{\psi}\rangle\right\| \ , \ \Lambda \in L, 0 \leq t_0 \leq \tau \leq t, \ \text{for almost all } \xi \in \Xi \ .$$

Moreover, $\left\|\big(\mathcal{U}_\Lambda(t,\tau)\,1\otimes P_{\Lambda'}\,\mathcal{U}_\Lambda(\tau) - \mathcal{U}_\Lambda(t)\big)1\otimes|\psi\rangle\right\|_{\mathcal{K}}^2$ is bounded from above by the mean value $\mathbb{E}\,\left\|C_\Lambda(t,\tau)\big(1-P_{\Lambda'}\big)C_\Lambda(\tau)|\tilde{\psi}\rangle\right\|^2$, and thus by $\langle\tilde{\psi}|\Theta(\tau)(1-P_{\Lambda'})(\Theta(t-\tau)\,1)(1-P_{\Lambda'})|\tilde{\psi}\rangle$. The last bound is $\Lambda$-independent and tends to zero as $\Lambda' \uparrow \Lambda_\infty$. Hence, from proposition 1, the operators $\mathcal{U}(t,t_0)$ satisfy all the axioms of RP but **(RP6)**. For any $|\psi\rangle, |\varphi\rangle \in \mathcal{H}$, the scalar product $\langle U_\Lambda(\tau)\varphi|A|U_\Lambda(\tau)\psi\rangle$ is bounded in absolute value by $\|A\|\,\|C(t)|\tilde{\varphi}\rangle\|\,\|C(t)|\tilde{\psi}\rangle\|$, which is in $L^2(\Xi,\mathbb{P})$. One then obtains (34) by applying the dominated convergence theorem. In particular,

$$\mathbb{E}\,\left\|U(\tau)|\psi\rangle\right\|^2 = \langle\psi|\Phi(\tau)\,1|\psi\rangle = \left\|\psi\right\|^2 \ , \ \tau \geq 0 \ .$$

This shows that $\mathcal{U}$ is a RP. $\square$

# 8 Conclusion

We have studied in this work a model describing dissipation in quantum systems by means of a random evolution in time. This model can be seen as a quantum generalization of a classical kinetic model, the classical collisions being replaced by quantum jumps. The input parameters of the models are: (1) a set of transition rates $\Gamma_{i\to j} \geq 0$, for all pairs $(|i\rangle, |j\rangle)$ of vectors of a given orthonormal basis $\{|i\rangle, i \in \Lambda_\infty\}$ of the system's Hilbert space $\mathcal{H}$; (2) some bounded operators $W_{i\to j}$ acting on $\mathcal{H}$, which describe the effect of quantum jumps and satisfy the assumption B of section 3.1. The random time evolution in the Hilbert space $\mathcal{H}$ is given by a set of independent Poisson processes, a different Poisson process, with parameter $\Gamma_{i\to j}$, being associated to each pair $(i,j)$. The evolution of the observables of the system averaged over the dynamical noise is given by quantum dynamical semigroup with a Lindblad generator $\mathcal{L} = \mathcal{L}_H + \mathcal{C}$ given by (32). Our main result is that under some exponential decay hypothesis on the transition rates $\Gamma_{i\to j}$ and on the matrix elements of the Hamiltonian $H$, together with some locality condition B on the jump operators $W_{i\to j}$, the stochastic evolution of the system is well-defined as some limit if $\mathcal{H}$ is infinite dimensional and the sum $\sum_{i,j}\Gamma_{i\to j}$ diverges. This result can be readily generalized to $N$-particles systems with one-particle Hilbert space, provided $N$ is finite. The limit of an infinite number of particles with a finite density (thermodynamic limit) requires, however, a more abstract algebraic approach [44]. For aperiodic solids like strongly disordered solids, one should define a stochastic dynamics on the $C^*$-algebra of the electronic observables in second quantization, which is the crossed product of a continuous field of $C^*$-algebras by a groupoid [36].

The use of Poisson processes is natural from a physical point of view, especially if the dissipation mechanism under study is due to absorption and emission of external particles by the system (phonons, photons,...). It is also convenient because of its mathematical simplicity. Unlike in the model defined by Dalibard *et al.* [6] and Carmichael [7], the random time evolution in our model linear, but not norm-preserving. The linearity simplifies greatly the mathematical analysis. The random evolution operators can be computed directly from formula (4). The model thus provides an example of quantum jump scheme which is given in terms of a classical stochastic processes and for which one can handle rigorously the case where infinitely many orthogonal wavefunctions $|i\rangle$ are coupled to the environment. The model can be applied to study electronic transport in disordered or aperiodic solids. A simple example was given in section 2.2. However, a theory of linear response similar to that elaborated in [25, 26, 27] is still lacking within our stochastic wavefunctions framework. Investigation in this direction will be the object of a separate publication [51].

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# A Appendix

We prove in this appendix the proposition 7 on the convergence of the stochastic Hamiltonian.

Let $I \subset \mathbb{R}$ be a compact interval. Define:

$$N_i(I) = \sum_{j\in\Lambda_\infty} N_{i\to j}(I) \quad , \quad N_i^*(I) = \sum_{j\in\Lambda_\infty} N_{j\to i}(I) \ . \tag{A.1}$$

The counting functions $N_i(t), N_i^*(t)$ are defined as in section 2.1. $(N_i(t))_{t\in\mathbb{R}}$ and $(N_i^*(t))_{t\in\mathbb{R}}$ are respectively Poisson processes of parameters $\Gamma_i$ and $\Gamma_i^*$, with:

$$\Gamma_i = \sum_{j\in\Lambda_\infty} \Gamma_{i\to j} \quad , \quad \Gamma_i^* = \sum_{j\in\Lambda_\infty} \Gamma_{j\to i} \ . \tag{A.2}$$

**Lemma 4** *Let assumption* (31) *of theorem 2 be satisfied. Let* $\nu \in \mathbb{N}^*$ *and* $i \mapsto z_i$ *be a map from* $\Lambda_\infty$ *into* $\mathbb{R}_+^*$ *such that* $\sum_{i\in\Lambda_\infty} z_i^{\delta-1} < \infty$ *for some* $\delta$*,* $0 < \delta < 1$*. Then:*

*(i) if* $\limsup_{i\in\Lambda_\infty} \Gamma_i > 0$ *(resp.* $\limsup_{i\in\Lambda_\infty} \Gamma_i^* > 0$*), then* $\sup_{i\in\Lambda_\infty} N_i^\nu(I)$ *(resp.* $\sup_{i\in\Lambda_\infty} N_i^*(I)^\nu$*) is infinite with probability* 1*.*

*(ii)* $\sum_{i\in\Lambda_\infty} N_i(I)^\nu \, z_i^{-1}$ *and* $\sum_{i\in\Lambda_\infty} N_i^*(I)^\nu \, z_i^{-1}$ *are finite with probability* 1*.*

*Proof:* Obviously, one needs only to prove *(i)* for $\nu = 1$. By hypothesis, there are $\beta > 0$ and a sequence $(i_m)_{m\in\mathbb{N}}$ in $\Lambda_\infty$ such that $\beta \le \Gamma_{i_m}$ for any $m \in \mathbb{N}$. Let $p$ be the probability that there exists an integer $n$ satisfying $N_i(I) < n$ for any $i \in \Lambda_\infty$. The Poisson processes $(N_i(t))_{t\in\mathbb{R}}$ being mutually independent,

$$p = \lim_{n\to\infty} \prod_{i\in\Lambda_\infty} \mathbb{P}(N_i(I) < n) \le \lim_{n\to\infty} \prod_{m=0}^{\infty} \Big(1 - \mathbb{P}(N_{i_m}(I) \ge n)\Big) \ . \tag{A.3}$$

Let $\gamma = \sup_{i\in\Lambda_\infty} \Gamma_i$. If $n > \gamma\,|I|$, then

$$\mathbb{P}(N_{i_m}(I) \ge n) = \sum_{l=n}^{\infty} e^{-\Gamma_{i_m}|I|} \, \frac{(\Gamma_{i_m}|I|)^l}{l!} \ \ge \ \sum_{l=n}^{\infty} e^{-\beta|I|} \, \frac{(\beta|I|)^l}{l!} > 0$$

for any $m \in \mathbb{N}$. Hence (A.3) implies $p = 0$, which proves *(i)*.

We now show *(ii)*. For any $m \in \mathbb{N}^*$, let $\chi_i^{(m)} = 1$ if $N_i(I)^\nu \geq z_i^{1/m}$ and 0 otherwise. By the Tchebychev inequality,

$$\mathbb{E} \sum_{i \in \Lambda_\infty} \chi_i^{(m)} = \sum_{i \in \Lambda_\infty} \mathbb{P}\big(N_i(I)^\nu \geq z_i^{1/m}\big) \leq \sum_{i \in \Lambda_\infty} \frac{\mathbb{E}\ N_i(I)^{m\nu}}{z_i} \ . \tag{A.4}$$

An explicit calculation shows that $\mathbb{E}\ N_i(I)^{m\nu}$ is a polynomial in $\Gamma_i|I|$ of degree $m\nu$, with coefficients independent of $i$. Since $0 \leq \Gamma_i \leq \gamma$ for all $i$'s, one has $\sup_{i \in \Lambda_\infty} \mathbb{E}\ N_i(I)^{m\nu} < \infty$. But $\sum_{i \in \Lambda_\infty} z_i^{-1} < \infty$, thus $\mathbb{E} \sum_{i \in \Lambda_\infty} \chi_i^{(m)}$ is finite. This implies that with probability one, $\chi_i^{(m)} = 0$ for all $i$'s but a finite number of them, say $i \in \Lambda^{(m)}$, with $\Lambda^{(m)}$ a random finite subset of $\Lambda_\infty$ (Borel Cantelli). We may choose $m \geq 1/\delta$, then:

$$\sum_{i \in \Lambda_\infty} \frac{N_i(I)^\nu}{z_i} < \sum_{i \in \Lambda^{(m)}} \frac{N_i(I)^\nu}{z_i} + \sum_{i \in \Lambda_\infty} z_i^{1/m-1} < \infty \tag{A.5}$$

with probability 1. Similar arguments hold for $N_i^*(I)$. $\square$

*Proof of Proposition 7.* Let $|\psi\rangle \in \mathcal{H}$ and $I$ be a finite interval containing $\operatorname{supp} g$. By hypothesis, $\|F'\|_0 = \sup_i \sum_l f'_{il}$ is finite. By writing the norm in terms of the matrix elements of $V_{i\to j}$ and using the assumption (53), one obtains:

$$\Big\| V_{i\to j} \sum_{n=-\infty}^{\infty} g(t^n_{i\to j})|\psi\rangle \Big\| \leq \mathit{const.}\ N_{i\to j}(I) \sum_{k \in \Lambda_\infty} \big|\langle k|\psi\rangle\big| \big(f'_{ik} + f'_{jk}\big)$$

with $\mathit{const.} = 2\,\|F'\|_0 \sup_t |g(t)|$. Thus:

$$\big\| < V_\Lambda, g > |\psi\rangle \big\| \leq \mathit{const.} \sum_{i \in \Lambda} \sum_{k \in \Lambda_\infty} \big(N_i(I) + N_i^*(I)\big) \big|\langle k|\psi\rangle\big| f'_{ik} \ . \tag{A.6}$$

Let $i \in \Lambda_\infty \mapsto z_i \in \mathbb{R}_+^*$ be a map on $\Lambda_\infty$ such that $b\,|i|^{d+\eta/4} \leq z_i \leq c\,|i|^{d+\eta/2}$ for sufficiently big $|i|$'s ($b$ and $c$ are positive constants). By assumption,

$$a_k^2 = \sum_{i \in \Lambda_\infty} f'_{ik}\, z_i < \infty \tag{A.7}$$

for any $k \in \Lambda_\infty$. The operator $A = \sum_k a_k |k\rangle\langle k|$ is self-adjoint, positive, and has a domain $\mathcal{D}$ dense in $\mathcal{H}$. It follows from (A.6) and from the Cauchy-Schwartz inequality that, if $|\psi\rangle \in \mathcal{D}$,

$$\big\| < V_\Lambda, g > |\psi\rangle \big\| \leq const. \, \big\|F'\big\|_0^{\frac{1}{2}} \Big( \sum_{i\in\Lambda_\infty} \big(N_i(I) + N_i^*(I)\big)^2 z_i^{-1} \Big)^{\frac{1}{2}} \big\| A|\psi\rangle \big\| \, . \qquad \text{(A.8)}$$

The map $i \mapsto z_i$ satisfies the assumptions of the lemma above if $\eta > 4\,\delta\, d/(1-\delta)$. Therefore the sum over $i$ is finite with probability 1. Given $\Lambda, \Lambda' \in L$, $\Lambda \subset \Lambda'$, a similar estimate gives:

$$\big\| (< V_{\Lambda'}, g > - < V_\Lambda, g >) |\psi\rangle \big\| \leq 2\, const. \, \big\|F'\big\|_0^{\frac{1}{2}} \Big( \sum_{i\in\Lambda'\setminus\Lambda} \big(N_i(I) + N_i^*(I)\big)^2 z_i^{-1} \Big)^{\frac{1}{2}} \big\| A|\psi\rangle \big\| \, . \qquad \text{(A.9)}$$

By the almost sure convergence of the series $\sum_{i\in\Lambda_\infty} (N_i(I) + N_i^*(I))^2 z_i^{-1}$, this proves that the left hand side tends to zero with probability one as $\Lambda, \Lambda' \uparrow \Lambda_\infty$. Therefore the net $(< V_\Lambda, g > |\psi\rangle)_{\Lambda \in L}$ is Cauchy and thus converges. The estimate (54) follows from (A.8). □

Let us show that $< V, g >$ may be unbounded with probability one. We assume that $V_{i\to j} = \pi\,|i\rangle\langle i|$ and that $\limsup_{i\in\Lambda_\infty} \Gamma_i > 0$ for any $i, j \in \Lambda_\infty$. If $g(t)$ a positive function equal to 1 on a finite interval $I \subset \mathbb{R}$, then:

$$\lim_{\Lambda\uparrow\Lambda_\infty} \big\| < V_\Lambda, g > \big\| \geq \sup_{k\in\Lambda_\infty} \Big\| \sum_{i,j\in\Lambda, n\in\mathbb{Z}} g\big(t - t_{i\to j}^n\big) V_{i\to j} |k\rangle \Big\| \geq \pi \sup_{k\in\Lambda_\infty} N_k(I) \, .$$

By the lemma above, the right hand side is infinite with probability one.

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