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A spectral sequence for the K-theory of tiling spaces

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Abstract. Let $\mathcal{T}$ be an aperiodic and repetitive tiling of $\mathbb{R}^d$ with finite local complexity. We present a spectral sequence that converges to the K-theory of $\mathcal{T}$ with page-2 given by a new cohomology that will be called PV in reference to the Pimsner–Voiculescu exact sequence. It is a generalization of the Serre spectral sequence. The PV cohomology of $\mathcal{T}$ generalizes the cohomology of the base space of a fibration with local coefficients in the K-theory of its fiber. We prove that it is isomorphic to the Čech cohomology of the hull of $\mathcal{T}$ (a compactification of the family of its translates).

1. *Main results*

Let $\mathcal{T}$ be an aperiodic and repetitive tiling of $\mathbb{R}^d$ with finite local complexity (Definition 3). The hull Ω is a compactification, with respect to an appropriate topology, of the family of translates of $\mathcal{T}$ by vectors of $\mathbb{R}^d$ (Definition 4). The tiles of $\mathcal{T}$ are given compatible Δ-complex decompositions (§4.2), with each simplex punctured, and the Δ-transversal Ξ_Δ is the subset of Ω corresponding to translates of $\mathcal{T}$ having the puncture of one of those simplices at the origin $0_{\mathbb{R}^d}$. The prototile space $\mathcal{B}_0$ (Definition 9) is built out of the prototiles of $\mathcal{T}$ (translational equivalence classes of tiles) by gluing them together according to the local configurations of their representatives in the tiling.

The hull is given a dynamical system structure via the natural action of the group $\mathbb{R}^d$ on itself by translation [**13**]. The C^*-algebra of the hull is isomorphic to the crossed product C^*-algebra $C(\Omega) \rtimes \mathbb{R}^d$.

There is a map $\mathfrak{p}_o$ from the hull onto the prototile space (Proposition 1)

$$\begin{array}{ccc} \Xi_\Delta & \hookrightarrow & \Omega \\ & & \downarrow \mathfrak{p}_0 \\ & & \mathcal{B}_0 \end{array}$$

which, thanks to a lamination structure on Ω (Remark 2), resembles (although is *not*) a fibration with base space $\mathcal{B}_0$ and fiber Ξ_Δ (Remark 4). The Pimsner–Voiculescu (PV) cohomology H^*_{PV} of the tiling (Definition 15) is a cohomology of the base space $\mathcal{B}_0$ with ‘local coefficients’ in the K-theory of the fiber Ξ_Δ (Remark 5).

THEOREM 1. *There is a spectral sequence that converges to the K-theory of the C^*-algebra of the hull*

$$E_2^{rs} \Rightarrow K_{r+s+d}(C(\Omega) \rtimes \mathbb{R}^d),$$

and whose page-2 is given by

$$E_2^{rs} \cong H^r_{\mathrm{PV}}(\mathcal{B}_0;\, K^s(\Xi_\Delta)).$$

By an argument using the Thom–Connes isomorphism [**21**] the K-theory of $C(\Omega) \rtimes \mathbb{R}^d$ is isomorphic to the topological K-theory of Ω (with a shift in dimension by d), and the above theorem can thus be seen formally as a generalization of the Serre spectral sequence [**67**] for a certain class of laminations $\Xi_\Delta \hookrightarrow \Omega \to \mathcal{B}_0$, which are foliated spaces [**52**] but not fibrations.

This result brings a different point of view on a problem solved earlier by Hunton and Forrest in [**27**]. They built a spectral sequence for the K-theory of a crossed product C^*-algebra of a $\mathbb{Z}^d$-action on a Cantor set. Such an action exists always for tilings of finite local complexity, but it is by no means canonical. Indeed, thanks to a result of Sadun and Williams [**64**], the hull of a repetitive tiling with finite local complexity is homeomorphic to a fiber bundle over a torus with fiber the Cantor set. These two results are sufficient to get the K-theory of the hull which, thanks to the Thom–Connes theorem [**21**], gives also the K-theory of the C^*-algebra of the tiling $\mathcal{C}(\Omega) \rtimes \mathbb{R}^d$. However, the construction of the hull through an inverse limit of branched manifolds, initiated by Anderson and Putnam [**1**] for the case of substitution tilings and generalized in [**13**] to all repetitive tilings with finite local complexity, suggests a different and more canonical construction. So far, however, it is not yet efficient for practical calculations.

One-dimensional repetitive tilings with finite local complexity are all Morita equivalent to a $\mathbb{Z}$-action on a Cantor set. The Pimsner–Voiculescu exact sequence [**55**] is then sufficient to compute the K-theory of the hull [**9**]. In the late nineties, before the paper by Forrest and Hunton was written, Mihai Pimsner suggested to one of the authors† a method to generalize the theorem to $\mathbb{Z}^d$-actions. This spectral sequence was used and described already in [**12**] and is a special case of the Kasparov spectral sequence [**40**] for KK-theory. We recall it here for completeness and to justify naming H^*_{PV} after Pimsner and Voiculescu.

THEOREM 2. *Let $\mathcal{A}$ be a C^*-algebra endowed with a $\mathbb{Z}^d$ action α by $*$-automorphisms. The PV complex is defined as $K_*(\mathcal{A}) \otimes \Lambda^*\mathbb{Z}^d \xrightarrow{d_{\mathrm{PV}}} K_*(\mathcal{A}) \otimes \Lambda^*\mathbb{Z}^d$ with*

$$d_{\mathrm{PV}} = \sum_{i=1}^{d} (\alpha_{i*} - \mathbf{1}) \otimes e_i \wedge,$$

where $\{e_1, \ldots, e_d\}$ is the canonical basis of $\mathbb{Z}^d$, $\alpha_i = \alpha_{e_i}$ is the restriction of α to the ith component of $\mathbb{Z}^d$, whereas $x\wedge$ is the exterior multiplication by $x \in \mathbb{Z}^d$.

There is a spectral sequence converging to the K-theory of $\mathcal{A}$

$$E_2^{rs} \Rightarrow K_{r+s+d}(\mathcal{A} \rtimes_\alpha \mathbb{Z}^d),$$

with page-2 isomorphic to the cohomology of the PV complex.

† J.B. is indebted to M. Pimsner for this suggestion.

Idea of the proof. The spectral sequence is built out of the cofiltration associated with the filtration of the mapping torus $M_\alpha(\mathcal{A})$ by ideals of functions vanishing on the skeleton of the torus. The differential for PV cohomology can be identified using the K-theory maps (Bott and boundary maps). The reader is referred to [**15**] and in particular to Proposition 10.4.1 where the problem for $d = 1$ is treated briefly. □

A more topological expression of this theorem consists in replacing $\mathbb{Z}^d$ by its classifying space, the torus $\mathbb{T}^d$, with a CW-complex decomposition given by an oriented (open) d-cube and all of its (open) faces in any dimension. Then the PV complex can be proved to be isomorphic to the following complex: the cochains are given by covariant maps $\varphi(e) \in K_*(\mathcal{A})$, where e is a cell of $\mathbb{T}^d$ and $\varphi(\overline{e}) = -\varphi(e)$ if $\overline{e}$ is the face e with opposite orientation. The covariance means that if two cells e, e' differ by a translation $a \in \mathbb{Z}^d$ then $\varphi(e') = \alpha^a \varphi(e)$. The differential is the usual one, namely $d\varphi(e) = \sum_{e' \in \partial e} \varphi(e')$. If $H^*_{\mathrm{PV}}(\mathbb{T}^d; K_*(\mathcal{A}))$ denotes the corresponding cohomology, this gives the following.

COROLLARY 1. *The PV cohomology group for the crossed product* $\mathcal{A} \rtimes_\alpha \mathbb{Z}^d$ *is isomorphic to* $H^*_{\mathrm{PV}}(\mathbb{T}^d; K_*(\mathcal{A}))$.

The spectral sequence used by Forrest and Hunton in [**27**] in the case $\mathcal{A} = \mathcal{C}(X)$ where X is the Cantor set coincides with the PV spectral sequence. The present paper generalizes this construction for tilings by replacing the classifying space $\mathbb{T}^d$, by the prototile space $\mathcal{B}_0$.

The hull can also be built out of a *box decomposition* [**13**]. Namely a *box* is a local product of the transversal (which is a Cantor set) by a polyhedron in $\mathbb{R}^d$ called the *base* of the box. Then the hull can be shown to be given by a finite number of such boxes together with identifications on the boundary of the bases. This is an extension of the mapping torus, for which there is only one box with base given by a *cube* [**67**]. Since $\mathcal{A}$ in the present case is the space of continuous functions on the Cantor set X, $K_0(\mathcal{A})$ is isomorphic to $\mathcal{C}(X, \mathbb{Z})$ whereas $K_1(\mathcal{A}) = 0$, leading to the present result. In §4.3 an example of an explicit calculation of the PV cohomology is proposed illustrating the way it can be used. Further examples, and methods of calculation of PV cohomology will be investigated in future research. However, as it turns out (see §4.1), the PV cohomology is isomorphic to other cohomologies used so far on the hull, such as the Čech cohomology [**1, 63**], the group cohomology [**27**] or the pattern-equivariant cohomology [**42, 43, 66**].

2. *Historic background*

Spectral sequences were introduced by Leray [**48**] during the Second World War, as a way to compute the cohomology of a sheaf. It was later put in the framework used today by Koszul [**44**]. One of the first applications of this method was performed by Borel and Serre [**16**]. Later, Serre, in his PhD Thesis [**67**], defined the notion of Serre fibration and built a spectral sequence to compute the singular homology or cohomology. This more or less led him to calculate the rational homotopy of spheres.

In the early fifties, Hirzebruch made an important step in computing the Euler characteristic of various complex algebraic varieties and complex vector bundles over them [**35**]. He showed that this characteristic can be computed from the Chern classes of the tangent bundle and of the vector bundle through universal polynomials [**36**] which coincides with the Todd genus in the case of varieties. It allowed him to show that the Euler characteristic is additive for extensions namely if E, E', E'' are complex vector bundles and if $0 \to E' \to E \to E'' \to 0$ is an exact sequence, then $\chi(E) = \chi(E') + \chi(E'')$. This additivity property led Grothendieck to define axiomatically an additive group characterizing this additivity relation, which he called the K-group [**32**]. It was soon realized by Atiyah and Hirzebruch [**2**] that the theory could be extended to topological spaces X and they defined the topological K-theory as a cohomology theory without the axiom of dimension. The K_0-group is the set of stable equivalence classes of complex vector bundles over X, while K_1 (and more generally K_n) is the set of stable equivalence classes of complex vector bundles over the suspension (respectively the nth suspension) of X. The Bott periodicity theorem reduces the number of groups to two only. Moreover, the Chern character was shown to define a natural map between the K-group and the integer Čech cohomology of X, and that it becomes an isomorphism when both groups are rationalized. In this seminal paper [**2**] Atiyah and Hirzebruch define a spectral sequence that will be used in the present work. It is a particular case of Serre spectral sequence for the trivial fibration of X by itself with fiber a point: it converges to the K-theory of X and its page-2 is isomorphic to the cohomology of X.

Almost immediately after this step, Atiyah and Singer extended the work of Hirzebruch to the *Index theorem* [**3, 4**] for elliptic operators. Such an operator is defined between two vector bundles. It is unbounded, in general, and with the correct domains of definition defines a Fredholm operator. The index can be interpreted as an element of the K-group and is calculated through a formula which generalizes the results of Hirzebruch for algebraic varieties. Eventually, Atiyah and Singer extended the theory to the equivariant K-theory valid if a compact Lie group G acts on the vector bundle. If P is an elliptic operator commuting with G its index gives an element of the covariant K-theory. In a programmatic paper [**70**], Singer proposed various extensions to elliptic operators with coefficients depending on parameters. As an illustration of such a program, Coburn, Moyer and Singer [**19**] gave an index theorem for elliptic operators with almost periodic coefficients (see also [**72**]). Eventually the index theorem became the cornerstone of Connes' program to build a *Non-commutative Geometry*. In a seminal paper [**20**] he defined the theory of *non-commutative integration* and showed that its first application was an index theorem for elliptic operators on a foliation (see also [**22, 24**]).

In the seventies, it was realized that the Atiyah–Hirzebruch K-theory could be expressed in algebraic terms through the C^*-algebra $\mathcal{C}(X)$ of continuous functions on the compact space X. The definition of the K-group requires the consideration of matrix-valued continuous functions $M_n(\mathcal{C}(X))$ instead for all n. The smallest C^*-algebra containing all of them is $\mathcal{C}(X) \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the C^*-algebra of compact operators on a separable Hilbert space. Therefore, since this later algebra is non-commutative, all the construction could be used for any C^*-algebra. Then Kasparov [**39, 40**] defined the notion of KK-theory generalizing even more the K groups to correspondences between two C^*-algebras.

The problem investigated in the present paper is the latest development of a program that was initiated in the early eighties [**5, 38**] when the first version of the *gap-labeling theorem* was proved (see [**6, 9, 13**] for later developments). At that time the problem was to compute the spectrum of a Schrödinger operator H in an aperiodic potential. Several examples where discovered of Schrödinger operators with a Cantor-like spectrum [**37, 53**]. Labeling the infinite number of gaps per unit length interval was a challenge. It was realized that the K-theory class of the spectral projection on spectrum below the gap was a proper way of doing so [**5**]. The first calculation was made on the Harper equation and gave an explanation for a result already obtained by Claro and Wannier [**18**], a result eventually used in the theory of the quantum Hall effect [**74**]. This problem was motivated by the need for a theory of aperiodic solids, in particular their electronic and transport properties. With the discovery of *quasicrystals* in 1984 [**68**], this question became crucial in Solid State Physics. The construction of the corresponding C^*-algebra became then the main issue and led to the definition of the *hull* [**6**]. It was proved that the hull is a compact metrizable space Ω endowed with an action of $\mathbb{R}^d$ via homeomorphisms. Then it was proved in [**6, 9**], that the resolvent of the Schrödinger operator H belongs to the C^*-algebra $\mathcal{A} = \mathcal{C}(\Omega) \rtimes \mathbb{R}^d$. With each $\mathbb{R}^d$-invariant probability measure $\mathbb{P}$ on Ω is associated a trace $\mathcal{T}_{\mathbb{P}}$ on this algebra. Following an argument described in [**5**], it was proved that a gap could be labeled by the value of the density of state, and that this value belongs to the image by the trace $\mathcal{T}_{\mathbb{P}}$ of the group $K_0(\mathcal{A})$. During the eighties several results went on to compute the set of gap labels [**6, 56**]. In one dimension, detailed results could be proved (see [**9**] for a review). The most spectacular result was given in the case of a discretized Laplacian with a potential taking on finitely many values (called *letters*): in such a case the set of gap labels is the $\mathbb{Z}$-module generated by the occurrence probability of all possible finite words found in the sequence defined by the potential. If this sequence is given by a *substitution*, these occurrence probabilities can be computed explicitly in terms of the incidence matrix of the substitution and of the associated substitution induced on the set of words with two letters [**9, 58**]. The key property in proving such results was the use of the Pimsner–Voiculescu exact sequence [**55**]. Soon after, van Elst [**26**] extended these results to the case of two-dimensional potentials, using the same method.

In the French version [**23**] of his book on *non-commutative geometry* [**24**], Connes showed how the general formalism he had developed could be illustrated with the special case of the Penrose tiling. Using the substitution rules for its construction, he introduced a C^*-algebra, which turns out to be AF, and computed its ordered K_0-group, using the classification of AF-algebras obtained by Bratelli in 1972 [**17**]. This result was an inspiration for Kellendonk, who realized that, instead of looking at the inflation rule as a source of non-commutativity, it was actually better to consider the space translations of the tiling [**41**], in the spirit of the formalism developed in [**5, 6, 9**] for aperiodic solids. He extended this latter construction of the hull to tilings and then this hull is called *tiling space*. In this important paper, Kellendonk introduced the notion of *forcing the border* for a general tiling, which appears today as an important property for the calculation of the K-groups and cohomology of a tiling space. This work gave a strong motivation to prove the *gap-labeling theorem in higher dimension* and to compute the K-group of the hull. It was clear that the method used in one dimension, through the Pimsner–Voiculescu exact sequence, could only be generalized through a spectral sequence. The first use of spectral sequences in computing the set of gap labels on the case of the two-dimensional octagonal tiling [**10**] was followed by a proof of the gap-labeling theorem for three-dimensional quasicrystals [**12**]. Finally, the work by Forrest and Hunton [**27**] used a classical spectral sequence to compute the full K-theory of the C^*-algebra for an action of $\mathbb{Z}^d$ on the Cantor set. It made possible the computation of the K-theory and the cohomology of the hull for quasicrystals in two and three dimensions [**28, 29**]. Other examples of tilings followed later [**65**].

In addition to Kellendonk's work, several important contributions helped to build tools to prove the higher-dimensional version of the gap labeling theorem. Among the major contributions was the work of Lagarias [**45–47**], who introduced a geometric and combinatoric aspect of tiling through the notion of a Delone set. This concept was shown to be conceptually crucial in describing aperiodic solids [**11**]. Through the construction of Voronoi, Delone sets and tilings become equivalent concepts, allowing for various intuitive point of views to study such problems.

Another important step was performed in 1998 by Anderson and Putnam [**1**], who proposed to build the tiling space of a substitution tiling through a CW-complex built from the prototiles of a tiling (called prototile space in this paper). The substitution induces a map from this CW-complex into itself and they showed that the inverse limit of such system becomes homeomorphic to the tiling space. A similar construction was proposed independently by Gambaudo and Martens in 1999 to describe dynamical systems. This latter case corresponds to one-dimensional repetitive tilings with finite local complexity, so that this latter construction goes beyond the substitution tilings. One interesting outcome of this work was a systematic construction of minimal dynamical systems, uniquely ergodic or not, and with positive entropy† (see [**30**]). Eventually, the Gambaudo–Martens construction led to the construction of the hull as an inverse limit of compact oriented branched flat Riemannian manifolds [**13**]‡ used in the present paper (although only their topological CW-complex structure is needed here). Equivalently, the hull can be seen as a *lamination* [**31**] of a *foliated space* [**52**]. The extension of this construction to include tilings without finite local complexity, such as the pinwheel model [**60**], was performed by Gambaudo and Benedetti [**14**] using the notion of solenoids (reintroduced by Williams in the seventies [**75**]).

3. *A mathematical reminder*

The preliminary definitions and results required for stating Theorem 1 are presented here. They are taken mostly from previous work of the second author in [**11, 13**] on tilings and Delone sets, and the reader is referred to those papers for complete proofs. Let $\mathbb{R}^d$ denote the usual Euclidean space of dimension d with Euclidean norm $\|\cdot\|$. First of all the definition of a repetitive tiling of $\mathbb{R}^d$ with finite local complexity is recalled, and then the hull and transversal of such a tiling are introduced. The connection with Delone sets is briefly mentioned as well as the definition of the groupoid of the transversal of a Delone set. The prototile space of a tiling and the ring of functions on the transversal are then built, and finally the PV cohomology is defined.

† This result circulated as a preprint in 1999 but was published only in 2006.
‡ This paper was posted on `arXiv.com math.DS/0109062` in its earlier version in 2001 but was eventually published in a final form in 2006 only.

3.1. *Tilings and their hulls.*

Definition 1.
(i) A *tile* of $\mathbb{R}^d$ is a compact subset of $\mathbb{R}^d$ which is homeomorphic to the unit ball.
(ii) A punctured tile is an ordered pair consisting of a tile and one of its points.
(iii) A *tiling* of $\mathbb{R}^d$ is a covering of $\mathbb{R}^d$ by a family of tiles whose interiors are pairwise disjoint. A tiling is said to be punctured if its tiles are punctured.
(iv) A *prototile* of a tiling is a translational equivalence class of tiles (including the puncture).
(v) Let r be the smallest distance between two punctures of $\mathcal{T}$. The first corona of a tile t is the union of the tiles of $\mathcal{T}$ that are within a distance r to t.
(vi) A *collared prototile* of $\mathcal{T}$ is the subclass of a prototile whose representatives have the same first corona up to translation.

A collared prototile is a prototile where a local configuration of its representatives has been specified: each representative has the same neighboring tiles.

In the following it is implicitly assumed that tiles and tilings are punctured. All tiles are assumed to be finite Δ-complexes which are particular CW-complex structures (see §4.2). They are also required to be compatible with the tiles of their first coronas, i.e. the intersection of any two tiles is itself a sub-Δ-complex of both tiles. In other words, tilings considered here are assumed to be Δ-complexes of $\mathbb{R}^d$.

Definition 2. Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$.
(i) A *patch* of $\mathcal{T}$ is a finite union of neighboring tiles in $\mathcal{T}$ that is homeomorphic to a ball. A patch is punctured by the puncture of one of the tiles that it contains. The radius of a patch is the radius of the smallest ball that contains it.
(ii) A *pattern* of $\mathcal{T}$ is a translational equivalence class of patches of $\mathcal{T}$.

The following notation will be used in what follows: prototiles and patterns will be written with a hat, for instance $\hat{t}$ or $\hat{p}$, to distinguish them from their representatives. Often the following convention will be implicit: if $\hat{t}$ is a prototile and $\hat{p}$ a pattern, then t and p will denote their respective representatives that have their punctures at the origin $0_{\mathbb{R}^d}$.

The results in this paper are valid for the class of tilings that are *repetitive* with *finite local complexity*.

Definition 3. Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$.
(i) $\mathcal{T}$ has *Finite Local Complexity* (FLC) if for any $R > 0$ the set of patterns of $\mathcal{T}$ whose representatives have radius less than R is finite.
(ii) $\mathcal{T}$ is *repetitive* if for any patch of $\mathcal{T}$ and every $\epsilon > 0$, there is an $R > 0$ such that for every x in $\mathbb{R}^d$ there exists *modulo an error* ϵ with respect to the Hausdorff distance, a translated copy of this patch belonging to $\mathcal{T}$ and contained in the ball $B(x, R)$.
(iii) For any x in $\mathbb{R}^d$, let $\mathcal{T} + x = \{t + x \mid t \in \mathcal{T}\}$ denote its translation; then $\mathcal{T}$ is *aperiodic* if there is no $x \neq 0$ in $\mathbb{R}^d$ such that $\mathcal{T} + x = \mathcal{T}$.

For tilings with FLC the repetitivity condition (ii) above can be stated more precisely: *a tiling $\mathcal{T}$ with FLC is repetitive if given any patch there is an $R > 0$ such that for every x in $\mathbb{R}^d$ there exists an* exact copy *of this patch in $\mathcal{T}$ contained in the ball $B(x, R)$.*

The class of repetitive tilings satisfying the FLC property is very rich and has been investigated for decades. It started in the 70s with the work of Penrose [**57**] and Meyer [**51**] and went on both from an abstract mathematical level and with a view towards applications, in particular to the physics of quasicrystals [**45, 46**]. It contains an important subclass of the class of *substitution tilings* that was reinvestigated in the 90s by Anderson and Putnam in [**1**], and also the class of tilings obtained by the *cut-and-projection* method, for which a comprehensive study by Hunton, Kellendonk and Forrest can be found in [**28**], and more generally it contains the whole class of *quasiperiodic tilings*, which are models for quasicrystals.

The Ammann aperiodic tilings are examples of substitution tilings (see [**33**, Ch. 10], Ammann's original work does not appear in the literature). In addition other examples include the octogonal tiling as well as the famous Penrose 'kite and darts' [**57**] tilings which are both substitution and cut-and-projection tilings (see [**33**, Ch. 10], and [**60**, Ch. 4]). However the so-called pinwheel tiling (see [**59**] and [**60**, Ch. 4]) is a substitution tiling but does not satisfy the FLC property given here since prototiles are defined here as equivalence classes of tiles under only translations and not more general isometries of $\mathbb{R}^d$ like rotations.

The first author has proposed in [**11**] a topology that applies to a large class of tilings (for which there exist an $r_0 > 0$ such that all tiles contain a ball of radius r_0). In the case considered here, where tiles are assumed to be finite CW-complexes, this topology can be adapted as follows.

Let $\mathcal{F}$ be a family of tilings whose tiles contain a ball of a fixed radius $r_0 > 0$ and have compatible CW-complex structures (the intersection of any two tiles is a subcomplex of both). Given an open set O in $\mathbb{R}^d$ with compact closure and an $\epsilon > 0$, a neighborhood of a tiling T in $\mathcal{F}$ is given by

$$U_{O,\epsilon}(T) = \left\{ T' \in \mathcal{F} \,\middle|\, \sup_{0 \le k \le d} d_H(O \cap T^k, O \cap T'^k) < \epsilon \right\},$$

where T^k and T'^k are the k-skeletons of T and T' respectively and d_H is the Hausdorff distance in $\mathbb{R}^d$.

Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$. The group $\mathbb{R}^d$ acts on the set of all translates of $\mathcal{T}$, the action (translation) is denoted T^a, $a \in \mathbb{R}^d$: $\mathrm{T}^a\mathcal{T} = \mathcal{T} + a = \{t + a \mid t \in \mathcal{T}\}$.

Definition 4.

(i) The *hull* of $\mathcal{T}$, denoted Ω, is the closure of $\mathrm{T}^{\mathbb{R}^d}\mathcal{T}$.

(ii) The canonical *transversal*, denoted Ξ, is the subset of Ω consisting of tilings that have the puncture of one of their tiles at the origin $0_{\mathbb{R}^d}$.

The hull of a tiling is seen as a dynamical system $(\Omega, \mathbb{R}^d, \mathrm{T})$ which, for the class of tilings considered here, has interesting properties that are now stated (see [**13**, §2.3]).

THEOREM 3. [**13, 45**] *Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$.*

(i) *$\mathcal{T}$ is repetitive if and only if the dynamical system of its hull $(\Omega, \mathbb{R}^d, \mathrm{T})$ is* minimal.
(ii) *If $\mathcal{T}$ has FLC, then its canonical transversal Ξ is totally disconnected.*
(iii) *If $\mathcal{T}$ is aperiodic, repetitive and has FLC, then Ξ is a Cantor set (perfect and totally disconnected).*

The minimality of the hull allows one to see any of its points as just a translate of $\mathcal{T}$.

Remark 1. A metric topology for tiling spaces has been used in the literature for historical reasons. Let $\mathcal{T}$ be a repetitive tiling or $\mathbb{R}^d$ with FLC. The orbit space of $\mathcal{T}$ under translation by vectors of $\mathbb{R}^d$, $\mathrm{T}^{\mathbb{R}^d}\mathcal{T}$, is endowed with a metric as follows (see [**13**, §2.3]). For T and T' in $\mathrm{T}^{\mathbb{R}^d}\mathcal{T}$, let A denote the set of ε in $(0, 1)$ such that there exist x and x' in $B(0, \varepsilon)$ for which $\mathrm{T}^x T$ and $\mathrm{T}^{x'} T'$ agree on $B(0, 1/\varepsilon)$, i.e. their tiles whose punctures lie in the ball are matching; then

$$\delta(T, T') = \begin{cases} \inf A & \text{if } A \neq \emptyset, \\ 1 & \text{if } A = \emptyset. \end{cases}$$

Hence the diameter of $\mathrm{T}^{\mathbb{R}^d}\mathcal{T}$ is bounded by 1 and the action of $\mathbb{R}^d$ is continuous.

For the class of repetitive tilings with FLC, the topology of the hull given in Definition 4 is equivalent to this δ-metric topology [**13**].

3.2. *Delone sets and groupoid of the transversal.* The notions for tilings given in the previous subsection can be translated for the set of their punctures in terms of Delone sets.

Definition 5. Let $\mathcal{L}$ be a discrete subset of $\mathbb{R}^d$.

(i) Given $r > 0$, $\mathcal{L}$ is *r-uniformly discrete* if any open ball of radius r in $\mathbb{R}^d$ meets $\mathcal{L}$ in at most one point.
(ii) Given $R > 0$, $\mathcal{L}$ is *R-relatively dense* if any open ball of radius R in $\mathbb{R}^d$ meets $\mathcal{L}$ in at least one point.
(iii) $\mathcal{L}$ is an *(r, R)-Delone set* if it is r-uniformly discrete and R-relatively dense.
(iv) $\mathcal{L}$ has *finite type* if $\mathcal{L} - \mathcal{L}$ is closed and discrete.
(v) $\mathcal{L}$ is repetitive if given any *finite* subset $p \subset \mathcal{L}$ *and any* $\epsilon > 0$, there is an $R > 0$ such that the intersection of $\mathcal{L}$ with any closed ball of radius R contains a copy (translation) of p *modulo an error of* ϵ *(with respect to the Hausdorff distance).*
(vi) $\mathcal{L}$ is *aperiodic* if there is no $x \neq 0$ in $\mathbb{R}^d$ such that $\mathcal{L} - x = \mathcal{L}$.

Langarias proved in [**45**] that condition (iv) is equivalent to finite local complexity.

Given a punctured tiling $\mathcal{T}$ if there exists $r, R > 0$ such that each of its tiles contains a ball a radius r and is contained in a ball of radius R, then its set of punctures $\mathcal{L}_{\mathcal{T}}$ is an (r, R)-Delone set, and it is repetitive and has FLC if and only if the tiling is repetitive and has FLC. Conversely, given a Delone set the Voronoi construction below gives a tiling, and they both share the same repetitivity or FLC properties.

Definition 6. Let $\mathcal{L}$ be an (r, R)-Delone set of $\mathbb{R}^d$. The Voronoi tile at $x \in \mathcal{L}$, is defined by

$$T_x = \{y \in \mathbb{R}^d \mid \|y - x\| \leq \|y - x'\|, \forall x' \in \mathcal{L}\},$$

with puncture the point x. The Voronoi tiling $\mathcal{V}$ associated with $\mathcal{L}$ is the tiling of $\mathbb{R}^d$ whose tiles are the Voronoi tiles of $\mathcal{L}$.

The tiles of the Voronoi tiling of a Delone set are (closed) convex polytopes that touch on common faces.

The hull and transversal of a Delone set are defined as follows [**11**].

Definition 7. Let $\mathcal{L}$ be a Delone set of $\mathbb{R}^d$. The hull Ω of $\mathcal{L}$ is the closure in the weak-$*$ topology of the set of translations of the Radon measure that has a Dirac mass at each point of $\mathcal{L}$

$$\Omega = \overline{\{\mathrm{T}^a \mu \mid a \in \mathbb{R}^d\}}^{w^*}, \qquad \mu = \sum_{x \in \mathcal{L}} \delta_x,$$

and the transversal Ξ of $\mathcal{L}$ is the subset

$$\Xi = \{\omega \in \Omega \mid \omega(\{0\}) = 1\}.$$

Given a general Delone set this topology of its hull is strictly coarser than the δ-metric topology for its Voronoi tiling given in Remark 1, but if it is repetitive and has FLC then they are equivalent [**13**]. The weak-$*$ topology is here also equivalent to the local Hausdorff topology on the set $\mathrm{T}^{\mathbb{R}^d}\mathcal{L}$ of translates of $\mathcal{L}$: given an open set O in $\mathbb{R}^d$ with compact closure and an $\epsilon > 0$, a neighborhood of ℓ is given by

$$U_{O,\epsilon}(\ell) = \{\ell' \in \mathrm{T}^{\mathbb{R}^d}\mathcal{L} \mid d_H(\ell \cap O, \ell' \cap O) < \epsilon\},$$

where d_H is the Hausdorff distance in $\mathbb{R}^d$.

The hull of a Delone set is also seen as a dynamical system under the homeomorphic action of $\mathbb{R}^d$ by translation. Just as in Proposition 3, the hull and transversal of repetitive Delone sets with FLC have similar interesting properties.

THEOREM 4. [**11**] *Let $\mathcal{L}$ be a Delone set of $\mathbb{R}^d$.*

(i) *$\mathcal{L}$ is repetitive if and only if the dynamical system of its hull* $(\Omega, \mathbb{R}^d, \mathrm{T})$ *is* minimal.

(ii) *If $\mathcal{L}$ has finite type, then its transversal Ξ is totally disconnected.*

(iii) *If $\mathcal{L}$ is aperiodic, repetitive and has finite type, then Ξ is a Cantor set (perfect and totally disconnected).*

The two approaches of tilings or Delone sets which are repetitive and have FLC are thus equivalent. Such tilings give rise to Delone sets (their sets of punctures), and conversely such Delone sets give rise to tilings (their Voronoi tilings) and their respective hulls share the same properties.

Recall that a groupoid is a small category in which every morphism is invertible [**20, 61**].

Definition 8. Let $\mathcal{L}$ be a Delone set of $\mathbb{R}^d$. The *groupoid of the transversal* is the groupoid Γ whose set of objects is the transversal, $\Gamma^0 = \Xi$, and whose set of arrows is

$$\Gamma^1 = \{(\xi, x) \in \Xi \times \mathbb{R}^d \mid \mathrm{T}^{-x}\xi \in \Xi\}.$$

Given an arrow $\gamma = (\xi, x)$ in Γ^1, its *source* is the object $s(\gamma) = \mathrm{T}^{-x}\xi \in \Xi$ and its *range* the object $r(\gamma) = \xi \in \Xi$.

The Delone sets considered here are repetitive with FLC and the groupoids of their transversals are *étale*, i.e. given any arrow γ with range (or source) object x, there exists an open neighborhood O_x of x in Γ^0 and a homeomorphism $\varphi : O_x \to \Gamma^1$ that maps x to γ. This implies that the sets of arrows having any given object as a source or range are discrete. This follows because for such Delone sets their sets of vectors are discrete and given any fixed $l > 0$ their sets of vectors of length less than l are finite.

3.3. *Prototile space of a tiling.* Let $\mathcal{T}$ be an aperiodic and repetitive tiling of $\mathbb{R}^d$ with FLC and assume its tiles are compatible finite CW-complexes (the intersection of two tiles is a subcomplex of both). A finite CW-complex $\mathcal{B}_0$, called prototile space, is built out of its prototiles by gluing them along their boundaries according to all the local configurations of their representatives in $\mathcal{T}$.

Definition 9. Let $\hat{t}_j$, $j = 1, \ldots, N_0$, be the prototiles of $\mathcal{T}$. Let t_j denote the representative of $\hat{t}_j$ that has its puncture at the origin. The *prototile space* of $\mathcal{T}$, $\mathcal{B}_0(\mathcal{T})$, is the quotient CW-complex

$$\mathcal{B}_0(\mathcal{T}) = \coprod_{j=1}^{N_0} t_j \,/\sim,$$

where two n-cells $e_i^n \in t_i^n$ and $e_j^n \in t_j^n$ are identified if there exists $u_i, u_j \in \mathbb{R}^d$ for which $t_i + u_i$ and $t_j + u_j$ are tiles of $\mathcal{T}$ such that $e_i^n + u_i$ and $e_j^n + u_j$ coincide on the intersection of their n-skeletons.

The *collared prototile space* of $\mathcal{T}$, $\mathcal{B}_0^c(\mathcal{T})$, is built similarly from the collared prototiles of $\mathcal{T}$: t_i^c, $i = 1, \ldots, N_0^c$.

The images in $\mathcal{B}_0(\mathcal{T})$ or $\mathcal{B}_0^c(\mathcal{T})$ of the tiles $t_j^{(c)}$ will be denoted τ_j and still be called tiles.

PROPOSITION 1. *There is a continuous map* $\mathfrak{p}_{0,\mathcal{T}}^{(c)} : \Omega \to \mathcal{B}_0^{(c)}(\mathcal{T})$ *from the hull onto the (collared) prototile space.*

Proof. Let $\lambda_0^{(c)} : \coprod_{j=1}^{N_0^{(c)}} t_j \to \mathcal{B}_0^{(c)}(\mathcal{T})$ be the quotient map. Also let $\rho_0^{(c)} : \Omega \times \mathbb{R}^d \to \coprod_{j=1}^{N_0^{(c)}} t_j^{(c)}$ be defined as follows. If x belongs to the intersection of k tiles $t^{\alpha_1}, \ldots, t^{\alpha_k}$, in ω, with $t^{\alpha_l} = t_{j_l}^{(c)} + u_{\alpha_l}(\omega)$, $l = 1, \ldots, k$, then $\rho_0^{(c)}(\omega, x) = \coprod_{l=1}^{k} x - u_{\alpha_l}(\omega)$ and lies in the disjoint union of the $t_{j_l}^{(c)}$.

The map $\mathfrak{p}_{0,\mathcal{T}}^{(c)}$ is defined as the composition: $\omega \mapsto \lambda_0^{(c)} \circ \rho_0^{(c)}(\omega, 0_{\mathbb{R}^d})$. The map $\rho_0^{(c)}(\cdot, 0_{\mathbb{R}^d})$ sends the origin of $\mathbb{R}^d$, which lies in some tiles of ω, to the corresponding tiles $t_j^{(c)}$ at the corresponding positions.

In $\mathcal{B}_0^{(c)}(\mathcal{T})$, points on the boundaries of two tiles $\tau_j^{(c)}$ and $\tau_{j'}^{(c)}$ are identified if there are neighboring copies of the tiles $t_j^{(c)}$ and $t_{j'}^{(c)}$ somewhere in $\mathcal{T}$ such that the two associated points match. This ensures that the map $\mathfrak{p}_{0,\mathcal{T}}^{(c)}$ is well defined, for if in $\mathbb{R}^d$ tiled by ω, the origin $0_{\mathbb{R}^d}$ belongs to the boundaries of some tiles, then the corresponding points in $\coprod_{j=1}^{N_0^{(c)}} t_j^{(c)}$ given by $\rho_0^{(c)}(\omega, 0_{\mathbb{R}^d})$ are identified by $\lambda_0^{(c)}$.

Let x be a point in $\mathcal{B}_0^{(c)}(\mathcal{T})$, and O_x an open neighborhood of x. Say x belongs to the intersection of some tiles $\tau_{j_1}^{(c)}, \ldots, \tau_{j_k}^{(c)}$. Let ω be a preimage of x: $\mathfrak{p}_{0,\mathcal{T}}^{(c)}(\omega) = x$. The preimage of O_x is the set of tilings ω' for which the origin lies in some neighborhood of tiles that are translates of $t_{j_1}^{(c)}, \ldots, t_{j_k}^{(c)}$, and this defines a neighborhood of ω in the hull. Therefore $\mathfrak{p}_{0,\mathcal{T}}^{(c)}$ is continuous. □

For simplicity, the prototile space $\mathcal{B}_0(\mathcal{T})$ is written $\mathcal{B}_0$, and the map $\mathfrak{p}_{0,\mathcal{T}}$ is written $\mathfrak{p}_0$.

The lift of the puncture of the tile τ_j in $\mathcal{B}_0$, denoted $\Xi(\tau_j)$, is a subset of the transversal called the *acceptance zone* of the prototile $\hat{t}_j$. It consists of all the tilings that have the puncture of a representative of $\hat{t}_j$ at the origin. The $\Xi(\tau_j)$ for $j = 1, \ldots, N_0$, form a clopen partition of the transversal, because any element of Ξ has the puncture of a unique tile at the origin which corresponds to a unique prototile. The $\Xi(\tau_j)$ are thus Cantor sets like Ξ.

Remark 2. Although those results will not be used here, it has been proven in [**13**] that the prototile space $\mathcal{B}_0$ (as well as the patch spaces $\mathcal{B}_p$ defined similarly in the next section, Definition 10) has the structure of a flat oriented Riemannian branched manifold. Also, the hull Ω can be given a *lamination* structure as follows: the lifts of the interiors of the tiles $B_{0j} = \mathfrak{p}_0^{-1}(\mathring{\tau}_j)$ are boxes of the lamination which are homeomorphic to $\mathring{t}_j \times \Xi(\tau_j)$ via the maps $(x, \xi) \mapsto \mathrm{T}^{-x}\xi$ which read as local charts.

3.4. *The hull as an inverse limit of patch spaces.* As in the previous section, let $\mathcal{T}$ be an aperiodic and repetitive tiling of $\mathbb{R}^d$ with FLC, and assume its tiles are compatible finite CW-complexes (the intersection of two tiles is a subcomplex of both). Let $\mathcal{L}_\mathcal{T}$ denote the Delone set of punctures of $\mathcal{T}$. It is repetitive and has FLC. Let $\mathcal{P}_\mathcal{T}$ denote the set of patterns of $\mathcal{T}$. As $\mathcal{T}$ has finite local complexity (hence finitely many prototiles), the set $\mathcal{P}_\mathcal{T}$ is countable, and for any given $l > 0$ the set of patterns whose representatives have radius less than l is finite.

A finite CW-complex $\mathcal{B}_p$, called a patch space, associated with a pattern $\hat{p}$ in $\mathcal{P}_\mathcal{T}$ is built from the prototiles of an appropriate subtiling of $\mathcal{T}$, written $\mathcal{T}_p$ below, in the same way that $\mathcal{B}_0$ was built from the prototiles of $\mathcal{T}$ in Definition 9. The construction goes as follows.

Let $\hat{p}$ in $\mathcal{P}_\mathcal{T}$ be a pattern of $\mathcal{T}$.

(i) Consider the sub-Delone set $\mathcal{L}_p$ of $\mathcal{L}_\mathcal{T}$ consisting of punctures of all the representative patches in $\mathcal{T}$ of $\hat{p}$. $\mathcal{L}_p$ is repetitive and has FLC.
(ii) The Voronoi tiling $\mathcal{V}_p$ of $\mathcal{L}_p$ is built and each point of $\mathcal{L}_\mathcal{T}$ is assigned to a unique tile of $\mathcal{V}_p$ as explained below.
(iii) Each tile v of $\mathcal{V}_p$ is replaced by the patch p_v of $\mathcal{T}$ made up of the tiles whose punctures have been assigned to v. This gives a repetitive tiling with FLC, $\mathcal{T}_p$, whose tiles are those patches p_v.
(iv) $\mathcal{B}_p$ is built out of the collared prototiles of $\mathcal{T}_p$, by gluing them along their boundaries according to the local configurations of their representatives in $\mathcal{T}_p$.

The second point needs clarifications since the tiles of $\mathcal{V}_p$ are Voronoi tiles (convex polytopes, see Definition 6) of $\mathcal{L}_p$ and not patches of $\mathcal{T}$. If a point of $\mathcal{L}_\mathcal{T}$ (a puncture of a tile of $\mathcal{T}$) lies on the boundary of some (Voronoi) tiles of $\mathcal{V}_p$, a criterion for assigning it to a specific one is required. To do so, let u be a vector of $\mathbb{R}^d$ that is not collinear to any of the faces of the tiles of $\mathcal{V}_p$ (such a vector exists since $\mathcal{V}_p$ has FLC, hence finitely many prototiles). A point x is said to be u-interior to a subset X of $\mathbb{R}^d$ if there exist an $\epsilon > 0$ such that $x + \epsilon u$ belongs to the interior of X. Since u is not collinear to any of the faces of the tiles of $\mathcal{V}_p$, if a point x belongs to the intersection of the boundaries of several (Voronoi) tiles of $\mathcal{V}_p$, it is u-interior to only one of them. This allows as claimed in (ii) to assign each point of $\mathcal{L}_\mathcal{T}$ to a unique tile of $\mathcal{V}_\mathcal{T}$. Now as explained in (iii), each Voronoi tile v can then be replaced by the patch of $\mathcal{T}$ which is the union of the tiles of $\mathcal{T}$ whose punctures are u-interior to v.

Each patch p_v is considered a tile of $\mathcal{T}_p$ and punctured by the puncture of the Voronoi tile v which is by construction the puncture of some representative patch of $\hat{p}$ in $\mathcal{T}$. As patches of $\mathcal{T}$, the p_v are also compatible finite CW-complexes as they are made up of tiles of $\mathcal{T}$ which are.

The prototiles of $\mathcal{T}_p$ are actually patterns of $\mathcal{T}$, and thus $\mathcal{T}_p$ is considered a subtiling of $\mathcal{T}$. From this remark it can be proven that there is a homeomorphism between the hull of $\mathcal{T}_p$ and Ω, that conjugates the $\mathbb{R}^d$-action of their associated dynamical systems (see [**13**, §2]).

Definition 10. The *patch space* $\mathcal{B}_p$ is the collared prototile space of $\mathcal{T}_p$ (Definition 9): $\mathcal{B}_p = \mathcal{B}_0^c(\mathcal{T}_p)$.

The images in $\mathcal{B}_p$ of the patches p_j (tiles of $\mathcal{T}_p$) will be denoted π_j and still be called patches. The map $\mathfrak{p}^c_{0,\mathcal{T}_p} : \Omega \to \mathcal{B}_p$, built in Proposition 1, is denoted $\mathfrak{p}_p$.

This construction of $\mathcal{B}_p$ from $\mathcal{T}_p$ is essentially the same as the construction of $\mathcal{B}_0$ from $\mathcal{T}$ given in Definition 9, the only difference being that collared prototiles of $\mathcal{T}_p$ (collared patches of $\mathcal{T}$) are used here instead. Such patch spaces are said to *force their borders*. This condition was introduced by Kellendonk in [**41**] for substitution tilings and was required in order to be able to recover the hull as the inverse limit of such spaces. It was generalized in [**13**] for branched manifolds of repetitive tilings with FLC and coincides here with the above definition.

The map $f_p = \mathfrak{p}_0 \circ \mathfrak{p}_p^{-1} : \mathcal{B}_p \to \mathcal{B}_0$ can be proven to be surjective and continuous [**13**]. It projects $\mathcal{B}_p$ onto $\mathcal{B}_0$ in the obvious way: a point x in $\mathcal{B}_p$ belongs to some patch π_j, hence to some tile, and f_p sends x to the corresponding point in the corresponding tile $\tau_{j'}$. More precisely, if $\tilde{x}$ in p_j is the point of $\mathbb{R}^d$ corresponding to x in π_j, then $\tilde{x}$ belongs to some tile, which is the translate of some $t_{j'}$ and $f_p(x)$ is the corresponding point in $\tau_{j'}$ in $\mathcal{B}_0$. If x belongs to the boundaries of say $\pi_{j_1}, \ldots, \pi_{j_k}$, in $\mathcal{B}_p$, then there are corresponding points $\tilde{x}_{j_1}, \ldots, \tilde{x}_{j_k}$, in $p_{j_1}, \ldots, p_{j_k}$, which are then on the boundaries of the copies of some tiles $t_{j'_1}, \ldots, t_{j'_{k'}}$. The boundaries of those tiles are identified by the map ρ_0 in the definition of $\mathfrak{p}_0$ and $f_p(x)$ is the corresponding point on the common boundaries of the $\tau_{j'_1}, \ldots, \tau_{j'_{k'}}$.

Recall the convention stated after Definition 2: if $\hat{p}$ is a pattern, then p denotes its representative that has its puncture at the origin. Given two patterns $\hat{p}$ and $\hat{q}$ with $q \subset p$, the composition $f_q^{-1} \circ f_p = \mathfrak{p}_q \circ \mathfrak{p}_p^{-1}$ might not exist, but if $\hat{p}$ is *zoomed out* of $\hat{q}$ as explained below, then it defines a continuous and surjective map $f_{qp} : \mathcal{B}_p \to \mathcal{B}_q$. Given three patterns that are zoomed out of each other $\hat{p}$, $\hat{q}$ and $\hat{r}$ with $r \subset q \subset p$ the following composition rule holds: $f_{rq} \circ f_{qp} = f_{rp}$. Moreover given two arbitrary patterns $\hat{q}$ and $\hat{r}$, there exists another pattern $\hat{p}$ such that p contains q and r (since $\mathcal{L}_q$ and $\mathcal{L}_r$ are repetitive sub-Delone sets of $\mathcal{L}_{\mathcal{T}}$). Hence the index set of the maps f_p is a directed set and $(\mathcal{B}_p, f_{qp})$ is a projective system.

As shown in [**13**], the hull Ω can be recovered from the inverse limit $\varprojlim(\mathcal{B}_p, f_{qp})$ under some conditions that are for convenience taken here to be directly analogous to those given in [**13**, §2.6]. Namely: the patch spaces $\mathcal{B}_p$ are required to *force their borders*, and only maps f_{qp} between patch spaces that are *zoomed out* of each other are allowed (Definition 11 below). The first condition, as mentioned above, is fulfilled here by the very definition of patch spaces given above, because they are built out of *collared patterns* as can be checked from the more general definition of a branched manifold that forces its border given in [**13**, Definition 2.43]. The second condition is stated as follows.

Definition 11. Given two patterns $\hat{p}$, $\hat{q}$ in $\mathcal{P}_{\mathcal{T}}$, with $q \subset p$, $\mathcal{B}_p$ is said to be *zoomed out* of $\mathcal{B}_q$ if the following two conditions hold.
(i) For all $i \in \{1, \ldots, N_p\}$, the patch p_i is the union of some copies of the patches q_j.
(ii) For all $i \in \{1, \ldots, N_p\}$, the patch p_i contains in its interior a copy of some patch q_j.

The first condition is equivalent to requiring that the tiles of $\mathcal{L}_p$ are patches of $\mathcal{L}_q$.

Given a patch space $\mathcal{B}_q$ it is always possible to build another patch space $\mathcal{B}_p$ that is zoomed out of $\mathcal{B}_q$: it suffices to choose a pattern $\hat{p}$ as a patch of $\mathcal{L}_q$ with $p \supset q$ and a radius large enough. This can be done by induction for instance: building $\mathcal{B}_p$ from patches of $\mathcal{L}_q$ (which are patches of $\mathcal{T}$) the same way $\mathcal{B}_q$ was built from patches of $\mathcal{T}$; if the radius of q is large enough, then each patch of $\mathcal{T}$ that $\mathcal{B}_q$ is composed of will contain a tile of $\mathcal{T}$ in its interior and so $\mathcal{B}_q$ will be zoomed out from $\mathcal{B}_0$.

Definition 12. A *proper sequence* of patch spaces of $\mathcal{T}$ is a projective sequence $\{\mathcal{B}_l, f_l\}_{l\in\mathbb{N}}$ where, for all $l \geq 1$, $\mathcal{B}_l$ is a patch space associated with a pattern $\hat{p}_l$ of $\mathcal{T}$ and $f_l = f_{p_{l-1}p_l}$, such that $\mathcal{B}_l$ is zoomed out of $\mathcal{B}_{l-1}$, with the convention that $f_0 = f_{p_1}$ and $\mathcal{B}_0$ is the prototile space of $\mathcal{T}$.

Note that the first patch space in a proper sequence can be chosen to be the prototile space of the tiling (i.e. made of uncollared prototiles), all that matters for recovering the hull by inverse limit as shown in the next theorem, is that the next patch spaces are zoomed out of each other, and built out of collared patches (i.e. force their borders).

THEOREM 5. *The inverse limit of a proper sequence $\{\mathcal{B}_l, f_l\}_{l\in\mathbb{N}}$ of patch spaces of $\mathcal{T}$ is homeomorphic to the hull of $\mathcal{T}$:*

$$\Omega \cong \varprojlim(\mathcal{B}_l, f_l).$$

Proof. The homeomorphism is given by the map $\mathfrak{p} : \Omega \to \varprojlim(\mathcal{B}_l, f_l)$, defined by $\mathfrak{p}(\omega) = (\mathfrak{p}_0(\omega), \mathfrak{p}_1(\omega), \ldots)$, with inverse $\mathfrak{p}^{-1}(x_0, x_1, \ldots) = \bigcap\{\mathfrak{p}_l^{-1}(x_l), l \in \mathbb{N}\}$.

The map $\mathfrak{p}$ is surjective, because the maps $\mathfrak{p}_l$, $l \in \mathbb{N}$, are surjective. For the proof of injectivity, consider $\omega, \omega' \in \Omega$ with $\mathfrak{p}(\omega) = \mathfrak{p}(\omega')$. For simplicity the metric δ defined in Remark 1 is used in this proof. Fix $\epsilon > 0$. To prove that $\delta(\omega, \omega') < \epsilon$ it suffices to show that the two tilings agree on a ball of radius $1/\epsilon$. For each l in $\mathbb{N}$, $\mathfrak{p}_l(\omega) = \mathfrak{p}_l(\omega')$ in some patch $\pi_{l,j}$ in $\mathcal{B}_l$. This means that the tilings agree on some translate of the patch $p_{l,j}$ that contains the origin. The definition of patch spaces made up from collared patterns (condition of *forcing the border* in [**13**]) implies that the tilings agree on the ball $B(0_{\mathbb{R}^d}, r_l)$, where r_l is the parameter of uniform discreteness of the Delone set $\mathcal{T}_{p_l}$. The assumption that the patch spaces are zoomed out of each other (condition (ii) in Definition 11) implies that $r_l > r_{l-1} + r$, where r is the parameter of uniform discreteness of $\mathcal{L}_{\mathcal{T}}$. Hence $r_l > (l+1)r$, and choosing l bigger than the integer part of $1/(r\epsilon)$ concludes the proof of the injectivity of $\mathfrak{p}$.

Condition (i) in Definition 11 implies that for every $l \geq 1$, $\mathfrak{p}_l^{-1}(x_l) \subset \mathfrak{p}_{l-1}^{-1}(x_{l-1})$. The definition of $\mathfrak{p}^{-1}$ then makes sense by compactness of Ω because any finite intersection of some $\mathfrak{p}_l^{-1}(x_l)$s is non-empty and closed.

A neighborhood of $x = (x_0, x_1, \ldots)$ in $\varprojlim(\mathcal{B}_l, f_l)$ is given by $U_n(x) = \{y = (y_0, y_1, \ldots) \mid y_i = x_i,\ i \leq n\}$ for some integer n. If ω is a preimage of x, $\mathfrak{p}(\omega) = x$, then the preimage of $U_n(x)$ is given by all tilings ω' such that $\mathfrak{p}_n(\omega') = \mathfrak{p}_n(\omega)$, i.e. those tilings agree with ω on some patch p_n around the origin; they form then a neighborhood of ω in Ω. This proves that $\mathfrak{p}$ and $\mathfrak{p}^{-1}$ are continuous. □

Another important construction which allows one to recover the hull by inverse limit has been given by Gähler in an unpublished work as a generalization of the construction of Anderson and Putnam in [**1**]. Instead of gluing together patches to form the patch spaces $\mathcal{B}_p$, Gähler's construction consists in considering 'multicollared prototiles' and the graph that link them according to the local configurations of their representatives in the tiling. This construction keeps track of the combinatorics of the patches that those 'multicollared tiles' represent and thus is enough to recover the hull topologically, which is sufficient for topological concerns (for cohomology or K-theory in particular). The construction given here, and taken from [**13**] where those patch spaces are proven to be branched manifolds, takes more structure into account and the homeomorphism between the hull and the inverse limit of such branched manifolds which is built in [**13**] is not only a topological conjugacy but yields a conjugacy of the dynamical systems' actions.

4. *Cohomology of tiling spaces*

4.1. *Tiling cohomologies.* Various cohomologies for tiling spaces have been used in the literature. For the class of tilings considered here (aperiodic, repetitive with FLC) they are all isomorphic.

First, the Čech cohomology of the hull was introduced for instance in [**1**] for substitution tilings (and in full generality for tilings on Riemannian manifolds in [**63**]). Hulls of such tilings are obtained by inverse limit of finite CW-complexes, and their Čech cohomology is obtained by direct limit.

For repetitive tilings with FLC, if a lamination structure is given to the hull as in [**13**], the cohomology of the hull is defined by direct limit of the simplicial cohomologies of branched manifolds that approximate the hull by inverse limit (a generalization of Theorem 5). This cohomology is isomorphic to the Čech cohomology of the hull, using the natural isomorphism between Čech and simplicial cohomologies that holds for such branched manifolds (which are CW-complexes) and passing to direct limit. It has been used to prove that the generators of the dth cohomology group are in one-to-one correspondence with invariant ergodic probability measures on the hull.

Another useful cohomology, the pattern-equivariant (PE) cohomology, has been proposed by Kellendonk and Putnam in [**42, 43**] for real coefficients and then generalized to integer coefficients by Sadun [**66**]. This cohomology has been used for proving that the Ruelle–Sullivan map (associated with an ergodic invariant probability measure on the hull) from the Čech cohomology of the hull to the exterior algebra of the dual of $\mathbb{R}^d$ is a ring homomorphism. Let $\mathcal{T}$ be a repetitive tiling with FLC. Assume its tiles are compatible CW-complexes, so that it gives a CW-complex decomposition of $\mathbb{R}^d$. The group of PE n-cochains C^n_{PE} is a subgroup of the group of integer singular n-cochains of $\mathbb{R}^d$ that satisfy the following property: an n-cochain φ is said to be PE if there exists a patch p of $\mathcal{T}$ such that $\varphi(\sigma_1) = \varphi(\sigma_2)$, for 2 n-simplices σ_1, σ_2 with image cells e_1, e_2, whenever there exists $x \in e_1,\ y \in e_2$ such that $\mathfrak{p}_p(\mathrm{T}^{-x}\mathcal{T}) = \mathfrak{p}_p(\mathrm{T}^{-y}\mathcal{T})$. The simplicial coboundary of a PE cochain is easily seen to be PE (possibly with respect to a patch of larger radius), and this defines the complex for integer PE cohomology. A PE cochain is a cochain that agrees on points which have the same local environments in $\mathcal{T}$ or equivalently is the pullback of a cochain on some patch space $\mathcal{B}_p$. Hence PE cohomology can be seen as the direct limit of the singular cohomologies of a proper sequence of patch spaces. Using the natural isomorphisms between cellular and Čech cohomologies that hold for those CW-complexes and taking direct limits, the integer PE cohomology turns out to be isomorphic to the Čech cohomology of the hull.

The PV cohomology described in §4.2 is also isomorphic to the Čech cohomology of the hull (see Theorem 7).

4.2. *The PV cohomology.* The definition of a Δ-*complex* structure is first recalled, following the presentation of Hatcher in his book on Algebraic Topology [**34**, §2.1].

Given $n+1$ points $v_0, \ldots, v_n$, in $\mathbb{R}^m$, $m > n$, that are not collinear, let $[v_0, \ldots, v_n]$ denote the n-simplex with vertices $v_0, \ldots, v_n$. Let Δ^n denote the standard n-simplex

$$\Delta^n = \left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \right\}$$

whose vertices are the unit vectors along the coordinate axes. An ordering of those vertices is specified and this allows us to define a canonical linear homeomorphism between Δ^n and any other n-simplex $[v_0, \ldots, v_n]$ that preserves the order of the vertices, namely, $(x_0, x_1, \ldots, x_n) \mapsto \sum x_i v_i$.

If one of the $n+1$ vertices of an n-simplex $[v_0, \ldots, v_n]$ is deleted, then the remaining n vertices span an $(n-1)$-simplex, called a face of $[v_0, \ldots, v_n]$. By convention the vertices of any subsimplex spanned by a subset of the vertices are ordered according to their order in the larger simplex.

The union of all the faces of Δ^n is the boundary of Δ, written $\partial\Delta^n$. The open simplex $\mathring{\Delta}^n$ is $\Delta^n \setminus \partial\Delta^n$, the interior of Δ^n (and Δ^0 is assumed to be its own interior).

A Δ-complex structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n depending on the index α, such that the following hold.

(i) The restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective, and each point of X is in the image of exactly one such restriction.

(ii) Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. The face of Δ^n is identified with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.

(iii) A set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

A Δ-complex X can be built as a quotient space of a collection of disjoint simplices by identifying various subsimplices spanned by subsets of the vertices, where the identifications are performed using the canonical linear homeomorphism that preserves orderings of the simplices. It can be shown that X is a Hausdorff space. Then condition (iii) implies that each restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is a homeomorphism onto its image, which is thus an open simplex in X. These open simplices $\sigma_\alpha(\mathring{\Delta}^n)$ are cells e_α^n of a CW-complex structure on X with the σ_α as characteristic maps.

It is now assumed that the tiles of $\mathcal{T}$ are compatible finite Δ-complexes: the intersection of two tiles is a sub-Δ-complex of both. In addition each cell $e_\alpha^n = \sigma_\alpha(\mathring{\Delta}^n)$ of each tile is punctured by, say, the image under σ_α of the barycenter of Δ^n. Hence $\mathcal{T}$ can be seen as a Δ-complex decomposition of $\mathbb{R}^d$, and this Δ-complex structure gives a 'refinement' of the tiling (each tile being decomposed into the union of the closures of the cells it contains). If $\mathcal{T}$ is the Voronoi tiling of a Delone set its tiles can be split into d-simplices since they are are polytopes: this gives a Δ-complex structures for which the maps σ_α are simply affine maps. The maps $\sigma_\alpha : \Delta^n \to \mathcal{B}_0$ of the Δ-complex structure of $\mathcal{B}_0$ will be called the *characteristic maps* of the n-simplices on $\mathcal{B}_0$ represented by its image.

By construction (Definition 9) as a quotient CW-complex, the prototile space $\mathcal{B}_0$ is a finite Δ-complex.

Definition 13. The Δ-*transversal*, denoted Ξ_Δ, is the subset of Ω consisting of tilings that have the puncture of one of their cells (of one of their tiles) at the origin $0_{\mathbb{R}^d}$.

Since $\mathcal{T}$ has finitely many prototiles and they have finite CW-complex structures, the set of the punctures of the cells of the tiles of $\mathcal{T}$ is a Delone set. This Delone set will be called the Δ-*Delone set* of $\mathcal{T}$, and denoted $\mathcal{L}_\Delta$. The Δ-transversal is thus the canonical transversal of $\mathcal{L}_\Delta$.

The Δ-transversal is not immediately related to the canonical transversal. Indeed the lift of the puncture of a cell does not belong to the transversal in general, unless this puncture coincides with the puncture of the tile of $\mathcal{B}_0$ that contains that cell.

The Δ-transversal is the lift of the punctures of the cells in $\mathcal{B}_0$. It is partitioned by the lift of the punctures of the n-cells, denoted Ξ_Δ^n which is the subset of Ω consisting of tilings that have the puncture of an n-cell at the origin. As for the transversal, the Δ-transversal is a Cantor set, and the Ξ_Δ^n give a partition by clopen sets. The ring of continuous integer-valued functions on the Δ-transversal, $C(\Xi_\Delta, \mathbb{Z})$, is thus the direct sum of the $C(\Xi_\Delta^n, \mathbb{Z})$ for $n = 0, \ldots, d$.

Given the characteristic map σ of an n-simplex e of $\mathcal{B}_0$, let $\Xi_\Delta(\sigma)$ denote the lift of the puncture of e, and χ_σ its characteristic function on Ξ_Δ. The subset $\Xi_\Delta(\sigma)$ is called the *acceptance zone* of σ. Since continuous integer-valued functions on a totally disconnected space are generated by characteristic functions of clopen sets, χ_σ belongs to $C(\Xi_\Delta, \mathbb{Z})$.

Consider the characteristic map $\sigma : \Delta^n \to \mathcal{B}_0$ of an n-simplex e of $\mathcal{B}_0$, and τ a face of σ (i.e. the restriction of σ to a face of Δ^n) with associated simplex f (a face of e). The simplices e and f in $\mathcal{B}_0$ are contained in some tile τ_j. Viewing e and f as subsets of the tile t_j in $\mathbb{R}^d$, it is possible to define the vector $x_{\sigma\tau}$ that joins the puncture of f to the puncture of e. Notice that since $\mathcal{B}_0$ is a *flat* branched manifold (Remark 2, and [**13**]), the vector $x_{\sigma\tau}$ is also well defined in $\mathcal{B}_0$ as a vector in a region containing the simplex e.

Definition 14.

(i) Let σ and τ be characteristic maps of simplices in $\mathcal{B}_0$. The operator $\theta_{\sigma\tau}$, on $C(\Xi_\Delta, \mathbb{Z})$, is defined by

$$\theta_{\sigma\tau} = \begin{cases} \chi_\sigma \mathrm{T}^{x_{\sigma\tau}} \chi_\tau & \text{if } \tau \subset \partial\sigma, \\ 0 & \text{otherwise,} \end{cases}$$

where $\tau \subset \partial\sigma$ means that τ is a face of σ. Here the translation acts by $\mathrm{T}^{x_{\sigma\tau}} f(\xi) = f(\mathrm{T}^{-x_{\sigma\tau}}\xi)$ whenever $f \in C(\Xi_\Delta(\tau), \mathbb{Z})$.

(ii) The *function ring of the transversal* $\mathcal{A}_{\Xi_\Delta}$ is the ring (finitely) generated by the operators $\theta_{\sigma\tau}$ and their adjoints $\theta^*_{\sigma\tau} = \chi_\tau \mathrm{T}^{-x_{\sigma\tau}} \chi_\sigma$ if $\tau \subset \partial\sigma$ and 0 otherwise, over all characteristic maps σ and τ of simplices in $\mathcal{B}_0$.

The operators $\theta_{\sigma\tau}$ satisfy the following properties:

$$\theta_{\sigma\tau}\theta^*_{\sigma\tau} = \chi_\sigma \quad \text{if } \tau \subset \partial\sigma, \tag{1a}$$

$$\sum_{\sigma:\partial\sigma\supset\tau} \theta^*_{\sigma\tau}\theta_{\sigma\tau} = \chi_\tau. \tag{1b}$$

The $\theta_{\sigma\tau}$ are thus *partial isometries*. The ring $\mathcal{A}_{\Xi_\Delta}$ is unital and its unit is the characteristic function of the Δ-transversal $1_{\mathcal{A}_{\Xi_\Delta}} = \chi_{\Xi_\Delta}$. The ring of continuous integer-valued functions on the Δ-transversal $C(\Xi_\Delta, \mathbb{Z})$ is a left $\mathcal{A}_{\Xi_\Delta}$-module.

Remark 3. It is important to notice that the operators $\theta_{\sigma\partial_i\sigma}$ are in one-to-one correspondence with a set of arrows that generate Γ^1_Δ, the arrows of the groupoid Γ_Δ of the Δ-transversal (groupoid of the transversal of the Delone set $\mathcal{L}_\Delta$, see Definition 8). The set of arrows Γ^1_Δ is indeed in one-to-one correspondence with the set of vectors joining points of $\mathcal{L}_\Delta$, and it is generated by the set of vectors joining a point of $\mathcal{L}_\Delta$ to a 'nearest neighbor' (a puncture of a face of the simplex of $\mathcal{T}$ whose puncture is that point, or of a simplex containing it on its boundary). Each vector can be decomposed into a sum of such generating vectors, and each arrow can be decomposed into a composition of such generating arrows (corresponding to those generating vectors). This generating set is finite, since $\mathcal{T}$ has FLC (in particular there are finitely many prototiles), and each of its vectors is an $x_{\sigma\partial_i\sigma}$ and thus corresponds to a unique operator $\theta_{\sigma\partial_i\sigma}$.

A 'representation by partial isometries' of the set of generators of Γ^1_Δ is given by

$$\theta(\gamma) = \chi_{\mathfrak{p}_0\circ\mathfrak{p}_0^{-1}(s(\gamma))}\, \mathrm{T}(\gamma)\, \chi_{\mathfrak{p}_0\circ\mathfrak{p}_0^{-1}(r(\gamma))},$$

where, if $\gamma = (\xi, x)$, then $\mathrm{T}(\gamma)$ is the translation operator T^x.

Let $\mathcal{S}_0^n$ be the set of the characteristic maps $\sigma : \Delta^n \to \mathcal{B}_0$ of the n-simplices of the Δ-complex decomposition of $\mathcal{B}_0$, and $\mathcal{S}_0$ the (disjoint) union of the $\mathcal{S}_0^n$. The group of simplicial n-chains on $\mathcal{B}_0$, $C_{0,n}$, is the free Abelian group with basis $\mathcal{S}_0^n$.

Definition 15. The *PV cohomology* of the hull of $\mathcal{T}$ is the homology of the complex $\{C_{\mathrm{PV}}^n, d_{\mathrm{PV}}^n\}$, where the following hold.

(i) The PV cochain groups are the groups of continuous integer valued functions on Ξ_Δ^n: $C_{\mathrm{PV}}^n = C(\Xi_\Delta^n, \mathbb{Z})$ for $n = 0, \dots, d$.

(ii) The PV differential, d_{PV}, is the element of $\mathcal{A}_{\Xi_\Delta}$ given by the sum over $n = 1, \dots, d$, of the operators

$$d_{\mathrm{PV}}^n : \begin{cases} C_{\mathrm{PV}}^{n-1} \longrightarrow C_{\mathrm{PV}}^n, \\ d_{\mathrm{PV}}^n = \displaystyle\sum_{\sigma \in \mathcal{S}_0^n} \sum_{i=0}^n (-1)^i \, \theta_{\sigma \partial_i \sigma}. \end{cases} \tag{2}$$

The 'simplicial form' of d_{PV}^n makes it clear that $d_{\mathrm{PV}}^{n+1} \circ d_{\mathrm{PV}}^n = 0$ for $n = 1, \dots, d-1$. Alternatively we shall call the PV cohomology of the hull of $\mathcal{T}$ simply the *PV cohomology of $\mathcal{T}$*.

Remark 4. Thanks to the lamination structure on Ω described in Remark 2 the map $\mathfrak{p}_0$

$$\begin{array}{ccc} \Xi_\Delta & \hookrightarrow & \Omega \\ & & \downarrow \mathfrak{p}_0 \\ & & \mathcal{B}_0 \end{array}$$

looks very much like a fibration of Ω with base space $\mathcal{B}_0$ and fiber Ξ_Δ. However because of the branched structure there are paths in $\mathcal{B}_0$ that cannot be lifted to any leaf of Ω. Hence this is not a fibration. Nevertheless the result in the present paper (Theorem 1) gives a spectral sequence analogous to the Serre spectral sequence for a fibration.

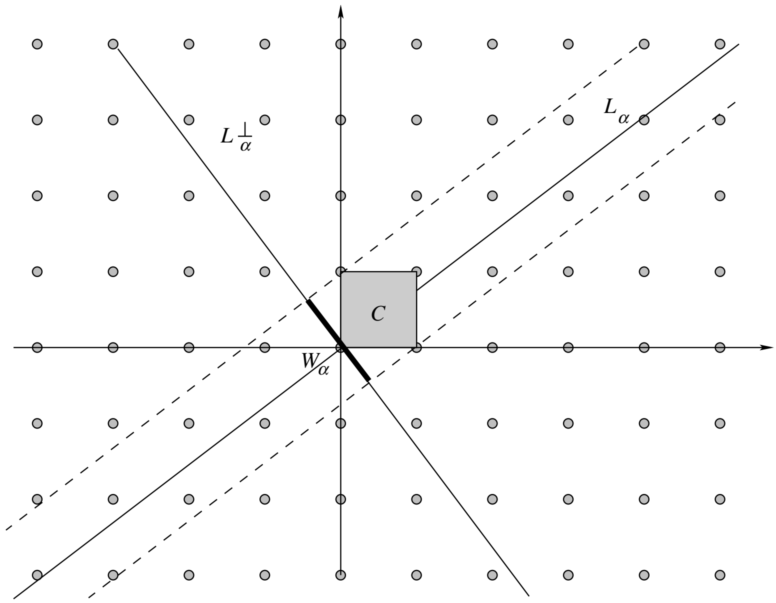
Remark 5. This PV cohomology of $\mathcal{T}$ is formally analogous to a cohomology with local coefficients (see [**34**, Ch. 3.H]) but in a more general setting. Let G_x be the group $C(\mathfrak{p}_0^{-1}(x), \mathbb{Z})$ for x in $\mathcal{B}_0$. As $\mathfrak{p}_0^{-1}(x)$ is a Cantor set, G_x is actually its K^0-group: $G_x = K^0(\mathfrak{p}_0^{-1}(x))$ while its K^1 group is trivial. The family of groups $(G_x)_{x \in \mathcal{B}_0}$ is analogous to a local coefficient system, with the operators $\theta_{\sigma\tau}$ in the role of group isomorphisms between the fibers, but they are not isomorphisms here (as they come from a groupoid). If σ in C_0^n is an n-chain, and φ in C_{PV}^{n-1} is a PV $(n-1)$-cochain, then the differential,

$$d_{\mathrm{PV}}^n \varphi(\sigma) = \sum_{i=0}^n (-1)^i \theta_{\sigma \partial_i \sigma} \varphi(\partial_i \sigma),$$

takes the restrictions of φ to the faces $\partial_i \sigma$, which are elements of the groups $G_{x_{\partial_i \sigma}}$ (where $x_{\partial_i \sigma}$ is the puncture of the image simplex of $\partial_i \sigma$), and pulls them to σ with the operators $\theta_{\sigma \partial_i \sigma}$ to get an element of the group G_{x_σ} (where x_σ is the puncture of the image simplex of σ).

A rigorous formulation of the points in this remark as well as in the previous Remark 3 will be investigated in further research.

4.3. *An example: PV cohomology in one-dimension.* This section presents the explicit calculation of PV cohomology of some one-dimensional tilings obtained by 'cut and projection'. This is to illustrate techniques that can be used to calculate PV cohomology, rather than showing its distinct features and differences compared to other cohomologies. Further examples in higher dimensions, and methods of calculation for PV cohomology, will be investigated in a future work.



Let $\mathbb{R}^2$ be the Euclidean plane with basis vectors e_1 and e_2, and let $\mathbb{Z}^2$ be the lattice of points of integer coordinates. Let $C = (0, 1] \times [0, 1)$ be the unit square 'open on the left and top'. Fix an irrational number α in $\mathbb{R}_+ \setminus \mathbb{Q}$. Let L_α be line of slope α (through the origin) and p_α the projection onto L_α. Let $L_\alpha^\perp$ be the orthogonal complement of L_α and $p_\alpha^\perp = 1 - p_\alpha$. Let $W_\alpha = p_\alpha^\perp(C)$, and $\Sigma = \{a \in \mathbb{Z}^2 \mid p_\alpha^\perp(a) \in W_\alpha\}$ be the set of points of $\mathbb{Z}^2$ that project to W_α. A lattice point $a = (a_1, a_2)$ in $\mathbb{Z}^2$ belongs to Σ if and only if $-\alpha \le a_2 - a_1\alpha < 1$. The projection of Σ onto L_α defines an aperiodic tiling $\mathcal{T}_\alpha$ whose tiles' edges are the projections of points of Σ. The tiling $\mathcal{T}_\alpha$ has two types of tiles that correspond to the projections of the vertical and horizontal of C. In particular $\mathcal{T}_\alpha$ is repetitive with FLC [**11, 45–47**].

Identifying $L_\alpha^\perp$ (oriented towards the 'northwest') with the real line, W_α can be seen as the interval $[-\alpha/(\sqrt{1+\alpha^2}), 1/(\sqrt{1+\alpha^2}))$. Let $\mathcal{A}_\alpha$ be the C^*-algebra generated by the characteristic functions $\chi_{[x_1,x_2)}$ for $x_1 < x_2$ in $p_\alpha^\perp(\Sigma)$. The transversal Ξ_α is defined to be the spectrum of $\mathcal{A}_\alpha$. It has been shown in [**11**] that it is a Cantor set (compact, perfect and totally disconnected), and can be seen as a completion of the interval W_α for a finer topology than the usual one, where the intervals $[x_1, x_2)$, for $x_1 < x_2$ in $p_\alpha^\perp(\Sigma)$, are open and closed.

We further identify W_α (rescaling it and identifying its endpoints) with the unit circle with the topology induced by Ξ_α. This defines the Cantor circle $\mathbb{S}^1_\alpha$. It admits a countable basis of open and closed sets $i_{lm} = [l\alpha/(1+\alpha), m\alpha/(1+\alpha)) \mod 1$ for integers l, m. We can alternatively define $\mathbb{S}^1_\alpha$ as the spectrum of the C^*-algebra generated by $\{\theta_\alpha^n \chi_\alpha, n \in \mathbb{Z}\}$ where χ_α is the characteristic function of the arc $[0, \alpha/(1+\alpha))$, and θ_α the rotation of angle $\alpha/(1+\alpha)$.

We give L_α an orientation towards the first quadrant (northeast), and puncture the tiles by their left vertices for convenience (not their barycenters). The prototile space $\mathcal{B}_0 = \mathbb{S}^1 \vee \mathbb{S}^1$ is the wedge sum of two circles corresponding to the two prototiles with their vertices identified. The difference with §4.2 is that the Ξ_Δ^n do not partition Ξ_Δ since there is only one vertex here, see Definition 13. Consequently the canonical transversal and Δ-transversal are identical here. The transversal Ξ_α is seen as the Cantor circle $\mathbb{S}^1_\alpha$. The arc $[-\alpha/(1+\alpha), 0)$ is the acceptance zone Ξ_a of the prototile a whose representatives are the projections of horizontal intervals of Σ, and the arc $[0, 1/(1+\alpha))$ is the acceptance zone Ξ_b of the prototile whose representatives are the projections of vertical intervals of Σ. The vectors $x_{\sigma\tau}$ in the definitions of the operators $\theta_{\sigma\tau}$ are here induced by the projections $p_\alpha^\perp(e_1)$ and $p_\alpha^\perp(e_2)$. Once rescaled to the unit circle the operator T^{x_a} becomes the rotation by $-1/(1+\alpha)$, and T^{x_b} the rotation by $\alpha/(1+\alpha)$, and are thus equal. The PV complex reads here simply

$$0 \longrightarrow C(\mathbb{S}^1_\alpha, \mathbb{Z}) \xrightarrow{d_{\mathrm{PV}} = \mathrm{id} - \theta_\alpha} C(\mathbb{S}^1_\alpha, \mathbb{Z}) \longrightarrow 0, \tag{3}$$

where θ_α is the rotation by $\alpha/(1+\alpha)$ on $\mathbb{S}^1_\alpha$, and is unitary.

PROPOSITION 2. *The PV cohomology of (the hull Ω_α of) $\mathcal{T}_\alpha$ is given by*

$$\begin{cases} H^0_{\mathrm{PV}}(\mathcal{B}_0; C(\mathbb{S}^1_\alpha, \mathbb{Z})) \cong \mathbb{Z}, \\ H^1_{\mathrm{PV}}(\mathcal{B}_0; C(\mathbb{S}^1_\alpha, \mathbb{Z})) \cong \mathbb{Z} \oplus \mathbb{Z}. \end{cases}$$

Proof. A function $f \in C(\mathbb{S}^1_\alpha, \mathbb{Z})$ reads as a finite sum $f = \sum n\chi_{I_n}$ where n is an integer and χ_{I_n} is the characteristic function of the clopen set $I_n = f^{-1}(n)$. The 0th cohomology group is the set of invariant functions under θ_α. Each I_n is a finite disjoint union of base clopen sets i_{lm}. Given two base clopens $i_{lm} \subset I_n$ and $i_{l'm'} \subset I_{n'}$,

$$\theta_\alpha^{m-m'} i_{l'm'} = [(l' + m - m')\alpha/(1+\alpha), m\alpha/(1+\alpha)) \mod 1,$$

so that $i_{lm} \cap \theta_\alpha^{m-m'} i_{l'm'} \neq \emptyset$. Hence if f is invariant under θ_α, then n must equal n' and therefore f must be constant.

The calculation of the first cohomology group, the group of coinvariants under θ_α, relies upon the fact that any function $f \in C(\mathbb{S}^1_\alpha, \mathbb{Z})$ can be written as $n_f \chi_{i_{01}} + m_f \chi_{\mathbb{S}^1_\alpha}$ modulo $(1-\theta_\alpha)C(\mathbb{S}^1_\alpha, \mathbb{Z})$, for some integers n_f, m_f, that are uniquely determined by the class of f. This technical point is tedious but elementary (it uses an encoding of the real numbers from the partial fraction decomposition of $\alpha/(1+\alpha)$). □

Remark 6.

(i) Using integration, $\int_{\mathbb{S}^1_\alpha}$, one gets a group isomorphism

$$H^1_{\mathrm{PV}}(\mathcal{B}_0;\, C(\mathbb{S}^1_\alpha,\mathbb{Z})) \cong \mathbb{Z} \oplus \frac{\alpha}{1+\alpha}\mathbb{Z},$$

but PV cohomology by itself cannot distinguish between different values of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

(ii) The above calculations yield also the K-groups of the tiling space:

$$\begin{cases} K^0(\Omega) \cong \check{H}^0(\Omega_\alpha;\mathbb{Z}) \cong H^0_{\mathrm{PV}}(\mathcal{B}_0;\, C(\mathbb{S}^1_\alpha,\mathbb{Z})), \\ K^1(\Omega) \cong \check{H}^1(\Omega_\alpha;\mathbb{Z}) \cong H^1_{\mathrm{PV}}(\mathcal{B}_0;\, C(\mathbb{S}^1_\alpha,\mathbb{Z})). \end{cases}$$

The isomorphism between Čech and PV cohomologies for the hull of tilings is proven in Theorem 7. The isomorphisms between K-theory and cohomology of the hull comes from the natural isomorphisms (Chern character) between K-theory and cohomology of CW-complexes of dimension less than three (a proof can be found in [**1**], proposition 6.2), and the fact that Ω_α is the inverse limit of such space (Theorem 5).

(iii) With the previous remark, the PV complex (3) can be viewed as the Pimsner–Voiculescu exact sequence [**55**] for the K-theory of the C^*-algebra $C(\mathbb{S}^1_\alpha) \rtimes_{\theta_\alpha} \mathbb{Z}$:

$$\begin{array}{ccccc} K_0(C(\mathbb{S}^1_\alpha)) & \xrightarrow{\mathrm{id}-\theta_\alpha} & K_0(C(\mathbb{S}^1_\alpha)) & \longrightarrow & K_0(C(\mathbb{S}^1_\alpha) \rtimes_{\theta_\alpha} \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(C(\mathbb{S}^1_\alpha) \rtimes_{\theta_\alpha} \mathbb{Z}) & \longleftarrow & K_1(C(\mathbb{S}^1_\alpha)) & \xleftarrow{\mathrm{id}-\theta_\alpha} & K_1(\mathbb{S}^1_\alpha) \end{array}$$

Indeed $C(\mathbb{S}^1_\alpha) \rtimes_{\theta_\alpha} \mathbb{Z}$ is the C^*-algebra of the groupoid of the transversal, which is Morita equivalent to the C^*-algebra of the hull of $\mathcal{T}_\alpha$ [**6**]. Also $K_0(C(\mathbb{S}^1_\alpha)) \cong C(\mathbb{S}^1_\alpha,\mathbb{Z})$ and $K_1(C(\mathbb{S}^1_\alpha)) \cong 0$ [**8**].

5. *Proof of Theorem 1*

The proof of Theorem 1 follows from the following two theorems, which will be proved separately for convenience and clarity.

It is important to notice that the first page of the spectral sequence of Theorem 1 is the PV complex (Definition 15) and therefore its second page is given by PV cohomology. However the direct identification of the PV differential on the first page is highly technical and will not be presented here. The proof will use instead an approximation of the hull (Theorem 5) as an inverse limit of patch spaces (Definition 10), and a direct limit spectral sequence for the K-theory of the hull (Theorem 6), with page-2 isomorphic to its Čech cohomology. Some abstract results on spectral sequences are also needed, in particular some properties of the Atiyah–Hirzebruch spectral sequence [**2**]. For the convenience of the reader, those known results are grouped together and proven in a separate section, §A.

Let $\mathcal{T}$ be an aperiodic and repetitive tiling of $\mathbb{R}^d$ with FLC (Definition 3), and assume its tiles are finite compatible Δ-complexes. Let Ω be its hull (Definition 4) and Ξ_Δ its Δ-transversal (Definition 13).

THEOREM 6. *There is a spectral sequence that converges to the K-theory of the C^*-algebra of the hull*

$$E_2^{rs} \Rightarrow K_{r+s+d}(C(\Omega) \rtimes \mathbb{R}^d),$$

and whose page-2 is isomorphic to the integer Čech cohomology of the hull

$$E_2^{rs} \cong \begin{cases} \check{H}^r(\Omega; \mathbb{Z}), & s \text{ even}, \\ 0, & s \text{ odd}. \end{cases}$$

THEOREM 7. *The integer Čech cohomology of the hull is isomorphic to the PV cohomology of the hull*

$$\check{H}^*(\Omega; \mathbb{Z}) \cong H^*_{\mathrm{PV}}(\mathcal{B}_0; C(\Xi_\Delta, \mathbb{Z})).$$

5.1. *Proof of Theorem 7.* The proof of Theorem 7 follows from Propositions 3 and 4 below. First a PV cohomology of $\mathcal{B}_p$, written $H^*_{\mathrm{PV}}(\mathcal{B}_0; C(\Sigma_p, \mathbb{Z}))$, is defined. It is proven to be isomorphic to the simplicial cohomology of $\mathcal{B}_p$ in Proposition 3, and then Proposition 4 establishes that the PV cohomology of the hull is isomorphic to the direct limit of the PV cohomologies of a proper sequence of patch spaces.

Let $\mathcal{S}_p^n$ be the set of the characteristic maps $\sigma_p : \Delta^n \to \mathcal{B}_p$ of the n-simplices of the Δ-complex decomposition of $\mathcal{B}_p$, and $\mathcal{S}_p$ the (disjoint) union of the $\mathcal{S}_p^n$. The group of simplicial n-chains on $\mathcal{B}_p$, $C_{p,n}$, is the free Abelian group with basis $\mathcal{S}_p^n$.

Remark 7. The map $f_p : \mathcal{B}_p \to \mathcal{B}_0$ preserves the orientations of the simplices (the ordering of their vertices).

Given a simplex σ_p on $\mathcal{B}_p$, let $\Xi_{p,\Delta}(\sigma_p)$ denote the lift of the puncture of its image in $\mathcal{B}_p$ and χ_{σ_p} its characteristic function. $\Xi_{p,\Delta}(\sigma_p)$ is called the *acceptance zone* of σ_p; it is a clopen set in Ξ_Δ.

LEMMA 1. *Given a simplex σ on $\mathcal{B}_0$, its acceptance zone is partitioned by the acceptance zones of its preimages σ_p on $\mathcal{B}_p$*

$$\Xi_\Delta(\sigma) = \coprod_{\sigma_p \in f_{p\#}^{-1}(\sigma)} \Xi_{p,\Delta}(\sigma_p)$$

where $f_{p\#} : C_{p,n} \to C_{0,n}$ denotes the map induced by f_p on the simplicial chain groups.

Proof. The union of the $\Xi_{p,\Delta}(\sigma_p)$ is equal to $\Xi_\Delta(\sigma)$, because each lift of the image simplex of a σ_p corresponds to the lift of the image simplex of σ that has a given local configuration (it is contained in some patch of $\mathcal{T}$ or some tile of $\mathcal{T}_p$, see §3.4). If σ_p and σ'_p are distinct in $f_{p\#}^{-1}(\sigma)$, then $\mathfrak{p}_p(\Xi_{p,\Delta}(\sigma_p)) \cap \mathfrak{p}_p(\Xi_{p,\Delta}(\sigma'_p))$ is empty, and thus so is $\Xi_{p,\Delta}(\sigma_p) \cap \Xi_{p,\Delta}(\sigma'_p)$. □

The maps in $f_{p\#}^{-1}(\sigma)$ are thus in one-to-one correspondence with the set of atoms of the partition of $\Xi_\Delta(\sigma)$ by the $\Xi_{p,\Delta}(\sigma_p)$, and the union over σ in $\mathcal{S}_0^n$ of the $f_{p\#}^{-1}(\sigma)$ is just $\mathcal{S}_p^n$.

The simplicial n-cochain group C_p^n is $\mathrm{Hom}(C_{p,n}, \mathbb{Z})$, the dual of the n-chain group $C_{p,n}$. It is represented faithfully on the group $C(\Xi_\Delta, \mathbb{Z})$ of continuous integer valued functions on the Δ-transversal by

$$\rho_{p,n} : \begin{cases} C_p^n & \longrightarrow & C(\Xi_\Delta^n, \mathbb{Z}), \\ \psi & \longmapsto & \sum_{\sigma_p \in \mathcal{S}_p^n} \psi(\sigma_p)\chi_{\sigma_p}. \end{cases} \tag{4}$$

The image of $\rho_{p,n}$ will be written $C(\Sigma_p^n, \mathbb{Z})$, to remind the reader that this consists of functions on the 'discrete transversal' Σ_p^n which corresponds to the atoms of the partitions of the $\Xi_\Delta(\sigma)$ by the $\Xi_{p,\Delta}(\sigma_p)$ for $\sigma_p \in f_{p\#}^{-1}(\sigma)$. The representation $\rho_{p,n}$ is a group isomorphism onto its image $C(\Sigma_p^n, \mathbb{Z})$; its inverse is defined as follows. Given $\varphi = \sum_{\sigma_p \in \mathcal{S}_p^n} \varphi_{\sigma_p} \chi_{\sigma_p}$, where φ_{σ_p} is an integer, $\rho_p^{n\ -1}(\varphi)$ is the group homomorphism from $C_{p,n}$ to $\mathbb{Z}$ whose value on the basis map σ_p is φ_{σ_p}.

Consider the characteristic map σ_p of an n-simplex e_p in $\mathcal{B}_p$. The simplex e_p is contained in some tile π_j. Viewing e_p as a subset of the patch p_j in $\mathbb{R}^d$, it is possible to define the vector $x_{\sigma_p\, \partial_i \sigma_p}$, for some i in $1, \ldots, n$, that joins the puncture of the ith face $\partial_i e_p$ (the simplex in $\mathcal{B}_p$ whose characteristic map is $\partial_i \sigma_p$) to the puncture of e_p. As a consequence of Remark 7, those vectors $x_{\sigma_p\, \partial_i \sigma_p}$ are identical for all σ_p in the preimage of the characteristic map σ of a simplex e on $\mathcal{B}_0$, and equal to the vector $x_{\sigma \partial_i \sigma}$ which defines the operator $\theta_{\sigma \partial_i \sigma}$ in Definition 14. By analogy, let $\theta_{\sigma_p\, \partial_i \sigma_p}$ be the operator $\chi_{\sigma_p} \mathrm{T}^{x_{\sigma \partial_i \sigma}} \chi_{\partial_i \sigma_p}$. With the relation $\mathrm{T}^a \chi_\Lambda = \chi_{\mathrm{T}^a \Lambda} \mathrm{T}^a$ for $\Lambda \subset \Omega$, $a \in \mathbb{R}^d$, and Lemma 1 it is easily seen that $\theta_{\sigma \partial_i \sigma}$ is the sum of the $\theta_{\sigma_p\, \partial_i \sigma_p}$ over all $\sigma_p \in f_{p\#}^{-1}(\sigma)$. Hence the PV differential given in equation (2) can be written

$$d_{\mathrm{PV}}^n = \sum_{\sigma_p \in \mathcal{S}_p^n} \sum_{i=0}^{n} (-1)^i \theta_{\sigma_p\, \partial_i \sigma_p}, \tag{5}$$

and is then well defined as a differential from $C(\Sigma_p^{n-1}, \mathbb{Z})$ to $C(\Sigma_p^n, \mathbb{Z})$.

Definition 16. Let $C_{\mathrm{PV}}^n(p) = C(\Sigma_p^n, \mathbb{Z})$, for $n = 0, \ldots, d$. The *PV cohomology* of the patch space $\mathcal{B}_p$, denoted $H_{\mathrm{PV}}^*(\mathcal{B}_0; C(\Sigma_p, \mathbb{Z}))$, is the homology of the complex $\{C_{\mathrm{PV}}^n(p), d_{\mathrm{PV}}^n\}$.

The notation $H_{\mathrm{PV}}^*(\mathcal{B}_0; C(\Sigma_p, \mathbb{Z}))$ requires some comments. The map $f_p : \mathcal{B}_p \to \mathcal{B}_0$ is a 'branched covering' with discrete 'fibers', the $\mathcal{S}_p^n$, that correspond to the 'discrete transversals' Σ_p^n:

$$\begin{array}{ccc} \Sigma_p & \hookrightarrow & \mathcal{B}_p \\ & & \downarrow \mathfrak{p}_p \\ & & \mathcal{B}_0 \end{array}$$

In analogy with Remark 5, the PV cohomology of $\mathcal{B}_p$ is analogous to a cohomology of the base space $\mathcal{B}_0$ with local coefficients in the K-theory of the 'fiber' Σ_p.

PROPOSITION 3. *The PV cohomology of the patch space $\mathcal{B}_p$ is isomorphic to its integer simplicial cohomology: $H_{\mathrm{PV}}^*(\mathcal{B}_0; C(\Sigma_p, \mathbb{Z})) \cong H^*(\mathcal{B}_p; \mathbb{Z})$.*

Proof. We actually prove a stronger statement: the complexes $C_{\mathrm{PV}}(p)^*$ and C_p^* are chain equivalent. As mentioned earlier, $\rho_{p,n}$ in equation (4) defines a group isomorphism between the simplicial cochain group C_p^n and the PV cochain group $C_{\mathrm{PV}}^n(p)$. Here they are both isomorphic to the direct sum $\mathbb{Z}^{|\mathcal{S}_p^n|}$, where $|\mathcal{S}_p^n|$ is the cardinality of $\mathcal{S}_p^n$, i.e. the number of n simplices on $\mathcal{B}_p$.

The differential of $\varphi \in C_{\mathrm{PV}}^{n-1}(p)$, evaluated on an n-simplex σ_p, reads using (5) $d_{\mathrm{PV}}^n \varphi(\sigma_p) = \sum_{\sigma_p \in \mathcal{S}_p^n} (d_{\mathrm{PV}}^n \varphi)_{\sigma_p} \chi_{\sigma_p}$, with $(d_{\mathrm{PV}}^n \varphi)_{\sigma_p} = \sum_{i=1}^n (-1)^i \varphi_{\partial_i \sigma_p}$. On the other hand the simplicial differential of $\psi \in C_p^{n-1}$ reads $\delta^n \psi(\sigma_p) = \sum_{i=1}^n (-1)^i \psi(\partial_i \sigma_p)$, and therefore $d_{\mathrm{PV}}^n \circ \rho_{p,n-1} = \rho_{p,n} \circ \delta^n$, $n = 1, \ldots, d$. Hence the $\rho_{p,n}$ give a chain map and conjugate the differentials, and thus yield isomorphisms $\rho_{p,n}^*$ between the nth cohomology groups. □

LEMMA 2. *Let $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ be a proper sequence of patch spaces of $\mathcal{T}$. The following holds: $\Xi_\Delta \cong \varprojlim(\mathcal{S}_l, f_l)$, and $C(\Xi_\Delta, \mathbb{Z}) \cong \varinjlim(C(\Sigma_l, \mathbb{Z}), f^l)$, where f^l is the dual map to f_l.*

Proof. This is a straightforward consequence of Theorem 5. □

PROPOSITION 4. *Let $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ be a proper sequence of patch spaces of $\mathcal{T}$. There is an isomorphism:*

$$H_{\mathrm{PV}}^*(\mathcal{B}_0; C(\Xi_\Delta, \mathbb{Z})) \cong \varinjlim(H_{\mathrm{PV}}^*(\mathcal{B}_0; C(\Sigma_l, \mathbb{Z})), f_l^*).$$

Proof. By the previous Lemma 2 the cochain groups C_{PV}^n of the hull are the direct limits of the cochain groups $C_{\mathrm{PV}}^n(l)$ of the patch spaces $\mathcal{B}_l$. Let $f_l^\# : C_{\mathrm{PV}}^n(l) \to C_{\mathrm{PV}}^n(l)$ denote the map induced by f_l on the PV cochain groups. Since the PV differential d_{PV} is the same for the complexes of each patch space $\mathcal{B}_l$, it suffices to check that the diagram,

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_{\mathrm{PV}}^{n-1}(l) & \xrightarrow{d_P^n} & C_{\mathrm{PV}}^n(l) & \longrightarrow \cdots \\
& \downarrow f_l^\# & & \downarrow f_l^\# & \\
\cdots \longrightarrow & C_{\mathrm{PV}}^{n-1}(l+1) & \xrightarrow{d_P^n} & C_{\mathrm{PV}}^n(l+1) & \longrightarrow \cdots
\end{array}$$

is commutative, which is straightforward using the relation $(f_l^\# \varphi)_{\partial_j \sigma_l} = \varphi_{f_{l\#}(\partial_j \sigma_l)}$. □

Proof of Theorem 7. By standard results in algebraic topology [**34, 71**], the Čech cohomology of the hull is isomorphic to the direct limit of the Čech cohomologies of the patch spaces $\mathcal{B}_i$, $\check{H}^*(\Omega; \mathbb{Z}) \cong \varinjlim(\check{H}^*(\mathcal{B}_l; \mathbb{Z}), f_l^*)$. By the natural isomorphism between Čech and simplicial cohomologies for finite CW-complexes [**71**], $\check{H}^*(\mathcal{B}_l; \mathbb{Z}) \cong H^*(\mathcal{B}_l; \mathbb{Z})$, and by Proposition 3, $H^*(\mathcal{B}_l; \mathbb{Z}) \cong H_{\mathrm{PV}}^*(\mathcal{B}_0; C(\Sigma_l, \mathbb{Z}))$, so that $\check{H}^*(\mathcal{B}_l; \mathbb{Z}) \cong H_{\mathrm{PV}}^*(\mathcal{B}_0; C(\Sigma_l, \mathbb{Z}))$. By Proposition 4, the direct limit of the $H_{\mathrm{PV}}^*(\mathcal{B}_0; C(\Sigma_l, \mathbb{Z}))$ is the PV cohomology of the hull. Therefore the integer Čech cohomology of the hull is isomorphic to the PV cohomology of the hull: $\check{H}^*(\Omega; \mathbb{Z}) \cong H_{\mathrm{PV}}^*(\mathcal{B}_0; C(\Xi_\Delta, \mathbb{Z}))$. □

5.2. *Proof of Theorem 6.* The proof of Theorem 6 follows from Propositions 5 and 6 below. First by Thom–Connes isomorphism [**21**], $K_{*+d}(C(\Omega) \rtimes \mathbb{R}^d) \cong K_*(C(\Omega))$, and therefore it suffices to build a spectral sequence that converges to the K-theory of the

C^*-algebra $C(\Omega)$. This is done by constructing a Schochet spectral sequence [**69**], $\{E_*^{r,s}\}$, associated with an appropriate filtration of $C(\Omega)$. It is shown in Proposition 5 that this spectral sequence is the direct limit of spectral sequences, $\{E_*^{r,s}(l)\}$, for the K-theory of the C^*-algebras $C(\mathcal{B}_l)$ of continuous functions on the patch spaces of a proper sequence. Then $\{E_*^{r,s}(l)\}$ is shown to be isomorphic to the Atiyah–Hirzebruch spectral sequence [**2**] for the topological K-theory of $\mathcal{B}_l$ in Proposition 6.

For basic definitions, terminology and results on spectral sequences, the reader is referred to [**50**]. In §A some technical results on exact couples used here are provided.

For $s = 0, \ldots, d$, let $\Omega^s = \mathfrak{p}_0^{-1}(\mathcal{B}_0^s)$ be the lift of the s-skeleton of the prototile space, and let $I_s = C_0(\Omega \setminus \Omega^s)$. I_s is a closed two-sided ideal of $C(\Omega)$, it consists of functions that vanish on the faces of dimension s of the boxes of the hull (see §3.3). This gives a filtration of $C(\Omega)$:

$$\{0\} = I_d \hookrightarrow I_{d-1} \hookrightarrow \cdots I_0 \hookrightarrow I_{-1} = C(\Omega). \tag{6}$$

Let $Q_s = I_{s-1}/I_s$, which is isomorphic to $C_0(\Omega^s \setminus \Omega^{s-1})$. Let $K(I)$ and $K(Q)$ be the respective direct sums of the $K_\epsilon(I_s)$ and $K_\epsilon(Q_s)$ over $\epsilon = 0, 1$, and $s = 0, \ldots, d$. The short exact sequences $0 \to I_s \xrightarrow{i_s} I_{s-1} \xrightarrow{\pi_s} Q_s \to 0$, lead, through long exact sequences in K-theory, to the exact couple $\mathfrak{P} = (K(I), K(Q), i, \pi, \partial)$, where i and π are the induced maps and ∂ the boundary map in K-theory. Its associated Schochet spectral sequence [**69**], $\{E_*^{r,s}\}$, converges to the K-theory of $C(\Omega)$:

$$\begin{cases} E_1^{r,s} \Rightarrow K_{r+s}(C(\Omega)), \\ E_1^{r,s} = K_{r+s}(Q_s). \end{cases} \tag{7}$$

Let $\mathcal{B}_p$ be a patch space associated with a pattern $\hat{p}$ of $\mathcal{T}$. Consider the filtration of the C^*-algebra $C(\mathcal{B}_p)$ by the closed two-sided ideals $I_s(p) = C_0(\mathcal{B}_p \setminus \mathcal{B}_p^s)$:

$$\{0\} = I_d(p) \hookrightarrow I_{d-1}(p) \hookrightarrow \cdots I_0(p) \hookrightarrow I_{-1}(p) = C(\mathcal{B}_p). \tag{8}$$

Let $Q_s(p) = I_{s-1}(p)/I_s(p)$, which is isomorphic to $C_0(\mathcal{B}_p^s \setminus \mathcal{B}_p^{s-1})$. Let $K(I(p))$ and $K(Q(p))$ be the respective direct sums of the $K_\epsilon(I_s(p))$ and $K_\epsilon(Q_s(p))$ over $\epsilon = 0, 1$, and $s = 0, \ldots, d$. The short sequences $0 \to I_s(p) \xrightarrow{i_{p,s}} I_{s-1}(p) \xrightarrow{\pi_{p,s}} Q_s(p) \to 0$, lead to the exact couple $\mathfrak{P}(p) = (K(I(p)), K(Q(p)), i_p, \pi_p, \partial)$, and its associated Schochet spectral sequence [**69**], $\{E_*^{r,s}(p)\}$, converges to the K-theory of $C(\mathcal{B}_p)$:

$$\begin{cases} E_1^{r,s}(p) \Rightarrow K_{r+s}(C(\mathcal{B}_p)), \\ E_1^{r,s}(p) = K_{r+s}(Q_s(p)). \end{cases} \tag{9}$$

PROPOSITION 5. *Given a proper sequence $\{\mathcal{B}_l, f_l\}_{l\in\mathbb{N}}$ of patch spaces of $\mathcal{T}$, the following holds:*

$$\{E_*^{r,s}\} \cong \varinjlim(\{E_*^{r,s}(l)\}, f_{l*}),$$

where $\{E_*^{r,s}(l)\}$ is the Schochet spectral sequence (9) for $\mathcal{B}_l$ corresponding to the patch $p = p_l$.

Proof. The map $f_l : \mathcal{B}_l \to \mathcal{B}_{l+1}$ induces a morphism of exact couples from $f_{l*} : \mathfrak{P}(l) \to \mathfrak{P}(l+1)$. Indeed, consider the following diagram:

$$\begin{array}{ccccccc}
K_\varepsilon(I_s(l)) & \xrightarrow{i_{l,s}} & K_\varepsilon(I_{s-1}(l)) & \xrightarrow{\pi_{l,s}} & K_\varepsilon(Q_s(l)) & \xrightarrow{\partial} & K_{\varepsilon+1}(I_s(l)) \\
\downarrow f_{l*} & & \downarrow f_{l*} & & \downarrow f_{l*} & & \downarrow f_{l*} \\
K_\varepsilon(I_s(l+1)) & \xrightarrow{i_{l+1,s}} & K_\varepsilon(I_{s-1}(l+1)) & \xrightarrow{\pi_{l+1,s}} & K_\varepsilon(Q_s(l+1)) & \xrightarrow{\partial} & K_{\varepsilon+1}(I_s(l+1))
\end{array}$$

The left and middle squares are easily seen to be commutative. To check the commutativity of the right square, recall that given a short exact sequence $0 \to I \to A \to A/J \to 0$ where A is a C^*-algebra and J a closed two-sided ideal, the boundary map of an element $[x] \in K_\varepsilon(A/J)$ is computed via a lift $z \in A \otimes \mathcal{K}$ of x. Let $[x] \in K_\varepsilon(Q_s(l))$. If $z \in I_{s-1}(l) \otimes \mathcal{K}$ is a lift of x, then $f_{l*}z \in I_{s-1}(l+1) \otimes \mathcal{K}$ is a lift of $f_{l*}x$ and the commutativity of the right square follows.

As a consequence of Theorem 5, $I \cong \varinjlim\{I(l), f_l^\#\}$ and $Q \cong \varinjlim\{Q(l), f_l^\#\}$, therefore $K(I) \cong \varinjlim\{K(I(l)), f_l^*\}$ and $K(Q) \cong \varinjlim\{K(Q(l)), f_l^*\}$. Hence $\{\mathfrak{P}(l), f_l^*\}$ is a direct system of exact couples and by Lemma A.2 $\mathfrak{P} \cong \varinjlim\{\mathfrak{P}(l), f_l^*\}$, and the same result on their associated spectral sequences follows by Corollary A.2. □

Given a finite CW-complex X, the Atiyah–Hirzebruch spectral sequence [**2**] for the topological K-theory of X is a particular case of Serre spectral sequence [**25, 50**] for the trivial fibration of X by itself with fiber a point:

$$\begin{cases} \underset{AH}{E}{}_2^{r,s} \Rightarrow K^{r+s}(X), \\ \underset{AH}{E}{}_2^{r,s} \cong H^r(X; K^s(\cdot)), \end{cases} \tag{10}$$

where $\cdot$ denotes a point, so $K^s(\cdot) \cong \mathbb{Z}$ if s is even, and 0 if s is odd.

It is defined on page-1 by $\underset{AH}{E}{}_2^{r,s} = K^r(X^s, X^{s-1})$ in [**2**], and then proven that the page-2 is isomorphic to the cellular cohomology of X. If X is a locally compact Hausdorff space, the spectral sequence can be rewritten algebraically, using the isomorphisms $K^*(Y) \cong K_*(C_0(Y))$ and $K^*(Y, Z) \cong K_*(C_0(Y)/C_0(Z))$ for locally compact Hausdorff spaces Y, Z. Consider the Schochet spectral sequence [**69**] for the K-theory of the C^*-algebra $C_0(X)$ associated with its filtration by the ideals $I_s = C_0(X, X^s)$ of functions vanishing on the s-skeleton. Then the spectral sequence associated with the cofiltration of $C_0(X)$ by the ideals $F_s = C_0(X)/I_s \cong C_0(X^s)$ turns out to be this algebraic form of Atiyah–Hirzebruch spectral sequence.

PROPOSITION 6. *The Schochet spectral sequence* $\{E_*^{r,s}(p)\}$ *for the* K*-theory of the* C^**-algebra* $C(\mathcal{B}_p)$ *is isomorphic to the Atiyah–Hirzebruch spectral sequence for the topological* K*-theory of* $\mathcal{B}_p$.

Proof. By Theorem A.2 (§A) the Schochet spectral sequences built from the filtration of $A(p) = C(\mathcal{B}_p)$ by the ideals $I_s(p)$, namely $\{E_*^{r,s}(p)\}$, and from the cofiltration of $A(p)$ by the quotients $F_s(p) = A(p)/I_s(p) \cong C_0(\mathcal{B}_p^s)$, are isomorphic. As remarked above this last spectral sequence is nothing but the Atiyah–Hirzebruch spectral sequence. □

Proof of Theorem 6. Let $\{\mathcal{B}_l, f_l\}_{l\in\mathbb{N}}$ be a proper sequence of patch spaces of $\mathcal{T}$. By Proposition 6, the page-2 of Schochet spectral sequence $\{E_*^{r,s}(l)\}$ is isomorphic to the simplicial cohomology of $\mathcal{B}_l$: $E_2^{r,s}(l) \cong \check{H}^r(\mathcal{B}_i; \mathbb{Z})$ for s even and 0 for s odd. By the natural isomorphism between simplicial and Čech cohomologies for CW-complexes [**71**], it follows that $E_2^{r,s}(l) \cong \check{H}^r(\mathcal{B}_i; \mathbb{Z})$ for s even and 0 for s odd. By standard results in algebraic topology [**34, 71**] $\check{H}^*(\Omega; \mathbb{Z}) \cong \varinjlim(\check{H}^*(\mathcal{B}_l; \mathbb{Z}), f_l^*)$, and therefore by Proposition 5 the page-2 of Schochet spectral sequence for the K-theory of $C(\Omega)$ is isomorphic to the integer Čech cohomology of the hull: $E_2^{r,s} \cong \check{H}^r(\Omega; \mathbb{Z})$ for s even and 0 for s odd. □

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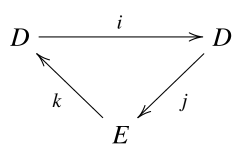
A. *Appendix. A reminder on spectral sequences*

Some technical results about exact couples are recalled here. The formalism allows to state Corollary A.1 which leads to the isomorphism between the Schochet spectral sequence associated with the filtration of a C^*-algebra and the Schochet spectral sequence associated with its corresponding cofiltration in Theorem A.2. The construction of a direct limit of spectral sequences is also recalled. A few elementary definitions are given to set the terminology. The original reference on exact couples is the work of Massey [**49**]. The link with spectral sequences is only mentioned, and the reader is referred to [**25, 50**] for further details.

An *exact couple* is a family $T = (D, E, i, j, k)$ where D and E are Abelian groups and i, j, k are group homomorphisms making the following triangle exact:



(A.1)

In more generality, D and E can be graded modules over a ring, and i, j, k module maps of various degrees. A morphism α between two exact couples T and T' is a pair of group homomorphisms (α_D, α_E) making the following diagram commutative:

$$\begin{array}{ccccccc} D & \xrightarrow{i} & D & \xrightarrow{j} & E & \xrightarrow{k} & D \\ \downarrow{\scriptstyle \alpha_D} & & \downarrow{\scriptstyle \alpha_D} & & \downarrow{\scriptstyle \alpha_E} & & \downarrow{\scriptstyle \alpha_D} \\ D' & \xrightarrow{i'} & D' & \xrightarrow{j'} & E' & \xrightarrow{k'} & D' \end{array} \tag{A.2}$$

The composition map $d = j \circ k : E \mapsto E$ is called the *differential* of the exact couple. Since (A.1) is exact, it follows that $d^2 = j \circ (k \circ j) \circ k = 0$. Let then $H_d(E)$ be the homology of the complex $E \xrightarrow{d} E$. Then the following theorem holds.

THEOREM A.1.

(i) *There is a* derived exact couple

$$\begin{array}{ccc} D^{(1)} = i(D) & \xrightarrow{\quad i^{(1)} \quad} & D^{(1)} = i(D) \\ & {\scriptstyle k^{(1)}} \nwarrow \qquad \swarrow {\scriptstyle j^{(1)}} & \\ & E^{(1)} = H_d(E) & \end{array} \tag{A.3}$$

defined by $i^{(1)} = i|_{i(D)}$, $j^{(1)}(i(x)) = j(x) + \mathrm{Im}(d)$ *and* $k^{(1)}(e + \mathrm{Im}(d)) = k(e)$.

(ii) *The derivation* $T \to T^{(1)}$ *is a functor on the category of exact couples (with morphisms of exact couples).*

The nth iterated derived couple of T is denoted $T^{(n)} = (D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)})$.

Remark A.1. Point (ii) in Theorem A.1 implies that a morphism of exact couples $\alpha : T \to T'$ induces derived morphisms $\alpha^{(n)}$ between the derived couples: $\alpha_D^{(n)} = \alpha_D^{(n-1)}|_{D^{(n)}}$ is the restriction of $\alpha_D^{(n-1)}$ to $D^{(n)} = i^{(n-1)}(D^{(n-1)})$, and $\alpha_E^{(n)} = (\alpha_E^{(n-1)})_*$ is the induced map of $\alpha_E^{(n-1)}$ on homology. Also α_E conjugates the differentials, namely $\alpha_E^{(n)} \circ d^{(n)} = d'^{(n)} \circ \alpha_E^{(n)}$ for all n.

The *spectral sequences* considered here come from exact couples. Given an exact couple (E, D, i, j, k), its associated spectral sequence is the family $(E_l, d_l)_{l\in\mathbb{N}}$ where $E_l = E^{(l)}$ is the derived E-term, and called the E_l*-page* or simply *page-l* of the spectral sequence, and $d_l = d^{(l)}$ is its differential. As noted above those E-terms are generally (bi)graded modules. A morphism of spectral sequence $\beta : (E_l, d_l)_{l\in\mathbb{N}} \to (E'_l, d'_l)_{l\in\mathbb{N}}$ is given by the α_E-morphism of the exact couples, and can be seen as a sequence of *chain maps* $\beta_l : E_l \to E'_l$ for each l, i.e. commute with the differentials: $d_l \beta_l = \beta_l d'_l$.

Definition A.1.

(i) An exact couple $T = (D, E, i, j, k)$ is said to be *trivial* if $E = 0$.

(ii) Two exact couples T and T' are said to be *equivalent* if there is a morphism $\alpha : T \to T'$ such that α_E is an isomorphism.

Remark A.2.

(i) If an exact couple is trivial, then it is of the form $(D, 0, \mathrm{id}, 0, 0)$.

(ii) A trivial exact couple is identical to its derived couple.

(iii) An equivalence between two exact couples is equivalent to an isomorphism between their associated spectral sequences.

Definition A.2.

(i) An exact couple is said to converge whenever there is an $L \in \mathbb{N}$ such that for $l \geq L$ the lth derived couples are trivial.

(ii) A spectral sequence is said to converge if its associated exact couple converges.

Given a morphism of exact couples $\alpha : T \to T'$, its kernel $\operatorname{Ker} \alpha = (\operatorname{Ker} \alpha_D, \operatorname{Ker} \alpha_E, i, j, k)$ and image $\operatorname{Im} \alpha = (\operatorname{Im} \alpha_D, \operatorname{Im} \alpha_D, i', j', k')$ are 3-periodic complexes (by commutativity of diagram (A.2)), but no longer exact couples in general.

LEMMA A.1. *Given an exact sequence of exact couples*

$$\cdots \to T_{m-1} \xrightarrow{\alpha_{m-1}} T_m \xrightarrow{\alpha_m} T_{m+1} \xrightarrow{\alpha_{m+1}} T_{m+2} \to \cdots ,$$

that is $\operatorname{Im} \alpha_{m-1} = \operatorname{Ker} \alpha_m$ *for all m, the following holds: if* T_{m-1} *and* T_{m+2} *are trivial, then* T_m *and* T_{m+1} *are equivalent.*

Proof. Using Remark A.2 the couples can be rewritten $T_{m-1} = (D_{m-1}, 0, \mathrm{id}, 0, 0)$ and $T_{m+2} = (D_{m+2}, 0, \mathrm{id}, 0, 0)$, and therefore $\alpha_{m-1} = (\alpha_{D_{m-1}}, 0)$, and $\alpha_{m+1} = (\alpha_{D_{m+1}}, 0)$. The exact sequence for the E terms then reads $\cdots \to 0 \to E_m \xrightarrow{\alpha_{E_m}} E_{m+1} \to 0 \to \cdots$, and therefore α_{E_m} is an isomorphism and gives an equivalence between T_m and T_{m+1}. □

COROLLARY A.1. *Given an exact triangle of exact couples, one of which is trivial, then the two others are equivalent.*

Definition A.3. A direct system of exact couples $\{T_l, \alpha_{lm}\}_I$, is given by a directed set I, and a family of exact couples T_l and morphisms of exact couples $\alpha_{lm} : T_l \to T_m$ for $l \leq m$ (with α_{ll} the identity), such that given $l, m \in I$ there exists $n \in I$, $n \geq l, m$ with $\alpha_{ln} = \alpha_{mn} \circ \alpha_{lm}$.

If T_l is written $(D_l, E_l, i_l, j_l, k_l)$, the definition of a direct system of associated spectral sequences is given similarly by keeping only the data of the E_l-terms, their differentials $j_l \circ k_l$ and the morphisms α_{E_l}.

Let R be a commutative ring. Recall that the direct limit of a directed system of R-modules, $\{M_l, \sigma_{lm}\}_I$ is given by the quotient of the direct product $\bigoplus_I M_l$ by the R-module generated by all elements of the form $a_l - \sigma_{lm}(a_l)$ for $a_l \in M_l$, where each R-module M_l is viewed as a submodule of $\bigoplus_I A_l$.

LEMMA A.2. *Let* $\{T_l, \alpha_{lm}\}_I$ *be a direct system of exact couples, and write the couples as* $T_l = (D_l, E_l, i_l, j_l, k_l)$*, and the morphisms as* $\alpha_{lm} = (\alpha_{D_{lm}}, \alpha_{E_{lm}})$*. There is a direct limit exact couple* $T = \varinjlim\{T_l, \alpha_{lm}\}$*, given by* $T = (D, E, i, j, l)$ *with* $D = \varinjlim\{D_l, \alpha_{D_lm}\}$*,* $E = \varinjlim\{E_l, \alpha_{E_lm}\}$*, and i, j, k the maps induced by the* i_l, j_l, k_l*.*

Proof. The direct limits D and E are well defined and it suffices to give the expressions of the module homomorphisms i, j, k and show that T is exact. Let d be in D, i.e. it is the class $[d_l]$ for some $d_l \in D_l$, then $i(d) = [i_l(d_l)]$. If $d = [d_m]$ for some $m \geq l$ then $d_m = \alpha_{D_{lm}}(d_l)$ and $[i_m(d_m)] = [i_m \circ \alpha_{D_{lm}}(d_l)] = [\alpha_{D_{lm}} \circ i_l(d_l)]$ because $\alpha_{D_{lm}}$ is a morphism of exact couple (commutativity of diagram (A.2)), and therefore $[i_m(d_m)] = [i_l(d_l)]$ and i are well defined. Similarly for $d = [d_l]$ in D, $j(d) = [j_l(d_l)]$ in E, and for $e = [e_l]$ in E, $k(e) = [k_l(e_l)]$ in D, and are well defined.

This proves also that T is a 3-periodic complex since the compositions $j \circ i$, $k \circ j$ and $i \circ k$ involve the compositions of the i_l, j_l, k_l, and are thus zero. Let $d \in \operatorname{Ker} j$ and $d = [d_l]$, then $j(d) = [j_l(d_l)] = 0$ and by exactness of T_l there exists $d'_l \in D_l$ such that $d_l = i_l(d'_l)$; let then $d' = [d'_l]$ in D to have $d = [i_l(d'_l)] = i(d')$ and therefore $\operatorname{Ker} j = \operatorname{Im} i$. The other two relations $\operatorname{Ker} i = \operatorname{Im} k$ and $\operatorname{Ker} k = \operatorname{Im} j$ are proven similarly, and this shows that T is exact. □

COROLLARY A.2. *The result of Lemma A.2 for the direct limit of exact couples also holds for the direct limit of associated spectral sequences.*

Let A be a C^*-algebra , and assume there is a finite filtration by closed two-sided ideals:

$$\{0\} = I_d \overset{i_d}{\hookrightarrow} I_{d-1} \overset{i_{d-1}}{\hookrightarrow} \cdots I_0 \overset{i_0}{\hookrightarrow} I_{-1} = A. \tag{A.4}$$

There is an associated cofiltration of A by the quotient C^*-algebras $F_p = A/I_p$:

$$A = F_d \overset{\rho_d}{\twoheadrightarrow} F_{d-1} \overset{\rho_{d-1}}{\twoheadrightarrow} \cdots F_0 \overset{\rho_0}{\twoheadrightarrow} F_{-1} = \{0\}. \tag{A.5}$$

Let $Q_p = I_{p-1}/I_p$ be the quotient C^*-algebra. There are short exact sequences:

$$0 \longrightarrow I_p \xrightarrow{i_p} I_{p-1} \xrightarrow{\pi_p} Q_p \longrightarrow 0, \tag{A.6a}$$

$$0 \longrightarrow Q_p \xrightarrow{j_p} F_p \xrightarrow{\rho_p} F_{p-1} \longrightarrow 0, \tag{A.6b}$$

$$0 \longrightarrow I_p \xrightarrow{l_p} A \xrightarrow{\sigma_p} F_p \longrightarrow 0. \tag{A.6c}$$

In (A.6a) i_p is the canonical inclusion, and π_p the quotient map $\pi_p(x) = x + I_p$. In (A.6c) $l_p = i_p \circ i_{p-1} \circ \cdots \circ i_0$ is the canonical inclusion, and σ_p the quotient map $\sigma_p(x) = x + I_p$. And in (A.6b) j_p is the canonical inclusion $j_p(x + I_p) = l_{p-1}(x) + I_p$, and ρ_p the quotient map $\rho_p(x + I_p) = x + I_{p-1}$.

Associated with the short exact sequences of C^*-algebras (A.6a) and (A.6b) there are a long (six-term periodic) exact sequences in K-theory that can be written in exact couples:

$$T_I : \begin{array}{ccc} K(I) & \xrightarrow{\quad i \quad} & K(I) \\ & {\scriptstyle \partial} \nwarrow \quad \swarrow {\scriptstyle \pi} & \\ & K(Q) & \end{array} \quad \text{with} \quad K(I) = \bigoplus_{p=-1}^{d} \bigoplus_{\varepsilon=0,1} K_\varepsilon(I_p), \tag{A.7a}$$

$$T_F : \begin{array}{ccc} K(F) & \xrightarrow{\quad \rho \quad} & K(F) \\ & {\scriptstyle j} \nwarrow \quad \swarrow {\scriptstyle \partial} & \\ & K(Q) & \end{array} \quad \text{with} \quad K(F) = \bigoplus_{p=-1}^{d} \bigoplus_{\varepsilon=0,1} K_\varepsilon(F_p), \tag{A.7b}$$

with $K(Q) = \bigoplus_{p=-1}^{d} \bigoplus_{\varepsilon=0,1} K_\varepsilon(Q_p)$, and i, π, j and ρ the induced maps on K-theory.

Since the filtration (A.4) and the cofiltration (A.5) are finite, both exact couples converge to the K-theory of A.

THEOREM A.2. *The spectral sequence for the K-theory of a C^*-algebra A, associated with a filtration of A by ideals I_p as in (A.4), and the spectral sequence associated with its corresponding cofiltration by quotients $F_p = A/I_p$ as in (A.5), are isomorphic.*

Proof. By Remark A.2(iii) it is sufficient to prove that the exact couples T_I (A.7a) and T_F (A.7b) are equivalent. By Corollary A.1 it is also sufficient to prove that T_I and T_F fit into an exact triangle with a trivial exact couple.

Let T_A be the the trivial exact couple $T_A = (K(A), 0, \mathrm{id}, 0, 0)$. We prove that the short exact sequences (A.6c) induce the following exact triangle of exact couples:

$$\begin{array}{ccc} T_I & \xrightarrow{\;\; l \;\;} & T_A \\ & {}_{\partial}\nwarrow \quad \swarrow_{\sigma} & \\ & T_F & \end{array} \tag{A.8}$$

The exactness of the triangle comes from the exactness of the K-theory functor, and one has now to verify that the applications l, σ, and ∂ define morphisms of exact couples.

Those proofs are similar. We give the details for $(l, 0) : T_I \to T_A$, where the application 0 is the induced quotient map of the trivial short exact sequence $0 \longrightarrow Q_p \xrightarrow{\mathrm{id}} Q_p \xrightarrow{0} 0 \longrightarrow 0$, i.e. we prove that the following diagram is commutative (ε is 0 or 1):

$$\begin{array}{ccccccc} K_\varepsilon(I_p) & \xrightarrow{i_{p*}} & K_\varepsilon(I_{p-1}) & \xrightarrow{\pi_{p*}} & K_\varepsilon(Q_p) & \xrightarrow{\partial} & K_{\varepsilon+1}(I_p) \\ \downarrow{\scriptstyle l_{p*}} & & \downarrow{\scriptstyle l_{p-1*}} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle l_{p*}} \\ K_\varepsilon(A) & \xrightarrow{\mathrm{id}_*} & K_\varepsilon(A) & \xrightarrow{0} & 0 & \xrightarrow{\partial} & K_{\varepsilon+1}(A) \end{array} \tag{A.9}$$

The middle square is commutative since the maps lead to 0. The commutativity of the left square comes from the functoriality of K-theory: since $l_p = l_{p-1} \circ i_p$, it follows that $\mathrm{id}_* l_{p*} = l_{p*} = l_{p-1*} i_{p*}$. For the right square $l_{p*} = \mathrm{id}_* l_{p*} = l_{p-1*} i_{p*}$ by commutativity of the left square, and thus $l_{p*}\partial = l_{p-1*} i_{p*}\partial = 0$ because $i_{p*}\partial = 0$ by exactness of the long exact sequence in K-theory induced by (A.6a). □

REFERENCES

[1] J. E. Anderson and I. F. Putnam. Topological invariants for substitution tilings and their associated C^*-algebra. *Ergod. Th. Dynam. Sys.* **18** (1998), 509–537.

[2] M. F. Atiyah and F. Hirzebruch. *Vector Bundles and Homogeneous Spaces (Proceedings of Symposia in Pure Mathematics, 3).* American Mathematical Society, Providence, RI, 1961, pp. 7–38.

[3] M. F. Atiyah and I. M. Singer. The index of elliptic operators on compact manifolds. *Bull. Amer. Math. Soc.* **69** (1963), 422–433.

[4] M. F. Atiyah, G. B. Segal and I. M. Singer. The index of elliptic operators. I, II, III. *Ann. of Math.* (2) **87** (1968), 484–530, 531–545, 546–604.

[5] J. Bellissard. *Schrödinger's Operators with an Almost Periodic Potential: An Overview (Lecture Notes in Physics, 153).* Springer, Berlin, 1982.

[6] J. Bellissard. K-theory of C^*-algebras in solid state physics. *Statistical Mechanics and Field Theory, Mathematical Aspects (Lecture Notes in Physics, 257).* Eds. T. C. Dorlas, M. N. Hugenholtz and M. Winnink. Springer, Berlin, 1986, pp. 99–156.

[7] J. Bellissard, B. Iochum, E. Scoppola and D. Testard. Spectral properties of one dimensional quasi-crystals. *Comm. Math. Phys.* **125** (1989), 527–543.

[8] J. Bellissard, A. Bovier and J. M. Ghez. Gap labelling theorem for one dimensional Schrödinger operators. *Rev. Math. Phys.* **4**(1) (1992), 1–37.

[9] J. Bellissard. Gap labelling theorems for Schrödinger's operators. *From Number Theory to Physics (Les Houches March 89)*. Eds. J. M. Luck, P. Moussa and M. Waldschmidt. Springer, Berlin, 1993, pp. 538–630.

[10] J. Bellissard, E. Contensou and A. Legrand. K-théorie des quasi-cristaux, image par la trace: le cas du réseau octogonal. *C. R. Acad. Sci. Paris Sér. I Math.* **326** (1998), 197–200.

[11] J. Bellissard, D. Hermmann and M. Zarrouati. Hull of aperiodic solids and gap labeling theorems. *Directions in Mathematical Quasicrystals (CRM Monograph Series, 13)*. Eds. M.B. Baake and R.V. Moody. American Mathematical Society, Providence, RI, 2000, pp. 207–259.

[12] J. Bellissard, J. Kellendonk and A. Legrand. Gap-labelling for three-dimensional aperiodic solids. *C. R. Acad. Sci. Paris, Sér. I Math.* **332** (2001), 521–525.

[13] J. Bellissard, R. Benedetti and J.-M. Gambaudo. Spaces of tilings, finite telescopic approximations and gap-labelling. *Comm. Math. Phys.* **261** (2006), 1–41.

[14] R. Benedetti and J.-M. Gambaudo. On the dynamics of $\mathbb{G}$-solenoids. Applications to Delone sets. *Ergod. Th. Dynam. Sys.* **23** (2003), 673–691.

[15] B. Blackadar. *K-Theory for Operator Algebras*, 2nd edn. Cambridge University Press, Cambridge, 1998.

[16] A. Borel and J.-P. Serre. Impossibilité de fibrer un espace euclidien par des fibres compactes. *C. R. Acad. Sci. Paris* **230** (1950), 2258–2260.

[17] O. Bratelli. Inductive limits of finite dimensional C^*-algebras. *Trans. Amer. Math. Soc.* **171** (1972), 195–234.

[18] F. H. Claro and G. H. Wannier. Closure of bands for bloch electrons in a magnetic field. *Phys. Status Solidi* b **88** (1978), K147–K151.

[19] L. A. Coburn, R. D. Moyer and I. M. Singer. C^*-algebra of almost periodic pseudo-differential operators. *Acta Math.* **130** (1973), 279–307.

[20] A. Connes. Sur la théorie non commutative de l'intégration. *Algèbres d'opérateurs (Sém., Les Plans-sur-Bex, 1978) (Lecture Notes in Mathematics, 725)*. Springer, Berlin, 1979, pp. 19–143.

[21] A. Connes. An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of R. *Adv. Math.* **39**(1) (1981), 31–55.

[22] A. Connes. A survey of foliations and operator algebras. *Operator Algebras and Applications, Part I, (Kingston, Ontario, 1980) (Proceedings of Symposia in Pure Mathematics, 38)*. American Mathematical Society, Providence, RI, 1982, pp. 521–628.

[23] A. Connes. *Géométrie Non Commutative*. InterEditions, Paris, 1990.

[24] A. Connes. *Noncommutative Geometry*. Academic Press, San Diego, 1994.

[25] J. F. Davis and P. Kirk. *Lecture Notes in Algebraic Topology (AMS Graduate Studies in Mathematics, 35)*. American Mathematical Society, Providence, RI, 2001.

[26] A. Van Elst. Gap-labelling theorems for Schrödinger operators on the square and cubic lattice. *Rev. Math. Phys.* **6** (1994), 319–342.

[27] A. H. Forrest and J. R. Hunton. The cohomology and K-theory of commuting homeomorphisms of the Cantor set. *Ergod. Th. & Dynam. Sys.* **19** (1999), 611–625.

[28] A. H. Forrest, J. R. Hunton and J. Kellendonk. Topological invariants for projection method patterns. *Mem. Amer. Math. Soc.* **159**(758) (2002).

[29] F. Gähler, J. Hunton and J. Kellendonk. Torsion in tiling homology and cohomology. arXiv.com math-phys/0505048, May 2005.

[30] J.-M. Gambaudo and M. Martens. Algebraic topology for minimal Cantor sets. *Ann. Henri Poincaré* **7** (2006), 423–446.

[31] E. Ghys. Laminations par surfaces de Riemann. *Dynamique et géométrie complexes, Panoramas et Synthèses* **8** (1999), 49–95.

[32] A. Grothendieck. La théorie des classes de Chern. *Bull. Soc. Math. France* **86** (1958), 137–154.

[33] B. Grünbaum and G. C. Shephard. *Tilings and Patterns*, 1st edn. W.H. Freemand and Co, New York, 1987.

[34] A. Hatcher. *Algebraic Topology*, 1st edn. Cambridge University Press, Cambridge, 2002 (and available online at http://www.math.cornell.edu/~hatcher/#ATI).

[35] F. Hirzebruch. Arithmetic genera and the theorem of Riemann–Roch for algebraic varieties. *Proc. Natl. Acad. Sci. USA* **40** (1954), 110–114.

[36] F. Hirzebruch. *Neue topologische Methoden in der algebraischen Geometrie (Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 9)*. Springer, Berlin, 1956 (*Topological Methods in Algebraic Geometry*, Reprint of the 3rd edn. Springer, Berlin, 1978).

[37] D. R. Hofstadter. Energy levels and wave functions of Bloch electrons in a rational or irrational magnetic field. *Phys. Rev.* B **14** (1976), 2239–2249.

[38] R. Johnson and J Moser. The rotation number for almost periodic potentials. *Comm. Math. Phys.* **84** (1982), 403–438.

[39] G. G. Kasparov. The operator K-functor and extensions of C^*-algebras. *Izv. Akad. Nauk SSSR* **44** (1980), 571–636.

[40] G. G. Kasparov. Equivariant KK-theory and the Novikov conjecture. *Invent Math.* **91** (1988), 147–201.

[41] J. Kellendonk. Noncommutative geometry of tilings and gap labelling. *Rev. Math. Phys.* **7** (1995), 1133–1180.

[42] J. Kellendonk. Pattern-equivariant functions and cohomology. *J. Phys. A: Math. Gen.* **36** (2003), 5765–5772.

[43] J. Kellendonk and I. F. Putnam. The Ruelle–Sullivan map for actions of $\mathbb{R}^n$. *Math. Ann.* **334** (2006), 693–711.

[44] J.-L. Koszul. Sur les opérateurs de dérivation dans un anneau. *C. R. Acad. Sci., Paris* **225** (1947), 217–219.

[45] J. C. Lagarias. Geometric models for quasicrystals I. Delone sets of finite type. *Discrete Comput. Geom.* **21** (1999), 161–191.

[46] J. C. Lagarias. Geometric models for quasicrystals. II. Local rules under isometries. *Discrete Comput. Geom.* **21** (1999), 345–372.

[47] J. C. Lagarias and P. A. B. Pleasants. Repetitive Delone sets and quasicrystals. *Ergod. Th. Dynam. Sys.* **23** (2003), 831–867.

[48] J. Leray. Structure de l'anneau d'homologie d'une représentation. *C. R. Acad. Sci., Paris* **222** (1946), 1419–1422.

[49] W. S. Massey. Exact couples in algebraic topology (Parts I & II). *Ann. Math.* **56** (1952), 363–396; Exact couples in algebraic topology (Parts III, IV & V). *Ann. Math.* **57** (1953), 248–286.

[50] J. McCleary. *A User's Guide to Spectral Sequences (Cambridge Studies in Advanced Mathematics, 58)*, 2nd edn. Cambridge University Press, Cambridge, 2001.

[51] Y. Meyer. *Algebraic Number and Harmonic Analysis*. North-Holland, Amsterdam, 1972.

[52] C. C. Moore and C. Schochet. *Global Analysis on Foliated Spaces (Mathematical Sciences Research Institute Publications, 9)*. Springer, New York, 1988.

[53] J. Moser. An example of a Schrödinger equation with almost periodic potential and nowhere dense spectrum. *Comment. Math. Helv.* **56** (1981), 198–224.

[54] R. Palais. *Seminar on the Atiyah–Singer Index Theorem (Annals of Mathematics Studies, 57)*. Princeton University Press, Princeton, NJ, 1965.

[55] M. Pimsner and D. Voiculescu. Exact sequences for K-groups and Ext groups of certain cross-product C^*-algebras. *J. Operator Theory* **4** (1980), 93–118.

[56] M. V. Pimsner. Ranges of traces on K_0 of reduced crossed products by free groups. *Operator Algebras and their Connections with Topology and Ergodic Theory (Buşteni, 1983) (Lecture Notes in Mathematics, 1132)*. Springer, Berlin, 1985, pp. 374–408.

[57] R. Penrose. The role of aesthetics in pure and applied mathematical research. *Bull. Inst. Math. Appl.* **10** (1974), 55–65.

[58] M. Queffelec. *Substitution Dynamical Systems-Spectral Analysis (Lecture Notes in Mathematics, 1294)*. Springer, Berlin, 1987.

[59] C. Radin. The pinwheel tiling of the plane. *Ann. of Math.* (2) **139** (1994), 257–264.

[60] C. Radin. *Miles of Tiles*. Vol. 1. American Mathematical Society, Student Mathematical Library, Providence, RI, 1999.

[61] J. Renault. *A Groupoid Approach to C^* (Lecture Notes in Mathematics, 793)*. Springer, Berlin, 1980.

[62] M. A. Rieffel. Morita equivalence for operator algebras. *Operator Algebras and Applications, Part 1 (Proceedings of Symposia in Pure Mathematics, 38)*. American Mathematical Society, Providence, RI, 1982, pp. 285–298.

[63] L. Sadun. Tiling spaces are inverse limits. *J. Math. Phys.* **44** (2003), 5410–5414.

[64] L. Sadun and R. F. Williams. Tiling spaces are Cantor set fiber bundles. *Ergod. Th. & Dynam. Sys.* **23** (2003), 307–316.

[65] L. Sadun. Explicit computation of the cohomology for the chair tiling, talk delivered at Banff Research Station, July 2005.

[66] L. Sadun. Pattern-equivariant cohomology with integer coefficients, arXiv.com math.DS/0602066, February 2006. Available at ftp://ftp.ma.utexas.edu/pub/papers/sadun/2006/integer4.pdf.

[67] J.-P. Serre. Homologie singulière des espaces fibrés. Applications. *Ann. of Math.* **54** (1951), 425–505.

[68] D. Shechtman, I. Blech, D. Gratias and J. V. Cahn. Metallic phase with long range orientational order and no translational symmetry. *Phys. Rev. Lett.* **53** (1984), 1951–1953.

[69] C. Schochet. Topological methods for C*-algebra I: spectral sequences. *Pacific J. Math.* **96**(1) (1981), 193–211.

[70] I. M. Singer. Future extensions of index theory and elliptic operators. *Prospects in Mathematics (Proc. Symp., Princeton University, NJ, 1970) (Annals of Mathematical Studies, 70)*. Princeton University Press, Princeton, NJ, 1971, pp. 171–185.

[71] E. Spanier. *Algebraic Topology*. McGraw-Hill, New York, 1966 (reprinted by Springer, Berlin).

[72] M. A. Subin. Spectral theory and the index of elliptic operators with almost-periodic coefficients (Russian). *Uspekhi Mat. Nauk* **34**(2(206)) (1979), 95–135.

[73] A. Süto. The spectrum of a quasi-periodic Schrödinger operator. *Comm. Math. Phys.* **111** (1987), 409–415.

[74] D. J. Thouless, M. Kohmoto, M. Nightingale and M. den Nijs. Quantum hall conductance in two dimensional periodic potential. *Phys. Rev. Lett.* **49** (1982), 405–408.

[75] R. F. Williams. Expanding attractors. *Publ. Math. IHES* **43** (1974), 169–203.