Anomalous Transport: results, conjectures and applications to Quasicrystals

J. Bellissard *

Institut Universitaire de France & Université Paul Sabatier,
I.R.S.A.M.C, 118, Route de Narbonne, F-31062 Toulouse Cedex, France
jeanbel@irsamc2.ups-tlse.fr

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Abstract

We review the main rigorous results accumulated during the last five years concerning the theory of transport in aperiodic media. Motivated by the transport properties of quasicrystals, we introduce the mathematical background needed to describe transport in aperiodic media. We then give the definitions and the properties of spectral and diffusion exponents with an emphasis on "anomalous transport". The last section is devoted to discussing the relevance of this theory for quasicrystals.

Keywords: Anomalous Quantum Transport, Spectral Exponents, Transports Exponents.

1. Transport in Quasicrystals

Prior to the discovery of quasicrystals (QC), several mechanisms had already been identified concerning the behaviour of the conductivity with respect to the temperature.
(i) For metals, the conductivity increases as the temperature decreases. Moreover, the Fermi liquid theory predicts [1, 2]

$$\sigma(T) \sim T^{-2}$$

over a large range of temperatures.
(ii) On the opposite side, thermally activated processes, namely whenever a gap arises in the spectrum near the Fermi level, give [3]

$$\sigma(T) \overset{T\downarrow 0}{\sim} e^{-\Delta/T}$$

(iii) For weakly disordered systems, such as strongly doped semiconductors or normal metals, there is a residual conductivity at low temperature, due to *quantum interferences* [3, 4]

$$\sigma(T) \overset{T\downarrow 0}{\sim} \sigma_0 > 0$$

(iv) In strongly disordered systems, such as for lightly doped semiconductors, and whenever the Density of States (DOS) is small at the Fermi level, one expects a *Mott variable range hopping* [3], namely in dimension D

$$\sigma(T) \overset{T\downarrow 0}{\sim} e^{-(T_0/T)^{1/(D+1)}}$$

Conductivity properties of QC's are rather anomalous when compared to the previous mechanisms.
(QC.1) Their conductivity is low to very low [5].
(QC.2) Their conductivity decreases enormously with the temperature, with

$$\frac{\sigma_{300K}}{\sigma_{4K}} \geq 300$$

in some cases.
(QC.3) The conductivity is not thermally activated.
(QC.4) The conductivity decreases with improved sample quality.
(QC.5) In the temperature range from $300K$ to $1000K$, the conductivity follows a scaling law [5].

$$\sigma \propto T^{\gamma} \qquad\qquad 1 < \gamma < 2$$

This behaviour is observed at lower temperature for $i-AlPdRe$ down to less than $10K$.

(QC.6) For most good quality icosahedral compounds, such as $AlCuFe$ or $AlPdMn$, the conductivity flattens below $300K$ to reach a residual conductivity $\sigma(t) \approx \sigma_0 > 0$ (see Fig.1).

(QC.7) In the icosahedral phase of $AlPdRe$, instead one observes some type of Mott variable range hopping with [7, 8]

$$\sigma \propto e^{-(T_0/T)^{1/4}}$$

in a range of temperature which varies from author to author within the interval 0 to $10K$.

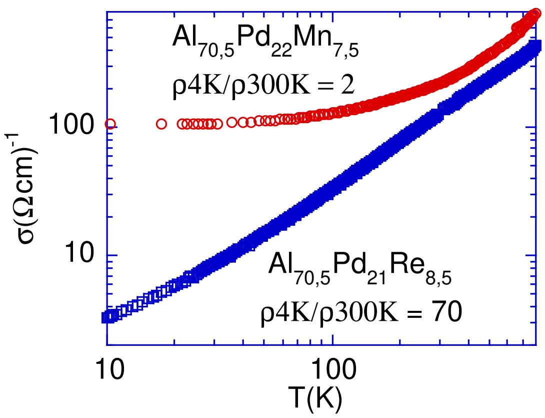

Fig.1 - Conductivity of QC's [6] -

The aim of this note is to provide a mathematical framework in which such qualitative behaviour can be found. We expect that such framework will prove suitable for a more quantitative description of transport properties of quasicrystals.

2. Mathematical Background

The framework we propose is valid within the one electron approximation. We will ignore explicitely the phonons and the electron-electron interactions as a first step. In this section only non dissipative transport properties are considered. These are dominated by interference effects due to Bragg reflections. Let us note that the framework applies to all kinds of aperiodic materials, including QC's, in any dimension d (in practice $d = 1, 2, 3$). Most of the results described in this section can be found in [9, 10]

More precisely, we assume that atomic sites are located on a discrete point set $\mathcal{R}$ contained in $\mathbb{R}^d$. Following Lagarias [11] we define the following hierarchy of properties:

(PS.1) $\mathcal{R}$ is *uniformly discrete* if there is $r > 0$, such any ball of radius r contains at most one point of $\mathcal{R}$. This means that there is a non zero minimum distance between points of $\mathcal{R}$.

(PS.2) $\mathcal{R}$ is a *Delone* (or Delaunay) *set* if, in addition, there is $R > 0$ such that any ball of radius R contains at least one points of $\mathcal{R}$.

(PS.3) $\mathcal{R}$ is *finitely generated* if the $\mathbb{Z}$-module generated by $\mathcal{R}$ in $\mathbb{R}^d$ is finitely generated. In this case points of $\mathcal{R}$ can be labelled by a finite number of integers.

(PS.4) $\mathcal{R}$ is a *Meyer set*, whenever both $\mathcal{R}$ and $\mathcal{R} - \mathcal{R}$ are Delone sets.

A point set in $\mathbb{R}^d$ randomly distributed with respect to a Poisson process, is discrete but not uniformly discrete with probability one. In practice, due to quantum mechanics, the equilibrium positions of atoms in any solid medium is a uniformly discrete set. Impurities in a semiconductor, distributed randomly, are located on a uniformly discrete set which is not a Delone set in general. This is also the case of zeolithes which may have empty holes of arbitrary size. However, most solids have their atoms on a Delone set. Random tilings built from a quasilattice, have their vertices on a finitely generated Delone sets. Quasiperiodic lattices are Meyer sets [12]. They may also be constructed either by inflation with matching rules or by cut-and-project method [5], by model sets [12] but also through covering clusters [13, 14, 15].

Given a uniformly discrete set $\mathcal{R}$ we associate the point measure

$$\nu^{(\mathcal{R})}(dx) = \sum_{y \in \mathcal{R}} \delta(x - y)$$

The space $\mathcal{M}(\mathbb{R}^d)$ of measures on $\mathbb{R}^d$ is endowed with the weak topology on the space $\mathcal{C}_c(\mathbb{R}^d)$ of continuous functions with compact support on $\mathbb{R}^d$. Then the family $\left(\nu^{(\mathcal{R}+a)}\right)_{a \in \mathbb{R}^d}$ of translated of $\nu^{(\mathcal{R})}$ has a compact closure Ω which is called the *Hull of* $\mathcal{R}$. The translation group $\mathbb{R}^d$ acts on Ω by homeomorphisms, so that $(\Omega, \mathbb{R}^d)$ defines a *topological dynamical system*. We will denote by τ^a the translation by $a \in \mathbb{R}^d$ acting on Ω. It turns out that given any $\omega \in \Omega$ one can find a uniformly discrete set $\mathcal{R}_\omega$ in $\mathbb{R}^d$ such that the measure ω coincides with $\nu^{(\mathcal{R}_\omega)}$. Moreover if $\mathcal{R}$ is a Delone set (respectively a finitely generated Delone set, a Meyer set) so are all the $\mathcal{R}_\omega$'s [10].

In practice, the point sets $\mathcal{R}_\omega$ can be obtained as limits of subsequences of translated of $\mathcal{R}$, where the limit means that given any finite hypercube Λ, the sequence looks eventually stationnary when restricted to Λ.

One can then see the Hull as a universal ideal object containing all possible finite size samples that an experimentalist could possibly produce. In QC's the Hull corresponds, as a set, to the acceptance zone.

We will then choose an $\mathbb{R}^d$-*invariant, ergodic probability measure* $\mathbf{P}$ on Ω. By the Birkhoff ergodic theorem we then have:

$$\int_{\omega \in \Omega} d\mathbf{P}(\omega) f(\omega) \;=\; \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{1}{|\Lambda \cap \mathcal{R}_{\omega_0}|} \sum_{x \in \Lambda \cap \mathcal{R}_{\omega_0}} f(x)$$

for $\mathbf{P}$-almost all ω's. Here Λ denotes hypercubes centered at the origin.

Such a probability measure can be seen as a precise procedure for taking space averages. In practical cases it is *given by the Gibbs measure describing the thermal equilibrium of atoms [16].*

For simplicity, we may assume that conduction electrons are spinless and can occupy only one atomic orbital per site. This simplification is not required for our framework to apply, but it is assumed here for pedagogical purposes. In this case, the physical Hilbert space of quantum states for such an electron can be taken as $\mathcal{H}_\omega = \ell^2(\mathcal{R}_\omega)$ namely the space of sequences $\psi = (\psi(x))_{x\in\mathcal{R}_\omega}$ whith $\|\psi\|^2 = \sum_{x\in\mathcal{R}_\omega} |\psi(x)|^2 < \infty$. The translation $T(a)\psi(x) = \psi(x-a)$ defines a unitary transformation between $\mathcal{H}_\omega$ and $\mathcal{H}_{\tau^a\omega}$.
A *covariant family of operators* is a map $\omega \in \Omega \mapsto A_\omega$ with A_ω being a bounded operator acting on $\mathcal{H}_\omega$, such that

$$T(a)A_\omega T(a)^{-1} \;=\; A_{\tau^a\omega}$$

for all $a \in \mathbb{R}^d$. Covariance means that matrix elements can be written as

$$\langle x|A_\omega|y\rangle \;=\; A(\tau^{-x}\omega; y-x) \qquad\qquad x, y \in \mathcal{R}_\omega$$

where A is a function of the pairs (ω, a) such that $\omega \in \Omega,\ a \in \mathbb{R}^d$ with $0 \in \mathcal{R}_\omega$and $0 \in \mathcal{R}_{\tau^{-a}\omega}$. The set of ω's with $0 \in \mathcal{R}_\omega$ is called the *canonical transversal* and will be denoted by Γ^0_Ω. It is a compact subset of Ω. The set of pairs (ω, a) with $\omega \in \Gamma^0_\Omega,\ a \in \mathbb{R}^d$ and $\tau^{-a}\omega \in \Gamma^0_\Omega$ is called the *groupoid of the transversal* and is denoted by Γ_Ω [17].
A covariant family of operators is *continuous* whenever the map $(\omega, a) \in \Gamma_\Omega \mapsto A(\omega, a) \in \mathbb{C}$ is continuous with compact support. An *observable* is any limit of sequences of continuous covariant families of operators under the norm $\|A\| = \sup_{\omega\in\Omega} \|A_\omega\|$. Endowed with the pointwise sum, adjoint and product, the set of observables is then a C^**-algebra* denoted by $\mathcal{A} = C^*(\Gamma_\Omega)$. The subset of continuous covariant families of operators will be denoted by $\mathcal{A}_0$ throughout this note.
The *trace per unit volume* $\mathcal{T}$ is defined for $A \in \mathcal{A}$ by

$$\begin{aligned}\mathcal{T}(A) \;&=\; \lim_{\Lambda\uparrow\mathbb{R}^d} \frac{1}{|\Lambda\cap\mathcal{R}_{\omega_0}|} \sum_{x\in\Lambda\cap\mathcal{R}_{\omega_0}} \langle x|A_\omega|x\rangle \\ &=\; \int_{\omega\in\Omega} d\mathbf{P}(\omega) A(\omega, 0)\end{aligned}$$

A differential is also defined by

$$\vec{\nabla} A(\omega, a) \;=\; \imath a A(\omega, a)$$

One remarks that if $\vec{X} = (X_1, \cdots, X_d)$ denotes the *position operator* acting on $\mathcal{H}_\omega$, then $\vec{\nabla} A_\omega = -\imath[\vec{X}, A_\omega]$.
The trace per unit volume and the differential are the non periodic analog of the integration and differentiation over the quasimomentum in the Brillouin zone [9].
In what follows, we will denote by H the Hamiltonian describing the one electron motion in the solid under consideration. Examples of such hamiltonian are given by tight binding operators of the form

$$H_\omega\psi(x) \;=\; \sum_{y\in\mathcal{R}_\omega} t(x-y)\psi(y) \qquad x \in \mathcal{R}_\omega,\ \ \psi \in \mathcal{H}_\omega$$

where t is a continuous function on $\mathbb{R}^d$ decreasing fast enough at infinity and with $t(-a) = \overline{t(a)}$ (to insure that H is selfadjoint). In all cases we will assume that $H \in \mathcal{A}$. The *current operator* $\vec{J}$ is obtained as the product of the charge q of the carriers by the velocity operator. Using the Heisenberg equations for the evolution of observables, it follows that $\vec{J}_\omega = q\, \imath[H_\omega, \vec{X}]$ so that $\vec{J} = q\, \vec{\nabla} H$ (units are such that $\hbar = 1$ here).

3. Spectral and Diffusion Exponents

The material contained in this section has been developped in full generality in [18]. We suppose $H = H^* \in \mathcal{A}$ is the one electron Hamiltonian. Its *local density of states* (LDoS), also called the *spectral measure* by mathematicians, is defined by

$$\int_{E_0}^{E_1} d\mu_{\omega,\phi} = \frac{1}{\pi}\Im m \int_{E_0}^{E_1} dE' \langle\phi|(E' - \imath 0^+ - H_\omega)^{-1}|\phi\rangle$$

for $\phi \in \mathcal{H}_\omega$. We define the local exponents by (for a more informations see [18] and references therein)

$$d^{\pm}_{\omega,\phi}(E) \;=\; \lim \left\{ \begin{array}{c} \sup \\ \inf \end{array} \right\}_{\varepsilon\downarrow 0} \frac{\ln \int_{E-\varepsilon}^{E+\varepsilon} d\mu_{\omega,\phi}}{\ln\varepsilon}$$

Given an energy interval Δ, we define $d^{\pm}_{\omega,\phi}(\Delta)$ by considering the supremum (infimum) of the $d^{\pm}_{\omega,\phi}(E)$ for $E \in \Delta$ modulo subsets of zero $\mu_{\omega,\phi}$-measure in Δ. Then we maximize (minimize) over $\phi \in \mathcal{H}_\omega$ to get the exponents $d^{\pm}_{\text{LDoS}}(\Delta)$ which happen to be independent of ω, $\mathbf{P}$-almost surely [18].
The exponents $d^{+}_{\text{LDoS}}(\Delta)$ *coincides also with the Hausdorff dimension of the* LDoS *in the energy interval* Δ .
The spectral properties of the LDoS can be rephrased as follows:
(SE.1) The LDoS is *absolutely continuous* in Δ whenever $d^{+}_{\text{LDoS}}(\Delta) = d^{-}_{\text{LDoS}}(\Delta) = 1$.
(SE.2) The LDoS is *pure point* in Δ, namely it is a sum of Dirac peaks, if $d^{+}_{\text{LDoS}}(\Delta) = d^{-}_{\text{LDoS}}(\Delta) = 0$.
(SE.3) If $0 < d^{\pm}_{\text{LDoS}}(\Delta) < 1$ the LDoS is *singular continuous.*
We also define the *fractal dimensions* by (here $p \in \mathbb{R}$)

$$D^{\pm}_{\Delta}(p) \;=\; \lim_{p'\to p} \frac{1}{p'-1} \lim \left\{ \begin{array}{c} \sup \\ \inf \end{array} \right\}_{\varepsilon\downarrow 0} \cdots$$

$$\cdots \frac{\ln\left(\int_{\omega\in\Omega} d\mathbf{P}(\omega) \int_\Delta d\mu_{\omega,\phi}(E) \left\{ \int_{E-\varepsilon}^{E+\varepsilon} d\mu_{\omega,\phi} \right\}^{p'-1} \right)}{\ln(\varepsilon)}$$

which does not depend upon the choice of ϕ.
We remark that if the LDoS is absolutely continuous on Δ, namely if for almost all $E \in \Delta$, there are $0 < c \leq C < \infty$ such that $c\varepsilon \leq \int_{E-\varepsilon}^{E+\varepsilon} d\mu_{\omega,\phi} \leq C\varepsilon$, then $D^{\pm}_{\Delta}(p) = 1$. On the other extreme, if the LDoS is pure point on Δ, we get $D^{\pm}_{\Delta}(p) = 0$. In general $D^{\pm}_{\Delta}(p)$ depends on p explicitely and $D^{-}_{\Delta}(p) < D^{+}_{\Delta}(p)$.

To define the *diffusion length* we consider the electron starting at some initial site $x \in \mathcal{R}_\omega$. One also forces it into some energy window $\Delta = [E_-, E_+]$, by setting P_Δ the corresponding eigenprojection of H and considering $P_{\Delta,\omega}|x\rangle$ as initial state instead. Then one measures the spreading of this wave function after time t, namely we consider the position operator after time t

$$\vec{X}_\omega(t) \;=\; e^{\imath t H_\omega} \vec{X} e^{-\imath t H_\omega}$$

so that $\langle x|P_{\Delta,\omega}|\vec{X}_\omega(t) - \vec{X}_\omega(0)|^p P_{\Delta,\omega}|x\rangle$ measures the spreading for $p \geq 1$. In order to avoid all kinds of fluctuations due to the choice of the initial point $x \in \mathcal{R}_\omega$ one averages over space and also over time by setting

$$L^{(p)}_{\Delta}(t)^p = \int_0^t \frac{ds}{t} \; \overline{\langle x|P_{\Delta,\omega}|\vec{X}_\omega(t) - \vec{X}_\omega(0)|^p P_{\Delta,\omega}|x\rangle}^{\text{space}}$$

As a matter of fact, using the definition of the trace per unit volume and of the differential, this *diffusion length* can also be written as

$$L^{(p)}_{\Delta}(t) \;=\; \left(\int_0^t \frac{ds}{t} \, \mathcal{T}\left(\, P_\Delta \, |\vec{\nabla} e^{\imath t H}|^p \right) \right)^{1/p}$$

In practice we mostly consider the case $p = 2$. The *diffusion exponents* $\beta_p(\Delta)$ are defined by

$$\beta^{\pm}_p(\Delta) \;=\; \lim \left\{ \begin{array}{c} \sup \\ \inf \end{array} \right\}_{t \uparrow \infty} \frac{\ln L^{(p)}_{\Delta}(t)}{\ln t}$$

We get then (β denotes either of $\beta^{\pm}_{p=2}(\Delta)$)

1. $\beta^{-}_p(\Delta) \leq \beta^{+}_p(\Delta)$
2. $p \leq p' \Rightarrow \beta^{\pm}_p(\Delta) \leq \beta^{\pm}_{p'}(\Delta)$
3. $\beta = 1$ corresponds to *ballistic motion.* This happens for example in perfect crystals.
4. $\beta = 0$ corresponds to *absence of diffusion.* This happens in particular whenever electrons are strongly localized.
5. $\beta = 1/2$ corresponds to *quantum diffusion.* Numerical computations show that this happens in weak localization regimes.
6. $\beta < 1$ corresponds to *subballistic regimes.* Such regimes have been observed in many models of motion on $1D, 2D$ quasicrystals [6].
7. $0 < \beta < 1/2$ corresponds to *subdiffusion.* Its is precisely expected to occur for $3D$ perfect quasicrystals within the one electron approximation.

Numerical simulation have shown examples of systems for which the diffusion exponent is not trivial. The Harper model [19] describing the motion of a 2D electron on a square lattice submitted to a uniform magnetic field, provides an example of quantum diffusion [20, 21]. The Fibonnacci chain [22, 23, 24], the kicked Harper model [25, 26, 27], the octagonal lattice [28] also give rise to non trivial diffusion exponents.

During the last ten years, a certain number of inequalities have been proved. The first one is the so called **Guarneri's inequality** [29, 30, 31]:

$$\beta^{\pm}_p(\Delta) \;\geq\; \frac{d^{\pm}_{\text{LDoS}}(\Delta)}{d}$$

where d is the dimension of the space in which the lattice lies. More recently this inequality has been improved in various ways. In the case of 1D chains, with nearest neighbour interactions only (the Hamiltonian then is a Jacobi matrix), a better lower bound was obtained on the ground of analytic arguments [32] or numerical computations [33, 34] and later proved rigorously [35, 36]. More recently this bound was extended to a general Hamiltonian [37] to produce the **BGT inequality**:

$$\beta^{\pm}_p(\Delta) \;\geq\; \frac{1}{d} \, D^{\pm}_{\Delta}(\frac{d}{d+p})$$

Can an absolutely continuous LDoS coexist with subdiffusive motion ?

This question may actually be relevant for systems with $d \geq 3$ as we will show below. Absolute continuity means that the LDoS is regular enough. In particular $d^{\pm}_{\text{LDoS}}(\Delta) = D^{\pm}_{\Delta}(p) = 1$. Thanks to Guarneri's and BGT's inequalities, this gives

- In 1D, absolute continuity of the LDoS implies ballistic motion.
- In 2D, absolute continuity of the LDoS can coexist with subballistic motion but not with subdiffusive one.
- For $d \geq 3$ subdiffusive motion is permitted provided $\beta \geq 1/d$.

Examples of such motion were provided in the literature. A class of models on an infinite dimensional hypercube [38], exhibits smooth LDoS and logarithmically slow diffusion, namely $L^{(p)}_{\Delta}(t) \sim \ln(t)^{1/2}$. Other models [39] on the basis of the *Labyrinth* proposed by C. Sire [40], exhibit absolutely continuous LDoS for $d \geq 3$ and subdiffusion, some of them having $\beta \leq .3343$.

4. Conductivity and Electronic Transport

In this section we consider now the possibility of dissipative mechanisms on top of the electronic motion in the lattice of atoms. Dissipation may occur from electron-phonon collisions (namely from the atomic motion coupled to electrons), from electron-electron interactions, or from any other source of coupling with the outside world. The main effect of dissipative mechanisms is the loss of phase coherence of the electronic wave packet, after a typical time $\tau_{\rm rel}$ which depends usually upon the temperature as

$$\tau_{\rm rel} \uparrow \infty \qquad \text{as} \qquad T \downarrow 0$$

Using an N-body approach one can even estimate this dependence. If the dissipation is due to phonons, we usually get [41]

$$\begin{aligned} \tau_{\rm rel}(T) &\sim T^{-1} \quad \text{for } T \text{ large} \\ \tau_{\rm rel}(T) &\sim T^{-5} \quad \text{for } T \text{ small } (\textbf{Bloch's law}) \end{aligned}$$

In [42, 43] a kinetic theory for a description of transport has been proposed. It is based upon the old idea of Drude [44].
In Drude's model, the electron is a classical free particle of mass m and charge q evolving in $\mathbb{R}^d$. It collides at Poissonian random times with colliders, so that its momentum after the collision becomes a random variable independent of the momentum just before the collision. Choosing its distribution according to the Maxwell-Boltzmann distribution with temperature T leads to Ohm's law for the conductivity and to Joule's laws for the electric power.

Our proposal was to mimick this model by assuming that the particle motion is quantized with a Hamiltonian H as in previous sections. Then collisions are represented by *quantum jumps* [43] from one eigenstate $|x>$ of H to another one $|y>$, occuring at random Poissonian times with parameter $\Gamma_{\omega;x\to y}$, namely the relaxation time between two such jumps is $\tau_{x\to y} = \Gamma^{-1}_{\omega;x\to y}$. If the state of the system is represented by the density matrix ρ just before the jump it will become $\kappa_{x\to y}(\rho)$ just after. Here $\kappa_{x\to y}$ is the *jump operator*: it should transform a density matrix into another one. The time evolution of the state is then given by the Hamiltonian H between jumps and by the jump operator at each jump. More details about such class of models will be published soon [43]. If we switch on an external force, such as a DC or AC uniform electric field $\vec{\mathcal{E}}$ at frequency $\tilde{\omega}$, the evolution of an observable A will be randomized by the collisions processes.
The *coherent part* of the current (controled by quantum interferences) is represented by the operator $\vec{J} = q\,d\vec{X}/dt$. In addition there is a *dissipative part* of the current produced by the quantum jump processes (e.g. phonon drag). The electric current observed in experiments however is an average of the current, obtained by averaging over the initial state, over the collisions, and over the time at the frequency of the external field. Since there are many electrons in the system, even if one considers them as independent, they behave like a Fermi gas, so that their equilibrium distribution is rather given by a *density matrix* according to the Fermi-Dirac statistics

$$\rho_{\beta,\mu} \;=\; (1+e^{\beta(H-\mu)})^{-1}$$

It then belongs to the observable algebra $\mathcal{A}$. Here $\beta = 1/k_B T$ is the inverse temperature, whereas μ is the chemical potential fixed by the requirement that

$$\mathcal{T}(\rho_{\beta,\mu}) \;=\; n_{\rm el}$$

where $n_{\rm el}$ is the electron density in the solid. Putting all contributions together, the averaged current $\vec{j}(\tilde{\omega})$ satisfies:

$$\vec{j}(\tilde{\omega}) = \sigma(\tilde{\omega})\vec{\mathcal{E}}(\tilde{\omega}) + O(\vec{\mathcal{E}}^2)) \qquad \text{(linear response)}$$

where $\sigma(\tilde{\omega})$ is the $d \times d$ *conductivity tensor*. One can prove rigorously, within this framework, that the error term is indeed $O(\vec{\mathcal{E}}^2)$. The formula that one gets for the conductivity is quite involved in general [43]. Within a good approximation one can reduce it to:

$$\sigma(\tilde{\omega})_{i,j} \;=\; \frac{q^2}{\hbar}\mathcal{T}\left(\partial_j \rho_{\beta,\mu}\, \frac{1}{\mathcal{C} + \mathcal{L}_H - \imath\tilde{\omega}}\, \partial_i H\right)$$

where $\mathcal{L}_H$ is the *Liouville operator* acting on $\mathcal{A}$ by $\mathcal{L}_H(A) = \imath[H,A]/\hbar$, whereas $\mathcal{C}$ is the *collision operator* acting on observables as

$$\mathcal{C}(A)_\omega \;=\; \sum_{x\to y} \Gamma_{\omega;x\to y}(A_\omega - \kappa^*_{x\to y}(A)_\omega)$$

where $\kappa^*_{x\to y}$ is the dual of the jump operator acting on observables instead of on states.
The *Relaxation Time Approximation* (RTA) consists in approximating the collision operator $\mathcal{C}$ by a multiple of the identity

$$\mathcal{C} \approx \frac{id}{\tau_{\rm rel}} \qquad \textbf{(RTA)}$$

where $\tau_{\rm rel}$ is a typical relaxation time for dissipative processes. It corresponds to the *coherence time* or to the *inelastic relaxation time* found in the literature. The RTA is usually valid at high enough temperature (typically at $T \geq 10-50K$ in many materials). One of the main results concerning the behaviour of the conductivity as $\tau_{\rm rel} \uparrow \infty$ (namely as $T \downarrow 0$) is the *anomalous Drude formula* proved in [42] on the basis of an argument initially due to D. Mayou [6] and completed by C. Sire [45]: if $\sigma_{||}$ denotes the parallel conductivity (i.e. $i = j$) at $\tilde{\omega} = 0$, one has

$$\sigma_{||} \overset{\tau_{\rm rel}\uparrow\infty}{\sim} \tau_{\rm rel}^{2\beta_{\rm F}-1} \qquad \textit{(anomalous Drude formula)} \qquad (1)$$

where $\beta_{\rm F} = \lim_{\varepsilon\downarrow 0} \beta_{p=2}(E_{\rm F}-\varepsilon, E_{\rm F}+\varepsilon)$ and $E_{\rm F}$ is the Fermi energy. This leads to the following discussion:

1. For ballistic motion, $\beta_{\rm F} = 1$, one recovers the usual Drude formula $\sigma_{||} \stackrel{\tau_{\rm rel}\uparrow\infty}{\sim} \tau_{\rm rel}$.

2. For absence of diffusion $\beta_{\rm F} = 0$, one gets an *anti-Drude* behaviour $\sigma_{||} \stackrel{\tau_{\rm rel}\uparrow\infty}{\sim} 1/\tau_{\rm rel}$. It is usual unobservable because at low temperature the RTA fails so that one sees all kinds of more complex effects due to the structure of the collision operator $\mathcal{C}$.

3. For a quantum diffusion $\beta_{\rm F} = 1/2$, the conductivity converges to a *residual* positive value that is usually observed in any weakly disordered system at low temperature.

4. For overdiffusive motion, where $1/2 < \beta_{\rm F} < 1$ the conductivity eventually diverges leading to a metallic behaviour.

5. For subdiffusive motion, where $0 < \beta_{\rm F} < 1/2$, the conductivity eventually converges to zero leading to an insulating behaviour. This is what is expected to arise in QC's.

6. If the Hamiltonian H contains two contributions corresponding to differents values of $\beta_{\rm F}$, the one with the *highest diffusion exponent dominates at low temperature.* Thus if a weak disorder is added to a subdiffusive system, there will be a residual conductivity a small temperature.

5. Application to QC's

The results found in the previous section are good candidates to explain the scaling behaviour of the conductivity in term of the temperature. In 1998, Fujiwara and Roche proposed using eq. (1) together with LMTO calculations on *AlCuCo* predicting $\beta_{\rm F} \approx .375$, to deduce a scaling law in terms of the temperature. To do so, they assumed that the Bloch law $\tau_{\rm rel} \propto T^{-5}$ is valid on a large range of temperature. If one accepts this as starting hypothesis, one obtains

$$\beta_{\rm F} \approx .375 \qquad \& \qquad \tau_{\rm rel} \propto T^{-5}$$
$$\Downarrow$$
$$\sigma_{||} \stackrel{T\downarrow 0}{\sim} T^{5(1-2\beta_{\rm F})} = T^{1.25}$$

a result compatible with experiments. This is the first known mechanism liable to lead to such a behaviour.
In QC's however such a scaling law is observed in the full range of temperature $300K \leq T \leq 1000K$. Even if one is ready to believe that electron-phonons coupling is the main source of dissipation, it raises the question of why Bloch's T^{-5}-law should hold at such high temperatures. If instead we accept the T^{-1}-law that is usually found at high temperatures, the previous argument fails to explain the experiments.
Nevertheless, this result may be of interest in the case of *AlPdRe* for which this law is followed down to $T \approx 10K$ (cf Fig.1). In this range of temperatures the Bloch law is likely to dominate. This is interesting because in metal, Bloch's law is hidden behind the Fermi liquid behaviour (see Section 1.). One explanation could be that in QC's the electron-electrons interactions should not lead to a Fermi liquid theory because the Fermi surface splits into an infinite number of tiny pockets due to the quasi periodicity of the lattice. If this argument holds, then the T^{-2}-behaviour of the conductivity that is observed in metal, should disappear in QC's. But there is another argument that may be used: even if the Fermi liquid theory holds, the usual Matthiessen rule $1/\tau = 1/\tau_1 + 1/\tau_2$ giving the relaxation time whenever two processes coexist, is replaced in QC's by an inverse Matthiessen rule [6] of the form $\tau^\alpha = \tau_1^\alpha + \tau_2^\alpha$ (with $\alpha = 1 - 2\beta_{\rm F}$) thanks to eq. (1). In such a case, the Bloch term $T^{5/4}$ dominates over the Fermi one proportional to $T^{1/2}$ here.
The low temperature behaviour for most QC's exhibits a residual conductivity, as can be seen in Fig.1. We have seen that this is what happens whenever $\beta_{\rm F} = 1/2$. Therefore in the light of the previous argument we may wonder why such a diffusion exponent should occur in this range of temperatures.
There are two mechanisms liable to lead to such a result. First of all, some weak disorder may occur in the solid. If it is weak it should become negligible at higher temperatures, leaving the previous scaling law dominant. But at smaller temperature the disorder cannot be neglected anymore, so that, by virtue of the point (6)at the end of Section 4. above, it will dominate the conductivity behaviour because the corresponding diffusion exponent is the highest one. The realistic QC's are indeed likely to exhibit some disorder. Most images in High Resolution Transmission Electron Microscopy (HRTEM) indicate that such disorder should be there. Moreover, the tunneling junction measurement of the DoS near the Fermi level [46], show that the pseudo gap at the Fermi level is partially filled with states which are likely to be related to disorder. Still, one does not know yet what is the origin of that disorder: random tiling, chemical or structural disorder ?
There may be another more subtle explanation however. Even if we assume that the atomic sites are sitting on a perfect quasilattice, we may have a low temperature diffusion exponent equal to 1/2 at the Fermi level. This is because the level spacing distribution should follow the prediction of the Random Matrix Theory GOE statistics. This has been observed in several numerical simulations for a tight-binding model on the Ammann-Beenker tiling (TBAB) of the plane [47, 48, 49, 50]. GOE statistics are usually associated to a weak localization regime in disordered systems, such as the Anderson model. The TBAB model however does not exhibit either weak or strong localization, as shown in [50], since the inverse participa-

tion ratio scales in a non classical way in term of the cluster size investigated. However in 3D one expects a *smoothening of the* LDoS [39] together with a GOE statistics at small energy scale. More generally, on the basis of [39], we **conjecture that for $d \geq 3$, any homogeneous Schrödinger operator at high enough energy or at low enough potential should have an absolutely continous spectral measure with level repulsion statistics**. Therefore such a mechanism entitles us to expect that $\beta_{\rm F} = 1/2$ in a perfect 3D QC. In addition, this conjecture may be violated if the effective Hamiltonian near the Fermi level is not a small perturbation of the free one. That may be a possible mechanism liable to separate *AlPdRe* from other QC's, form the point of view of the low temperature conductivity [52]. However if this last mechanism is competing with some disorder, there is a need to know at which energy scale it can actually be observed [52]. Because then we could not understand why a residual conductivity does not appear in *AlPdRe*. On the contrary, at low enough temperature, several experiments show some Mott variable range hopping [7, 8]. Together with the tunneling junction experiments, showing a deeper pseudogap for *AlPdRe* than for other QC's (see [46]), this is an indication that there is a low amount of "impurities", creating a low density of states at the Fermi level in the pseudo gap. Thus these states should be far enough from each other to be localized. Since the conducting electrons have energy close to $E_{\rm F}$ at low temperature, they should be trapped in localized states, leading to conduction by variable range hopping.

6. Conclusions

To summarize, we have shown that

(C.1) The formalism of the *Non Commutative Brillouin Zone* gives a mathematical framework allowing to describe anomalous coherent transport, thanks to spectral and diffusion exponents.

(C.2) In dimensions $d \geq 3$, we expect the quantum spectral measure (LDoS) to be absolutely continuous, at least in the perturbative regime, with subdiffusive behaviour.

(C.3) Using a kinetic model for describing dissipation processes, one gets a Kubo formula and one can derive an *anomalous Drude formula*.

(C.4) The anomalous Drude formula is likely to explain the scaling laws observed in quasicrystals for the conductivity in term of the temperature. However problems remain to understand the Bloch law for the electron-phonon relaxation time at high temperature.

(C.5) A pseudo gap near the Fermi level occurs in the energy scale of $50 meV$ [46], with DOS vanishing like $\sqrt{|E - E_F|}$. It is probably due to the residual electron-electron Coulomb interaction (the Altshuler-Aronov mechanism [51]). As for semiconductors, this implies that the low temperature conductivity may be dominated by "impurities". Then one can interpret the weak localization effect by a larger density of impurities than in the case of the *AlPdRe* which should then be in the localized regime. This could explain why the scaling laws are seen down to very low temperatures and should also explain why Mott's variable range hopping is observed in such compounds. However, the very low value of the Mott temperature T_0 (sometimes as low as $1\, mK$ [8]), raises the question of the validity of Mott's argument in such cases.

(C.6) The occurence of level repulsion observed in numerical computations may be an alternative explanation: if this regime dominates then a residual conductivity should occur. Otherwise the quantum interferences due to Bragg reflexions should dominate. That could be the source of the difference between *AlPdRe* and other QC's [52].

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