Non-Commutative Geometry and Quantum Hall Effect *

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Abstract

A mathematical framework based on Non-Commutative Geometry is proposed to described the Integer Quantum Hall Effect. It takes localization effects into account. It permits to prove rigorously that the Hall conductivity is quantized and that plateaus occur when the Fermi energy varies in a region of localized states.

1 Introduction

In 1880, E.H. Hall [14] undertook the classical experiment which led to the so-called Hall effect. A century later, von Klitzing and his co-workers [17] showed that the Hall conductivity was quantized at very low temperatures as an integer multiple of the universal constant e^2/h. Here e is the electron charge whereas h is Planck's constant. This is the Integer Quantum Hall Effect (IQHE). This discovery led to a new accurate measurement of the fine structure constant and a new definition of the standard of resistance [21].

On the other hand, during the seventies, A. Connes [8, 10] extended most of the tools of differential geometry to non-commutative C^*-algebras, thus creating a new branch of mathematics called *Non-Commutative Geometry*. The main new result obtained in this field was the definition of cyclic cohomology and the proof of an index theorem for elliptic operators on a foliated manifold. He recently extended this theory to what is now called *Quantum Calculus* [11].

After the works by Laughlin [19] and especially by Kohmoto, den Nijs, Nightingale and Thouless [23] (called TKN_2 below), it became clear that the quantization of the Hall conductance at low temperature had a geometric origin.

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The universality of this effect had then an explanation. Moreover, as proposed by Prange [20, 16], Thouless [22] and Halperin [15], the Hall conductance plateaus, appearing while changing the magnetic field or the charge-carrier density, are due to localization. Neither the original Laughlin paper nor the TKN_2 one however could give a description of both properties in the same model. Developing a mathematical framework able to reconcile topological and localization properties at once was a challenging problem. Attempts were made by Avron et al. [2] who exhibited quantization but were not able to prove that these quantum numbers were insensitive to disorder. In 1986, H. Kunz [18] went further on and managed to prove this for disorder small enough to avoid filling the gaps between Landau levels.

But in [3, 5, 4], we proposed to use Non-Commutative Geometry to extend the TKN_2 argument to the case of arbitrary magnetic field and disordered crystal. It turned out that the condition under which plateaus occur was precisely the finiteness of the localization length near the Fermi level. This work was rephrased later on by Avron et al. [1] in terms of charge transport and relative index, filling the remaining gap between experimental observations, theoretical intuition and mathematical frame.

It is our aim in this talk to describe the main steps of this construction. The reader interested by details of the physical phenomena or of the mathematical proofs is kindly invited to look into the recent work [7].

Acknowledgements. This work has benefited from many contacts during the last ten years. It is almost impossible to give the list of all colleagues who contributed to these discussions. I want to address my special thanks to Alain Connes for his outstanding contribution, continuous support and warm encouragements. Also I want to acknowledge Y. Avron, R. Seiler, B. Simon, have been for their constant interest in shading light on this difficult subject. Let me thank my recent and young collaborators Hermann Schulz-Baldes and Andreas van Elst, whose enthusiasm and competence permitted to write the main report which is the basis of this work.

2 IQHE: experiments and theories

Let us consider a very flat conductor, considered as two-dimensional, placed in a constant uniform magnetic field B in the z direction perpendicular to the plane Oxy of the plate (see Fig.1). If we force a constant current $\vec{j}$ in the x direction, the electron fluid will be submitted to the Lorentz force perpendicular to the current and the magnetic field creating an electric field $\vec{\mathcal{E}}$ along the y axis. In a stationary state, writing that the total force acting on the charge vanishes leads to the relation $\vec{j} = \boldsymbol{\sigma}\vec{\mathcal{E}}$ with a 2×2 antidiagonal antisymmetric matrix with matrix element $\pm\sigma_H$ given by

$$\sigma_H = \nu \frac{e^2}{h} \ , \qquad \nu = \frac{nh}{eB} \ ,$$

where n is the two-dimensional density of charge carriers, h is Planck's constant,

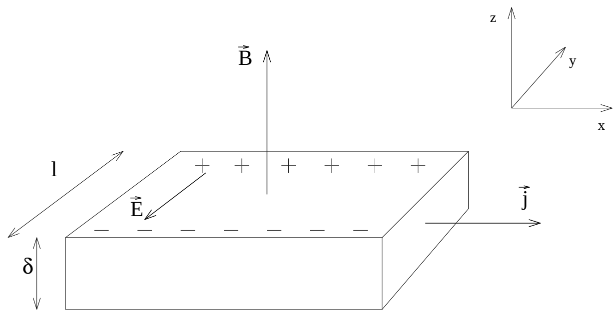

Figure 1: *The classical Hall effect: the sample is a thin metallic plate of width δ. The magnetic field is uniform and perpendicular to the plate. The current density $\vec{j}$ parallel to the x-axis is stationary. The magnetic field pushes the charges as indicated creating the electric field $\vec{\mathcal{E}}$ along the y direction. The Hall voltage is measured between opposite sides along the y-axis*

e is the electron charge and ν is called the *filling factor*. We remark that the sign of σ_H depends upon the sign of the carrier charge. In particular, the orientation of the Hall field will change when passing from electrons to holes. This observation is commonly used nowadays to determine which kind of particles carries the current. The quantity $R_H = h/e^2$ is called the Hall resistance. It is a universal constant with value $R_H = 25812.80\Omega$. R_H can be measured directly with an accuracy better than 10^{-8} in QHE experiments. Since January 1990, this is the new standard of resistance at the national bureau of standards [21].
Lowering the temperature below $1K$ leads to the observation of plateaus for integer values of the Hall conductance (see Fig.2). The accuracy of the Hall conductance on the plateaus is better than 10^{-8}. For values of the filling factor corresponding to the plateaus, the direct conductivity $\sigma_{//}$, namely the conductivity along the current density axis, vanishes: the sample becomes insulating. To summarize:

(i) At very low temperature, in the limit of large sample size, and provided the system can be considered as two-dimensional, Hall plateaus appear at integer values of the Hall conductance in unit of the inverse Hall resistance.

(ii) On plateaus the sample is an insulator. This is due to disorder in the sample which produce the localization of charge carriers wave functions.

(iii) for the Hall plateaus with large index (namely indices ≥ 2) one can ignore the Coulomb interaction between charge carriers without too much error.

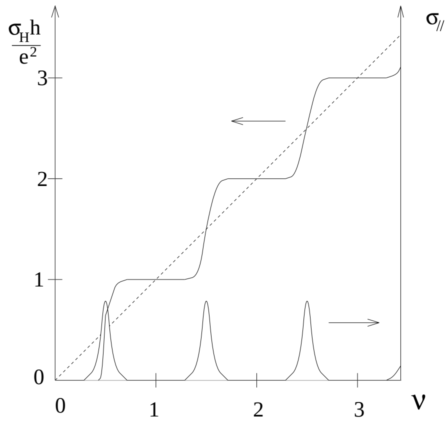

Figure 2: *Schematic representation of the experimental observations in the IQHE. The Hall conductivity σ_H is drawn in units of e^2/h versus filling factor ν. The dashed line shows the Hall conductivity of the Landau Hamiltonian without disorder. The direct conductivity $\sigma_{//}$ is shown in arbitrary units.*

3 The Kubo-Chern formula

Since we can ignore Coulomb interactions between particles, the fermion fluid made of the charge carriers is entirely described by the one-particle theory. The quantum motion can be derived from the data of a self adjoint operator called the Hamiltonian of the system. A typical example of one-particle Hamiltonian involved in the QHE for spinless particles, is given by

$$H_\omega = \frac{(\vec{P} + e\vec{A})^2}{2m_*} + V_\omega(\vec{x}) \ , \tag{1}$$

where where $\vec{P}$ is the 2D momentum operator and m_* is the effective mass of the particle, $\vec{A} = (A_1, A_2)$ is the vector potential given by the magnetic field while $V_\omega(\vec{x})$ describes the potential created by disorder in the plate. Here ω, which denotes the configuration of disorder, can be seen as a point in a compact metrizable Hausdorff space Ω on which the translation group $\mathbf{R}^2$ acts by homeomorphisms. Then the covariance condition $V_\omega(\vec{x} - \vec{a}) = V_{T^{\vec{a}}\omega}(\vec{x})$ expresses that moving the sample or changing the reference axis backward are equivalent.

Such a model is typical but may be replaced by others, such as lattice approximants, or particle with spin. In any cases, the one-particle Hamiltonian describing the fermion fluid satisfies the following general properties:

(i) The translation group $\mathcal{G}$ acting on the sample is $\mathbf{R}^2$ or $\mathbf{Z}^2$. It acts by homeomorphism $T^a, a \in \mathcal{G}$ on the space Ω of the disorder configurations. It also acts by unitary projective representation $T(a), a \in \mathcal{G}$, on the one-particle Hilbert space $\mathcal{H}$.

(ii) The one-particle Hamiltonian is a norm resolvent strongly continuous family $(H_\omega)_{\omega\in\Omega}$ of selfadjoint operators on $\mathcal{H}$, bounded from below and satisfying the covariance condition $T(a)H_\omega T(a)^{-1} = H_{T^a\omega}$.

Such Hamiltonian actually generalizes the case of a periodic operator, namely the case for which there is a sufficiently large discrete subgroup of $\mathcal{G}$ leaving the Hamiltonian invariant. In this latter case, the Bloch theorem permits to describe the quantum motion in term of quasi-momenta k belonging to the so-called *Brillouin zone*, which is a manifold diffeomorphic to a torus. Both magnetic field and disorder break this translation symmetry in a non trivial way, so that the notion of Brillouin becomes meaningless in the classical sense. Actually it is still possible to describe such a manifold in term of Non Commutative Geometry, by replacing the algebra of continuous functions over the Brillouin zone by a Non Commutative C^*-algebra. In our case this C^*-algebra is nothing but the one generated by the bounded functions of the H_ω's, $\omega \in \Omega$. It turns out that it is a closed subalgebra of the twist crossed product $\mathcal{A} = C^*(\Omega, \mathcal{G}, B) = \mathcal{C}(\Omega) \times_B \mathcal{G}$ [6] where the product is twisted by a module defined by the magnetic field. A differential and integral calculus exists on such an algebra making it a non commutative differential manifold that we have proposed to called the *Non Commutative Brillouin zone.* More precisely in our two dimensional situation, we can define two derivations ∂_i $(i = 1, 2)$ through using the position operators $X_i, i = 1, 2$ as

$$(\partial_i A)_\omega = \imath[X_i, A_\omega] \ , \qquad\qquad A \in \mathcal{A} \ .$$

Thus $\mathcal{C}^1$ elements of $\mathcal{A}$ are well defined.

The integral depends upon the choice of a $\mathcal{G}$-invariant ergodic probability $\mathbf{P}$ on Ω. It is then given by the *trace per unit area* $\mathcal{T}_P$ namely

$$\mathcal{T}_P(A) = \lim_{\Lambda\uparrow\mathcal{G}} \frac{1}{|\Lambda|} \mathrm{Tr}_\Lambda(A_\omega) \ , \qquad\qquad A \in \mathcal{A} \ , \text{for } \mathbf{P}\text{-almost all } \omega\text{'s} \tag{2}$$

where Λ denotes a sequence of squares in $\mathcal{G}$ centered at the origin and covering $\mathcal{G}$ and Tr_Λ is the restriction to Λ of the usual trace.

Due to the Fermi statistics obeyed by the charge carriers (electrons or holes), two different particles of the fluid must occupy different quantum eigenstates of the Hamiltonian H_ω. In the limit of zero temperature they occupy the levels of lowest energy, namely all eigenstates with energy lower than some maximal one E_F called the *Fermi level.* We will denote by $P_{F,\omega}$ the corresponding eigenprojection of the Hamiltonian.

Standard results in transport theory permit to compute the conductivity in term of the linear response of the fermion fluid under the influence of an external field. This is the famous *Green-Kubo* formula. In the QHE-limit, namely in the limit of (i) zero temperature, (ii) infinite sample size, (iii) negligible collision processes,

(iv) vanishingly small electric fields, the direct conductivity either vanishes or is infinite, whereas the transverse conductivity, when defined, is given by

$$\sigma_H = \frac{e^2}{h}\mathbf{Ch}(P_F) = \frac{e^2}{h} 2\imath\pi \ \mathcal{T}_{\mathbf{P}}(P_F[\partial_1 P_F, \partial_2 P_F]) \ . \tag{3}$$

It turns out that **Ch** is nothing but the non commutative analog of a Chern character. Thus Kubo's formula gives rise to a Chern character in the QHE limit. This is why we propose to call eq (3) the *Kubo-Chern formula*, associating Japan with China.

The main properties of the non commutative Chern character are the following

(i) *homotopy invariance*: given two equivalent $\mathcal{C}^1$ projections P and Q in $\mathcal{A}$, namely such that there is $U \in \mathcal{C}^1(\mathcal{A})$ with $P = U^*U$ and $Q = UU^*$, then $\mathbf{Ch}(P) = \mathbf{Ch}(Q)$. This is actually what happens if P and Q are homotopic in $\mathcal{C}^1(\mathcal{A})$.

(ii) *additivity*: given two $\mathcal{C}^1$ orthogonal projections P and Q in $\mathcal{A}$, namely such that $PQ = QP = 0$ then $\mathbf{Ch}(P \oplus Q) = \mathbf{Ch}(P) + \mathbf{Ch}(Q)$.

In particular, the homotopy invariance shows that $\mathbf{Ch}(P_F)$, when it is defined, is a topological quantum number. One of the main results of Non Commutative Geometry is that this Chern character is an integer provide it is well defined. Thus thank to eq. (3) we get the Hall conductance quantization. We will see in Section 5 below that this Chern character is well defined precisely whenever the Fermi level lies in a region of localized states. Moreover changing the value of the filling factor produces the moving of the Fermi level, which does not change the Chern character as long as the localization length stay bounded.

4 The four traces way

In this section we use four different traces that are technically needed to express the complete results of this theory. The first one is the usual trace on matrices or on trace-class operators. The second one, introduced in the Section 3 above, is the *trace per unit volume*. The third one is the *graded trace* or *supertrace* introduced in this Section below. This is the first technical tool proposed by A. Connes [8] to define the cyclic cohomology and constitutes the first important step in proving quantization of the Hall conductance [4]. The last one is the *Dixmier trace* defined by Dixmier in 1964 [12] and of which the importance for Quantum Differential Calculus was emphasized by A. Connes [9, 10, 11]. It will be used in connection with Anderson's localization.

Let $\mathcal{H}$ be the physical one-particle Hilbert space of Section 3. We then built the new Hilbert space $\hat{\mathcal{H}} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with $\mathcal{H}_\pm = \mathcal{H}$. The grading operator $\hat{G}$ and the "Hilbert transform" F are defined as follows:

$$\hat{G} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad F = \begin{pmatrix} 0 & \frac{X}{|X|} \\ \frac{\overline{X}}{|X|} & 0 \end{pmatrix} , \tag{4}$$

where $X = X_1 + \imath X_2$ (here the dimension is $D = 2$). It is clear that F is selfadjoint and satisfies $F^2 = \mathbf{1}$. An operator T on $\hat{\mathcal{H}}$ will said to be of degree 0 if it commutes with $\hat{G}$ and of degree 1 if it anticommutes with $\hat{G}$. The graded commutator (or supercommutator) of two operators and the graded differential dT are defined by

$$[T, T']_S = TT' - (-)^{deg(T)deg(T')} T'T , \qquad dT = [F, T]_S .$$

Then, $d^2T = 0$. The graded trace Tr_S (or supertrace) is defined by

$$\mathrm{Tr}_S(T) = \frac{1}{2}\mathrm{Tr}_{\hat{\mathcal{H}}}(\hat{G}F[F, T]_S) = \mathrm{Tr}_{\mathcal{H}}(T_{++} - uT_{--}\overline{u}) , \tag{5}$$

where $u = X/|X|$ and T_{++} and T_{--} are the diagonal components of T with respect to the decomposition of $\hat{\mathcal{H}}$. It is a linear map on the algebra of operators such that $\mathrm{Tr}_S(TT') = \mathrm{Tr}_S(T'T)$. However, this trace is not positive. Observables in $\mathcal{A}$ will become operators of degree 0, namely $A \in \mathcal{A}$ will be represented by $\hat{A}_\omega = A_\omega \oplus A_\omega$.

Given a Hilbert space $\mathcal{H}$, the characteristic values $\mu_1, \ldots, \mu_n, \ldots$ of a compact operator T are the eigenvalues of $|T| = (TT^*)^{1/2}$ labeled in the decreasing order. The Mačaev ideals $\mathcal{L}^{p+}(\mathcal{H})$is the set of compact operators on $\mathcal{H}$ with characteritic values satisfying

$$\|T\|_{p+} = \sup_{N\to\infty} \frac{1}{\ln N} \sum_{n=1}^{N} \mu_n^p \; < \; \infty \; .$$

Let Lim be a positive linear functional on the space of bounded sequences $l^\infty_+(\mathbf{N})$ of positive real numbers which is translation and scale invariant. For $T \in \mathcal{L}^{1+}(\mathcal{H})$ its Dixmier trace is defined by

$$\mathrm{Tr}_{\mathrm{Dix}}(T) = \mathrm{Lim}(\frac{1}{\ln N} \sum_{n=1}^{N} \mu_n) \; .$$

Remark that $T \in \mathcal{L}^{1+}$ if and only if $\mathrm{Tr}_{\mathrm{Dix}}(|T|) < \infty$. Moreover, if the sequence $(\frac{1}{\ln N}\sum_{n=1}^{N} \mu_n)$ converges, then all functionals Lim of the sequence are equal to the limit and the Dixmier trace is given by this limit. From this definition, one can show that $\mathrm{Tr}_{\mathrm{Dix}}$ is a trace [12, 11].

The first important result is provided by a formula that was suggested by a result of A. Connes [9]. Namely if $A \in \mathcal{C}^1(\mathcal{A})$ and if $\vec{\nabla} = (\partial_1, \partial_2)$ we have [7]:

$$\mathcal{T}_{\mathbf{P}}(|\vec{\nabla} A|^2) = \frac{1}{\pi}\mathrm{Tr}_{\mathrm{Dix}}(|dA_\omega|^2) , \qquad \text{for } \mathbf{P}\text{-almost all } \omega \; . \tag{6}$$

Let now $\mathcal{S}$ denote the closure of $\mathcal{C}^1(\mathcal{A})$ under the non commutative *Sobolev norm* $\|A\|^2_{\mathcal{S}} = \mathcal{T}_{\mathbf{P}}(A^*A) + \mathcal{T}_{\mathbf{P}}(\vec{\nabla} A^* \vec{\nabla} A)$. The eq. (6) shows that for any element $A \in \mathcal{S}$, dA_ω belongs to $\mathcal{L}^{2+}(\hat{\mathcal{H}})$ $\mathbf{P}$-almost surely.

The following formula, valid for $A_0, A_1, A_2 \in \mathcal{C}^1(\mathcal{A})$, is the next important result proved in [8, 4, 1, 7]:

$$\int_\Omega d\mathbf{P}(\omega)\mathrm{Tr}_S(\hat{A}_{0,\omega}d\hat{A}_{1,\omega}d\hat{A}_{2,\omega}) = 2\imath\pi\mathcal{T}_{\mathbf{P}}(A_0\partial_1 A_1\partial_2 A_2 - A_0\partial_2 A_1\partial_1 A_2) \ . \qquad (7)$$

Thanks to eq. (6) this formula extends to $A_i \in \mathcal{S}$.

Applying these formulæ to the Fermi projection, the Chern character $\mathbf{Ch}(P_F)$ is well defined provide $P_F \in \mathcal{S}$ and

$$\mathbf{Ch}(P_F) = \int_\Omega d\mathbf{P}(\omega)\mathrm{Tr}_S(\hat{P}_{F,\omega}d\hat{P}_{F,\omega}d\hat{P}_{F,\omega}) \ . \qquad (8)$$

The last step is a consequence of the Fedosov formula [13] namely the operator $P_\omega F^{+-}|_{P_\omega\mathcal{H}_-}$ is Fredholm and its index is an integer given by:

$$n(\omega) = \mathrm{Ind}(P_\omega F^{+-}|_{P_\omega\mathcal{H}_-}) = \mathrm{Tr}_S(\hat{P}_{F,\omega}d\hat{P}_{F,\omega}d\hat{P}_{F,\omega}) \ . \qquad (9)$$

It remains to show that this index is $\mathbf{P}$-almost surely constant. By the covariance condition $P_{T^a\omega}F^{+-}|_{P_{T^a\omega}\mathcal{H}_-}$ and $P_\omega T(a)^{-1}F^{+-}T(a)|_{P_\omega\mathcal{H}_-}$ are unitarily equivalent, so that they have same Fredholm index. Moreover $P_\omega T(a)^{-1}F^{+-}T(a)|_{P_\omega\mathcal{H}_-} - P_\omega F^{+-}|_{P_\omega\mathcal{H}_-}$ is easily seen to be compact so that $P_{T^a\omega}F^{+-}|_{P_{T^a\omega}\mathcal{H}_-}$ have the same index as $P_\omega F^{+-}|_{P_\omega\mathcal{H}_-}$. In other words, $n(\omega)$ is a $\mathcal{G}$-invariant function of ω. The probability $\mathbf{P}$ being $\mathcal{G}$-invariant and ergodic, $n(\omega)$ is $\mathbf{P}$-almost surely constant. Consequently, since $F^{+-} = u$, if $P_F \in \mathcal{S}$:

$$\mathbf{Ch}(P_F) = \mathrm{Ind}(P_{F,\omega}u|_{P_{F,\omega}\mathcal{H}}) \ \in \mathbf{Z} \ , \qquad \mathbf{P}\text{-almost surely} \ .$$

5 Localization

It remains to show how the condition $P_F \in \mathcal{S}$ is related to the Anderson's localization. The easiest way to define the localization length consist in measuring averaged square displacement of a wave packets on the long run. Let Δ be an interval. We denote by P_Δ the eigenprojection of the Hamiltonian corresponding to energies in Δ. Then, if X is the position operator in $\mathcal{G}$ we set $X_{\Delta,\omega}(t) = e^{\frac{\imath}{\hbar}H_\omega t}P_{\Delta,\omega}XP_{\Delta,\omega}e^{-\frac{\imath}{\hbar}H_\omega t}$. Then we define the Δ-localization length as:

$$l^2(\Delta) = \limsup_{T\to\infty}\int_0^T \frac{dt}{T}\int_\Omega d\mathbf{P}(\omega) < 0|(X_{\Delta,\omega}(t) - X_{\Delta,\omega}(0))^2|0 > \ .$$

In [7] we have shown that equivalently

$$
\begin{aligned}
l^2(\Delta) &= \limsup_{T\to\infty} \int_0^T \frac{dt}{T} \mathcal{T}_{\mathbf{P}}(|\vec{\nabla}(e^{-\frac{i}{\hbar}Ht}P_\Delta)|^2) && (10)\\
&= \sup_{\mathcal{P}} \sum_{\Delta'\in\mathcal{P}} \mathcal{T}_{\mathbf{P}}(|\vec{\nabla}P_{\Delta'}|^2)\ . && (11)
\end{aligned}
$$

where $\mathcal{P}$ runs in the set of finite partitions of Δ by Borel subsets. Moreover we have also shown [7] that $\ell^2(\Delta) < \infty$ *implies that the spectrum of* H_ω *is pure point in* Δ, $\mathbf{P}$*-almost surely.*

The density of states is the positive measure $\mathcal{N}$ on $\mathbf{R}$ defined by $\int_{\mathbf{R}} d\mathcal{N}(E) f(E) = \mathcal{T}_{\mathbf{P}}(f(H))$ for f a continuous function with compact support. It turns out [7] that if $l^2(\Delta) < \infty$ one can find a positive $\mathcal{N}$-square integrable function ℓ on Δ such that

$$
l^2(\Delta') = \int_{\Delta'} d\mathcal{N}(E)\ l(E)^2\ , \tag{12}
$$

for any subinterval Δ' of Δ. We propose to call $\ell(E)$ the *localization length* at energy E.

We can now conclude. Thanks to eq. (11) the finiteness of the localization length in the interval Δ implies that [7]

(i) $P_F \in \mathcal{S}$ whenever the Fermi level E_F lies in Δ,

(ii) $E_F \in \Delta \mapsto P_F \in \mathcal{S}$ is continuous (for the Sobolev norm) at every regularity point of $\mathcal{N}$.

(iii) $\mathbf{Ch}(P_F)$ is constant on Δ, leading to existence of plateaus for the transverse conductivity.

(iv) If the Hamiltonian is changed continuously (in the norm resolvent topology), $\mathbf{Ch}(P_F)$ stay constant as long as the localization length remains finite at the Fermi level.

As a Corollary, we notice that between two Hall plateaus with different indices, the localization length must diverge [15, 18]. The reader will find in [7] how to compute practically the Hall index using homotopy (property (iv)) and explicit calculation for simple models.

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