The HULL

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Sponsoring

Grant no. 0901514

CRC 701, Bielefeld

Main References

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Content

I - Uniformly Discrete Sets

J. Bellissard, D. Hermmann, M. Zarrouati, in *Directions in Mathematical Quasicrystals*, CRM Monograph Series, **13**, 207-259, M.B. Baake & R.V. Moody Eds., AMS Providence, (2000).

UD-sets

Motivation:

- **Pointlike Nuclei:** The *atomic nuclei* in a solid are located on a discrete subset of $\mathbb{R}^3$. These nuclei can be considered as pointlike.

- **Exclusion Principle:** Due to the electron-electron repulsion, produced by the Pauli's *exclusion principle*, there is a *minimum distance* between nuclei. Hence the nuclei positions make up a uniformly discrete subset of $\mathbb{R}^3$.

- **Homogeneity:** All solids considered are *homogeneous*, namely their large scale physical properties are invariant by translation

UD-sets

- A discrete subset $\mathcal{L} \subset \mathbb{R}^d$ is called *uniformly discrete* whenever there is $r > 0$ such that $\#\{B(x;r) \cap \mathcal{L}\} = 0, 1$ for any $x \in \mathbb{R}^d$
- Associated with $\mathcal{L}$ is the *Radon measure*

$$\nu^{\mathcal{L}} = \sum_{y \in \mathcal{L}} \delta_y$$

- $\nu^{\mathcal{L}}$ is characterized by two properties
  - If B is any bounded Borel subset of $\mathbb{R}^d$ then $\nu^{\mathcal{L}}(B) \in \mathbb{N}$
  - For all $x \in \mathbb{R}^d$ then $\nu^{\mathcal{L}}(B(x;r)) \in \{0, 1\}$
- Let UD_r be the set of such measures on $\mathbb{R}^d$.

UD-sets

- The space $\mathfrak{M}(\mathbb{R}^d)$ of *Radon measures* on $\mathbb{R}^d$ is the dual to the space $C_c(\mathbb{R}^d)$ of continuous functions with compact support. It will be endowed with the *weak*-topology*. $\mathbb{R}^d$ acts on it and this action is weak*-continuous. Then T will denote this action.
- **Theorem:**
  - *A Radon measure μ belongs to* UD_r *if and only if it has the form* $\nu^{\mathcal{L}}$ *with* $\mathcal{L}$ *being r-uniformly discrete*
  - UD_r *is invariant by the translation group* $\mathbb{R}^d$
- **Theorem:** *For any $r > 0$, the space* UD_r *is compact*

The Hull

- **Hull of μ:** if $\mu \in UD_r$ its Hull is the closure of its translation orbit.

$$\mathrm{Hull}(\mu) = \overline{\{T^a \mu \,;\, a \in \mathbb{R}^d\}}$$

- It follows immediately that the *Hull* is *compact* and that $\mathbb{R}^d$ acts on it by *homeomorphisms*. Hence

$(\mathrm{Hull}(\mu), \mathbb{R}^d, T)$ is a *topological dynamical system*

The Canonical Transversal

- If $\mu \in \mathrm{UD}_r$ its *canonical transversal* is the subset $\mathrm{Trans}(\mu)$ defined by those elements $\xi \in \mathrm{Hull}(\mu)$ with $\xi(\{0\}) = 1$

- If $\xi \in \mathrm{Trans}(\mu)$ and if $a \in \mathbb{R}^d$ is small enough and nonzero $0 < |a| < r$ then $\mathrm{T}^a \xi \notin \mathrm{Trans}(\mu)$

- $\mathrm{Trans}(\mu)$ is also compact. The *groupoid* induced by the action of $\mathbb{R}^d$ has discrete fibers and is *étale*

- $\xi \in \mathrm{Trans}(\mu)$ then its *fiber* is the set of points $\mathcal{L}_\xi \subset \mathbb{R}^d$ such that $a \in \mathcal{L}_\xi \;\Rightarrow\; \mathrm{T}^{-a}\xi \in \mathrm{Trans}(\mu)$. Hence $\mathcal{L}_\xi$ is nothing but the *support* of ξ

Atomic Potentials

- Let $v \in L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. The *atomic potential* associated with $\mu \in \mathrm{UD}_r$ is defined by

$$V_\mu(x) = \sum_{y \in \mathcal{L}_\mu} v(x - y)$$

- **Theorem:**
  - *If $H_\mu = -\Delta + V_\mu$ is the corresponding Schrödinger operator, then the map $\mu \in \mathrm{UD}_r \mapsto (zI - H_\mu)^{-1}$ is strongly continuous and covariant*
  - *The Hull of H_μ is homeomorphic to the Hull of μ*

Localization

- The space $C_c(\mathbb{R}^d)$ of continuous functions with compact support, can be seen as the direct limit of the spaces $C_0(U)$ whenever U runs through the set of *bounded open* subsets of $\mathbb{R}^d$. Here $C_0(U)$ is endowed with the uniform norm.

- Let $\mathfrak{M}_U$ be the space of Radon measures on U, namely the Banach space dual to $C_0(U)$.

Localization

- **Theorem:**

  1. *Let $U \subset V$ be a pair of bounded open sets. If $\rho \in \mathfrak{M}_V$ then the linear map $f \in C_0(U) \mapsto \rho(f) \in \mathbb{C}$ defines a Radon measure $\pi_{U \leftarrow V}(\rho)$ on U, the* restriction *of ρ to U*
  2. *The* restriction map *$\pi_{U \leftarrow V} : \rho \in \mathfrak{M}_V \mapsto \pi_{U \leftarrow V}(\rho) \in \mathfrak{M}_V$ is weak*-continuous*
  3. *The map $\pi_{U \leftarrow V}$ maps $\mathrm{UD}_r(V)$ to $\mathrm{UD}_r(V)$*
  4. *The spaces $\mathfrak{M}(\mathbb{R}^d)$ and UD_r can be seen as the inverse limits*

$$\mathfrak{M}(\mathbb{R}^d) = \varprojlim(\mathfrak{M}_U, \pi_{U \leftarrow V}) \qquad \mathrm{UD}_r = \varprojlim(\mathrm{UD}_r(U), \pi_{U \leftarrow V})$$

II - Repetitivity

M. Queffélec, *Substitution dynamical systems-spectral analysis*, Lecture Notes in Math, **1294**, (1987).

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Patches

- Let $\mathcal{L}$ be UD_{r_0}. A patch of radius r is a finite set of the form

$$p = (\mathcal{L} - y) \cap \overline{\mathrm{B}}(0; r) \qquad \textit{for some} \quad y \in \mathcal{L}$$

- Let $\mathcal{P}_r$ be the set of patches of radius r in $\mathcal{L}$. It can be topologized by identifying it with the measure ν^p. Let $\mathcal{Q}_r$ be the weak*-closure of $\mathcal{P}_r$
- **Theorem:**
  - $\mathcal{Q}_r \subset \mathfrak{M}(B(0; r))$ *is a compact space*
  - *if* $r' > r$ *let* $\pi_{r \leftarrow r'}$ *be the restriction map from* $\mathfrak{M}(B(0; r'))$ *onto* $\mathfrak{M}(B(0; r))$. *Then* $\pi_{r \leftarrow r'}$ *maps continuously* $\mathcal{Q}_{r'}$ *onto* $\mathcal{Q}_r$

Tiling Space

- Let $\mathcal{L}$ be UD. Its *tiling space* is defined by the inverse limit

$$\Xi = \varprojlim(\Omega_r, \pi_{r \leftarrow r'})$$

- **Theorem:** *if* $\mathcal{L}$ *is* UD_{r_0} *then its tiling space is homeomorphic to its transversal*

$$\Xi \simeq \mathrm{Trans}(\nu^{\mathcal{L}})$$

Delone sets

- A measure $\mu \in \mathrm{UD}_r$ is *Delone* if there is $R \geq r$ so that

$$\mu(\overline{\mathrm{B}}(x;R)) \geq 1 \qquad\qquad \forall x \in \mathbb{R}^d$$

A similar definition for UD-sets holds

- Let $\mathrm{Del}_{r,R}$ denotes the set of such measures: it is weak$*$-compact and $\mathbb{R}^d$-invariant. In particular,

$$\textit{if } \mu \in \mathrm{Del}_{r,R} \quad \textit{then} \quad \mathrm{Hull}(\mu) \subset \mathrm{Del}_{r,R}$$

Repetitivity

- A UD-set $\mathcal{L}$ is called *repetitive* if for any patch p and any $\epsilon > 0$, there is an $R_{p,\epsilon} > 0$ such that in each ball of radius $R_{p,\epsilon}$ there is a translated copy of a patch p' such that the Hausdorff distance from p to p' is less than ϵ.

- **Proposition:** *Any repetitive* UD-*set is Delone*

- **Theorem:** *(see Queffélec '87, Radin-Wolff '92)*
  A UD-*set is repetitive if and only if the* $\mathbb{R}^d$ *action on its Hull is minimal*

III - Finite Local Complexity

J. E. ANDERSON, I. F. PUTNAM, Ergod. Th, & Dynam. Sys., **18**, (1998), 509-537.

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Finite Local Complexity

- A Delone set $\mathcal{L}$ has *finite local complexity (FLC)*, if the set $\mathcal{L} - \mathcal{L}$ is discrete and closed *(Lagarias '99)*, where

$$\mathcal{L} - \mathcal{L} = \{y - z\,;\, y, z \in \mathcal{L}\}$$

- If $\mathcal{L}$ is FLC and if $R' > 0$, then for each $y \in \mathcal{L}$, there is only a *finite number* of choices for the vectors $z \in \mathcal{L}$ with $|z - y| \leq R'$. Hence FLC is a mathematical counterpart for *rigidity*

- **Proposition:** *$\mathcal{L}$ has FLC if and only if $\mathcal{Q}_r = \mathcal{P}_r$ is finite for all $r > 0$*

Finite Local Complexity

- **Theorem:** *Let* $\mathcal{L}$ *be FLC. Then for any* $\omega \in \mathrm{Hull}(\nu^{\mathcal{L}})$ *the sets* $\mathcal{L}_\omega - \mathcal{L}_\omega = \mathcal{L} - \mathcal{L}$ *coincide*

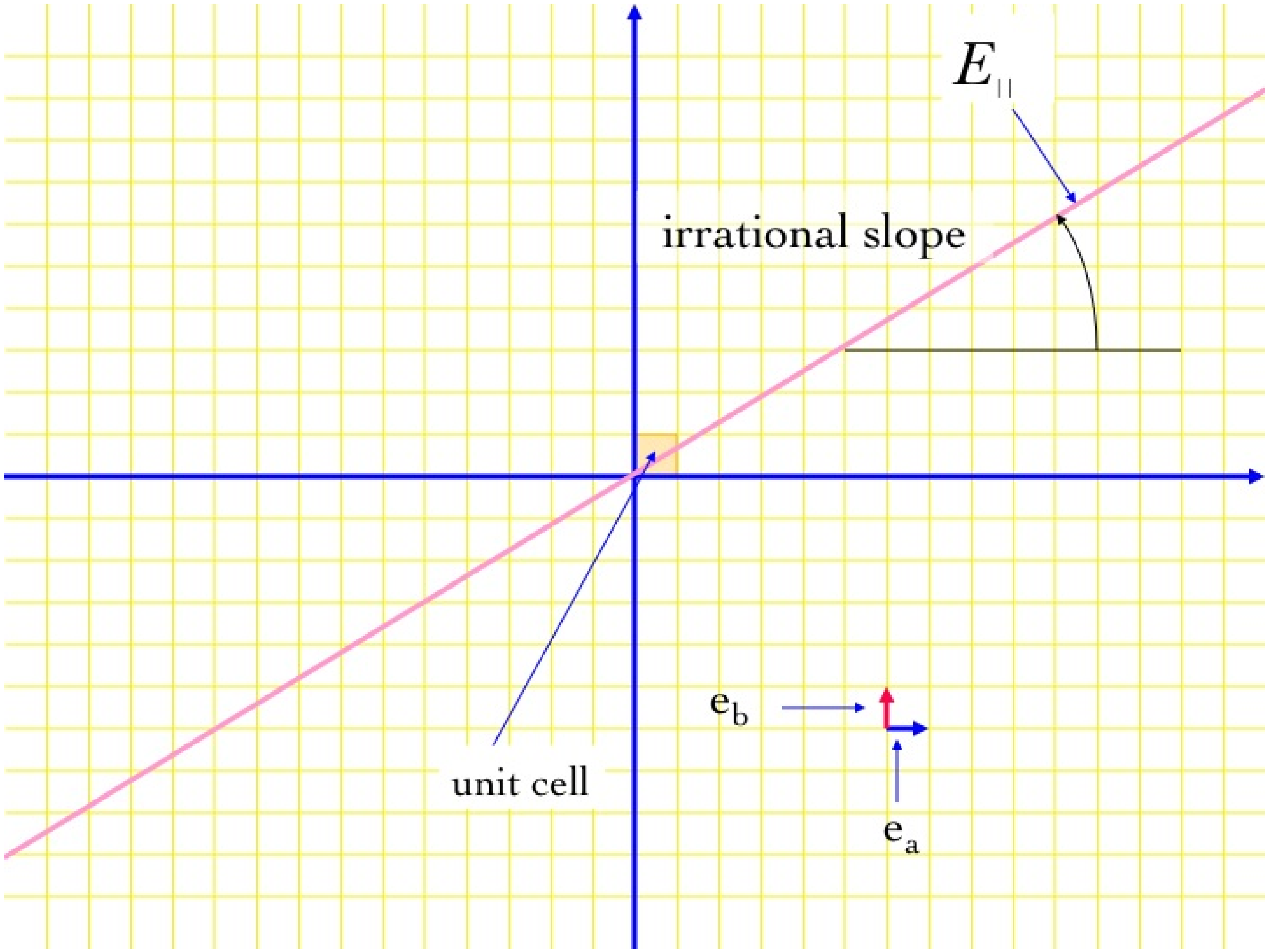
- **Theorem:** *(Kellendonk '97)*
  Let $\mathcal{L}$ *be Delone and FLC. Then* $\mathrm{Trans}(\nu^{\mathcal{L}})$ *is completely disconnected*

- **Theorem:** *(Lagarias '99)*
  Let $\mathcal{L}$ *be Delone and FLC. Then the free group generated by* $\mathcal{L} - \mathcal{L}$ *has finite rank*

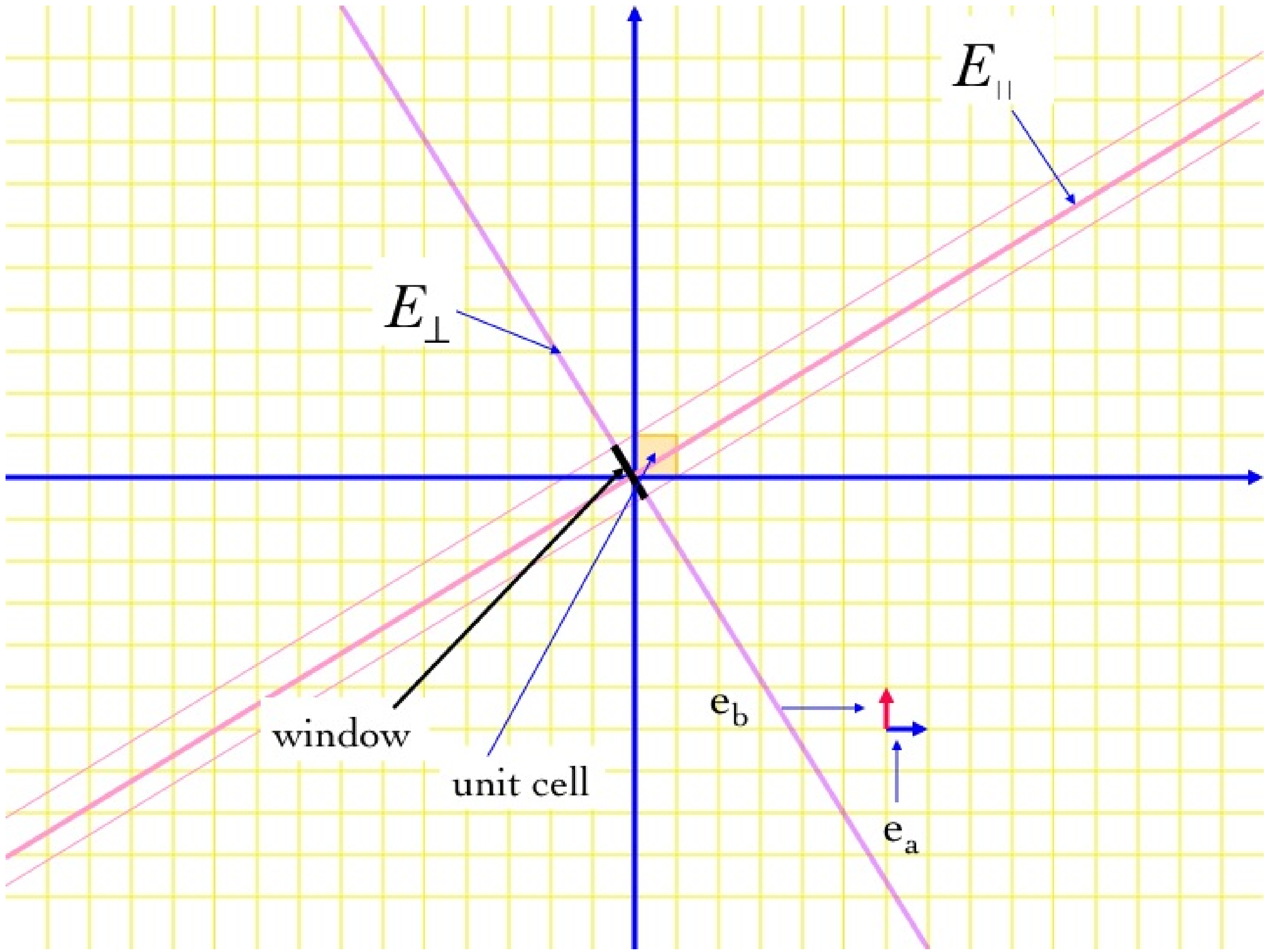
Meyer sets

- $\mathcal{L}$ is a *Meyer set* whenever both $\mathcal{L}$ and $\mathcal{L} - \mathcal{L}$ are Delone.
- **Example:** *Quasicrystal* are described by Meyer sets
- Meyer sets are obtained from a *cut-and-project* construction.

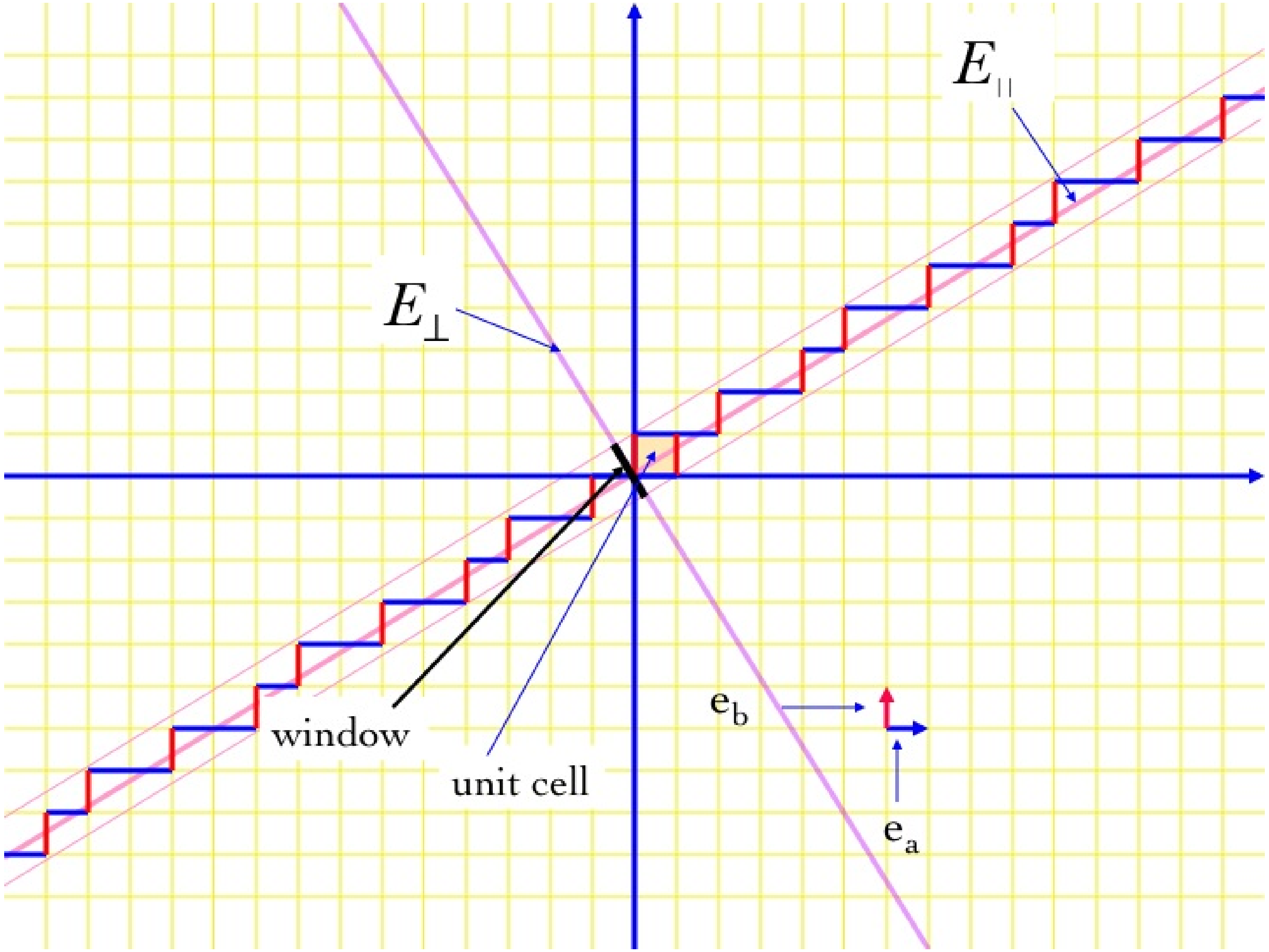
Cut-and-Projection Construction



Cut-and-Projection Construction



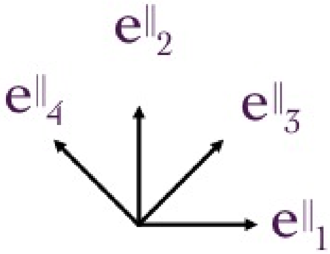
Cut-and-Projection Construction

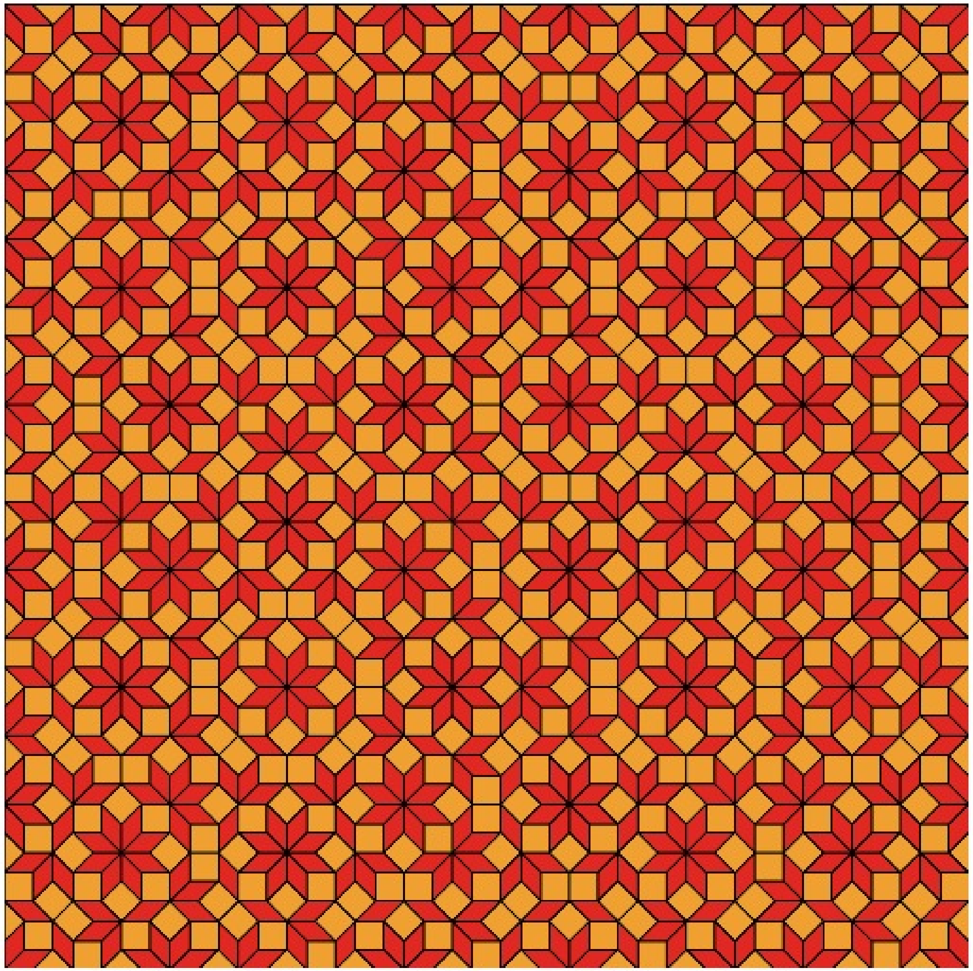


The Octagonal Tiling

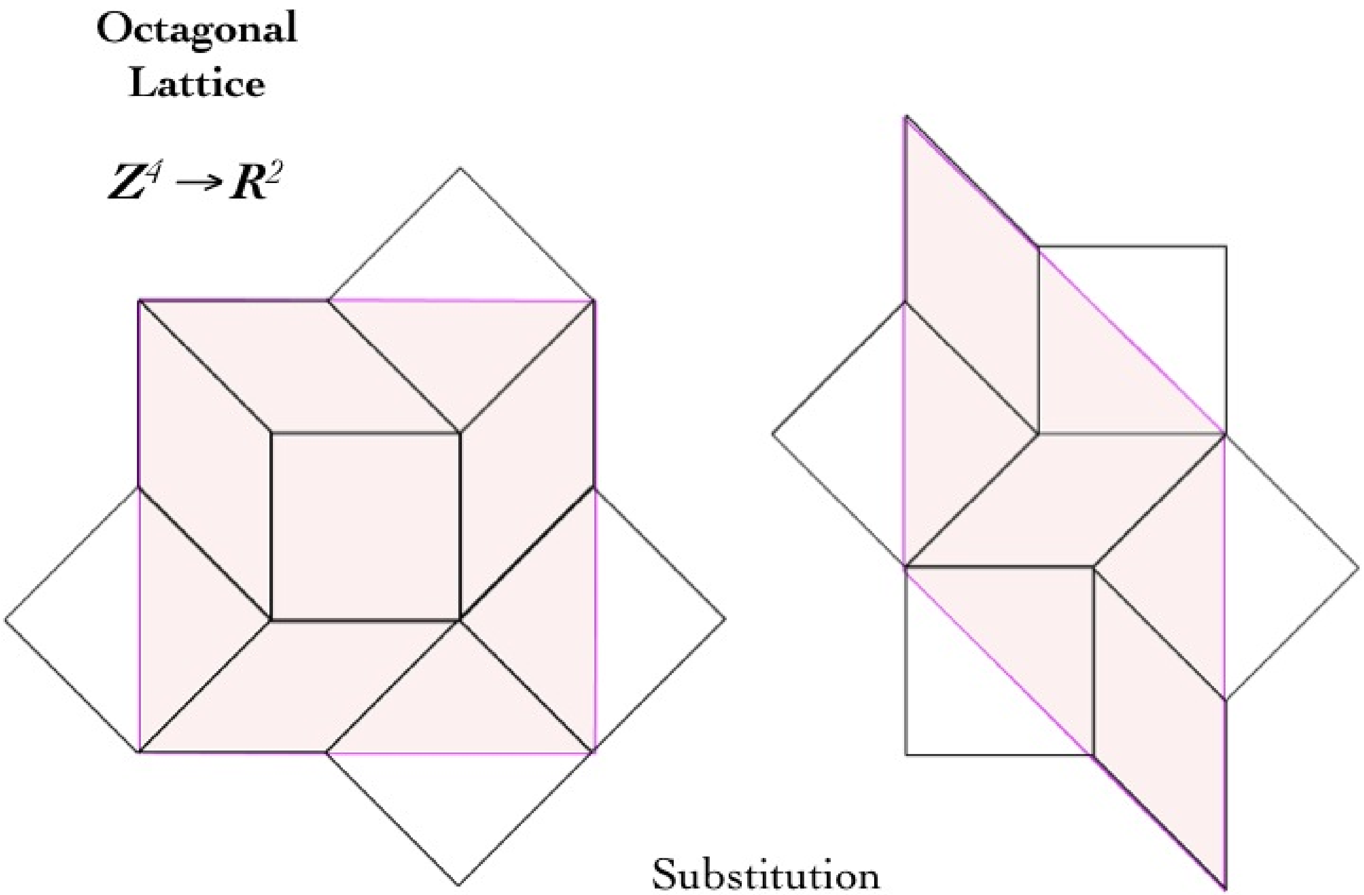
Octagonal
Lattice

$Z^4 \to R^2$





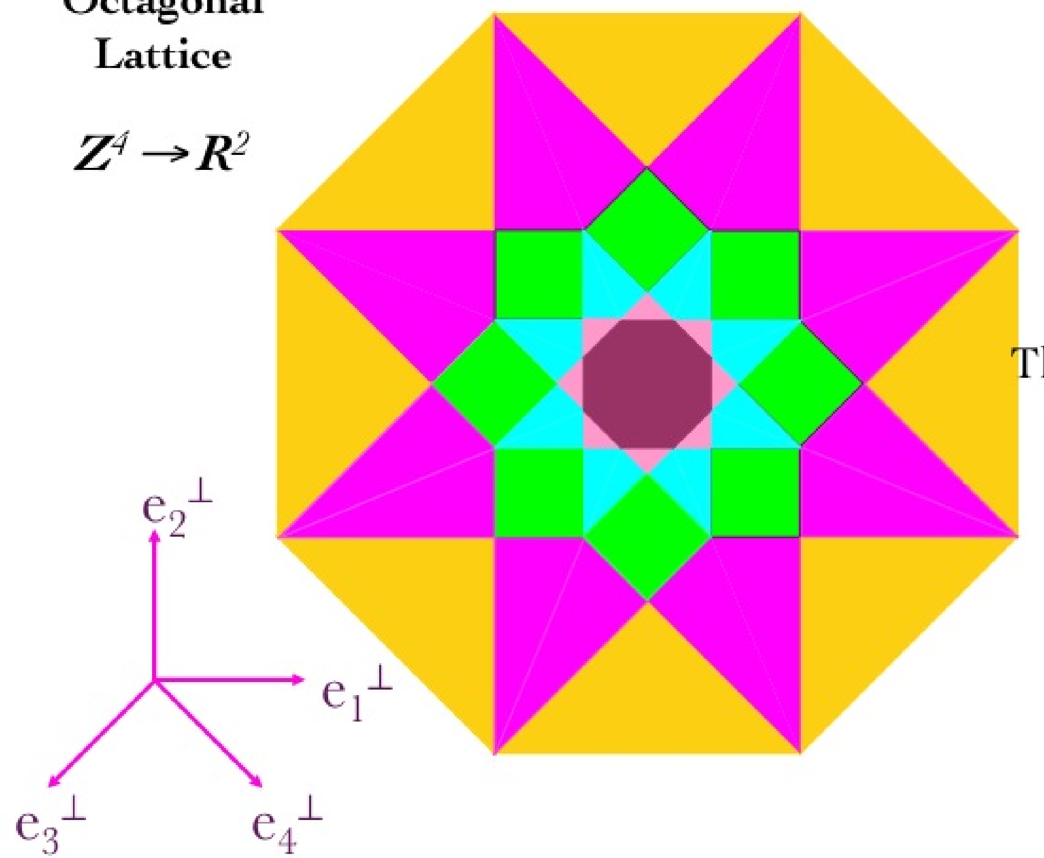
The Octagonal Tiling



The Octagonal Tiling

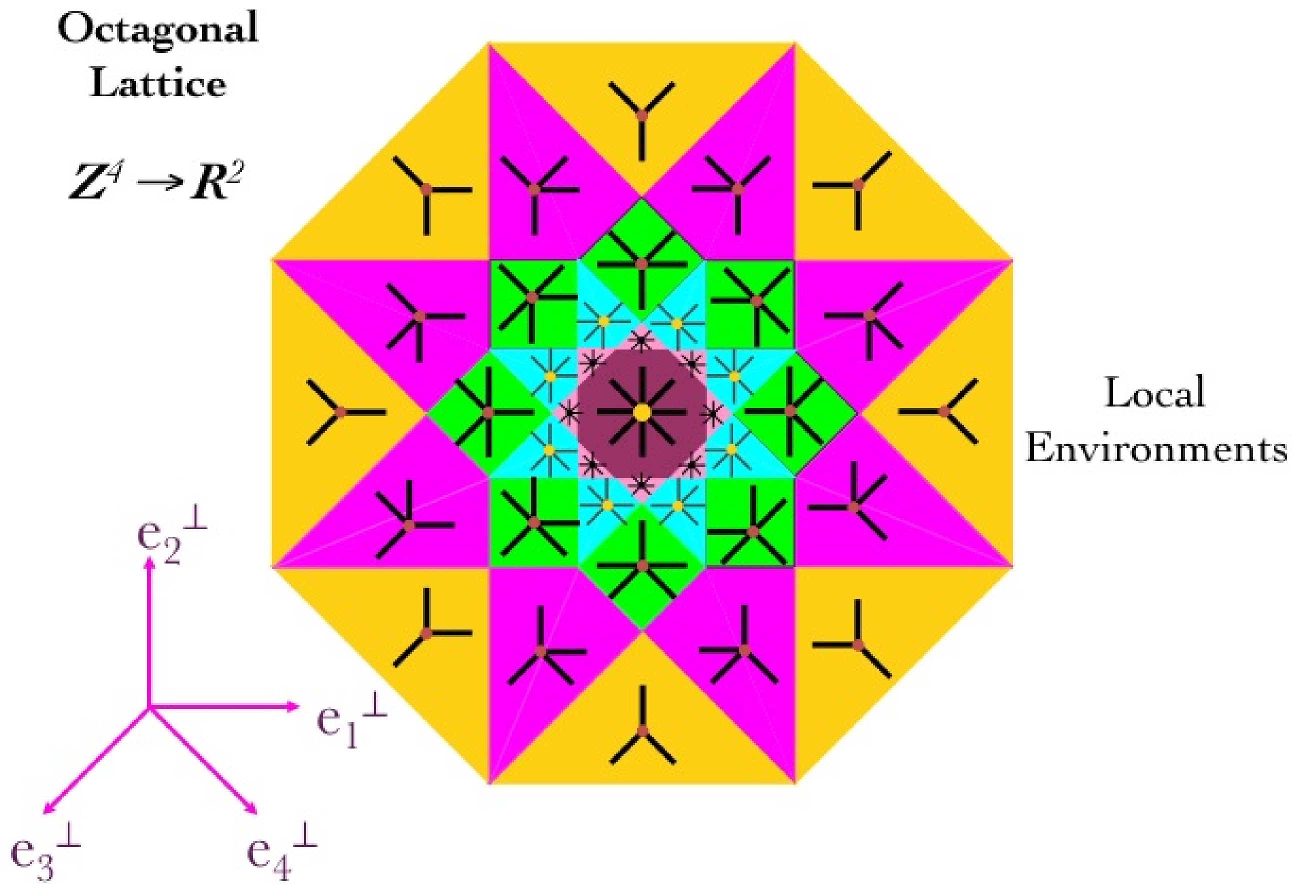
Octagonal Lattice

$Z^4 \rightarrow R^2$



The Transversal or Window

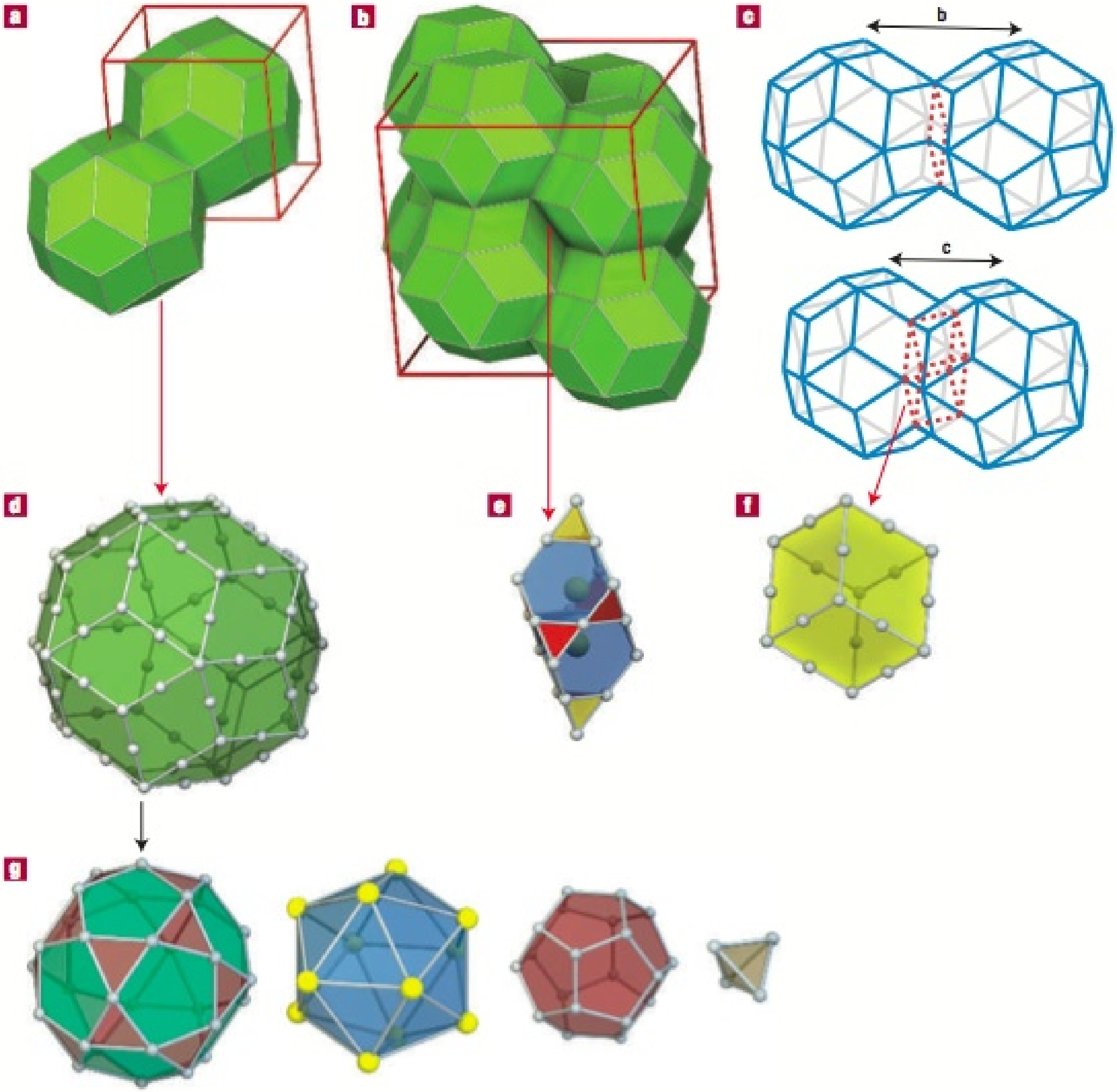
The Octagonal Tiling



Quasicrystals

1. Stable Ternary Alloys *(icosahedral symmetry)*
   - High Quality: **AlCuFe** ($Al_{62.5}Cu_{25}Fe_{12.5}$)
   - Stable Perfect: **AlPdMn** ($Al_{70.}Pd_{22}Mn_{7.5}$)
     AlPdRe ($Al_{70.}Pd_{21}Re_{8.5}$)

2. Stable Binary Alloys
   - Periodic Approximants: **$YbCd_6$, $YbCd_{5.8}$**
   - Icosahedral Phase **$YbCd_{5.7}$**

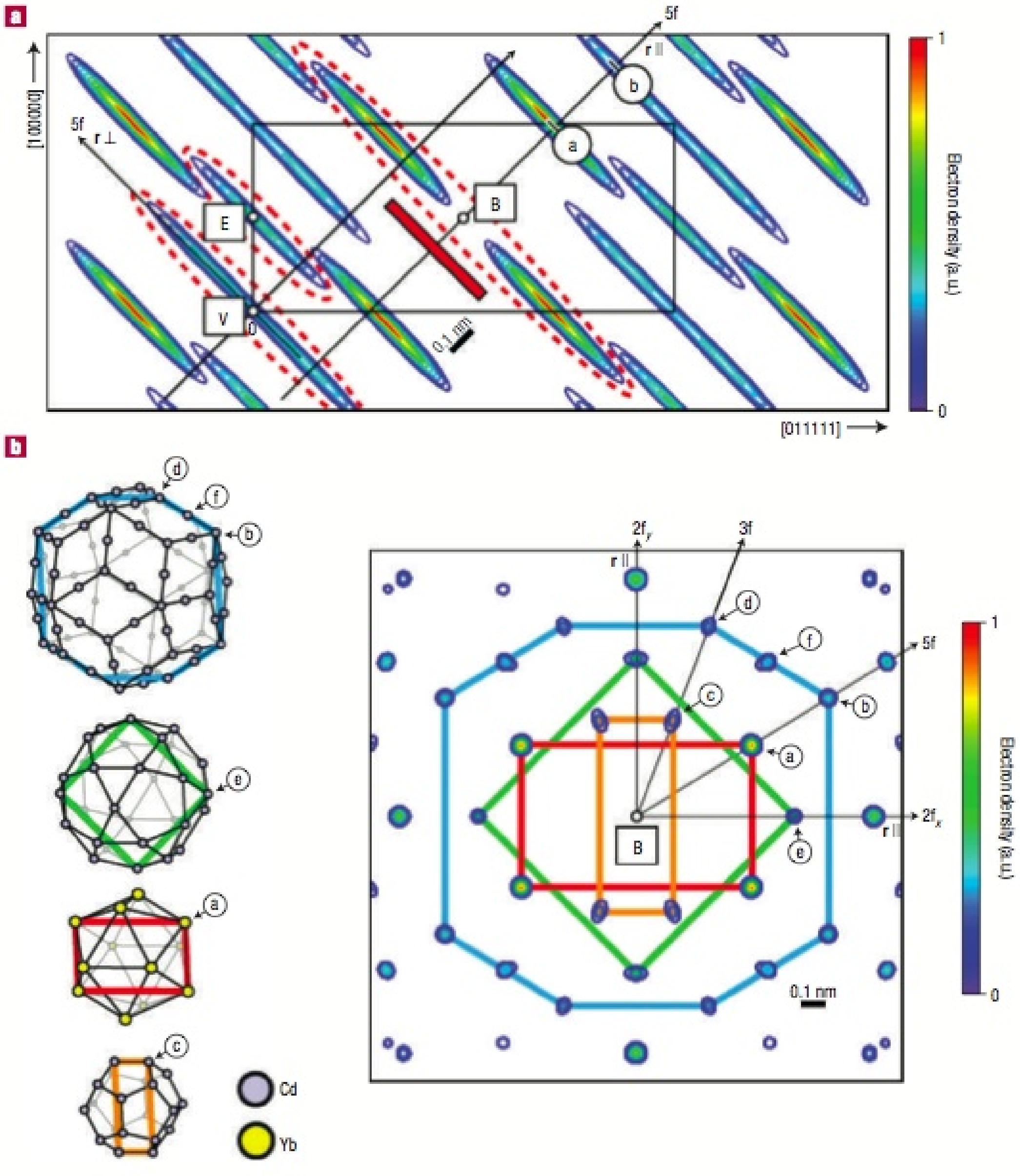


Clusters in

YbCd-approximants.

(Cd in grey, Yb in yellow)

H. Takakura, et al., Nat. Mat. '04

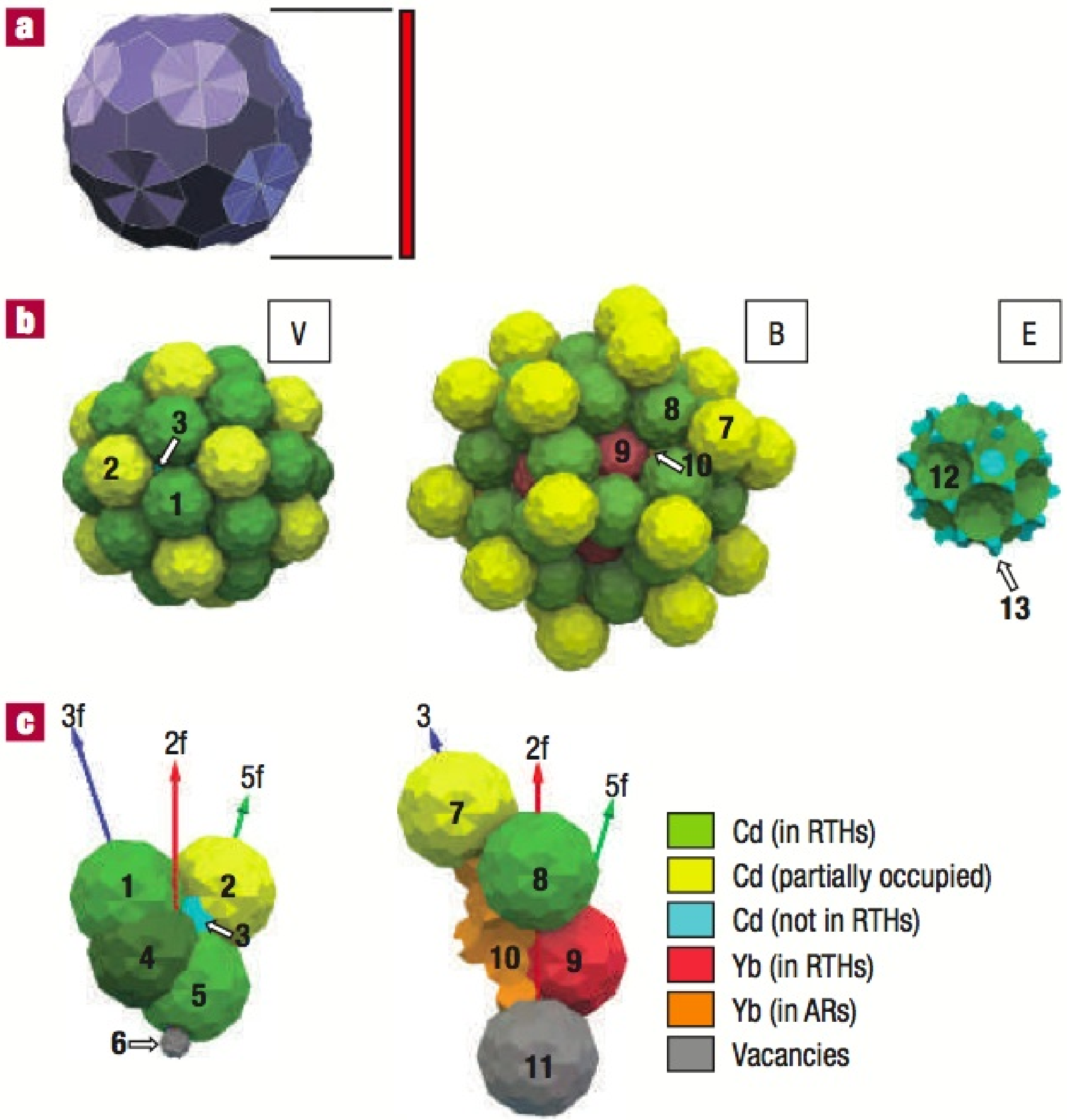


Clusters in

i-YbCd.

(Cd in grey, Yb in yellow)

H. Takakura, et al., Nat. Mat. '04



Acceptance domains

in *Tiling Space*

H. Takakura, et al., Nat. Mat. '04

IV - The Anderson-Putnam Complex

J. E. ANDERSON, I. F. PUTNAM, Ergod. Th, & Dynam. Sys., **18**, (1998), 509-537.

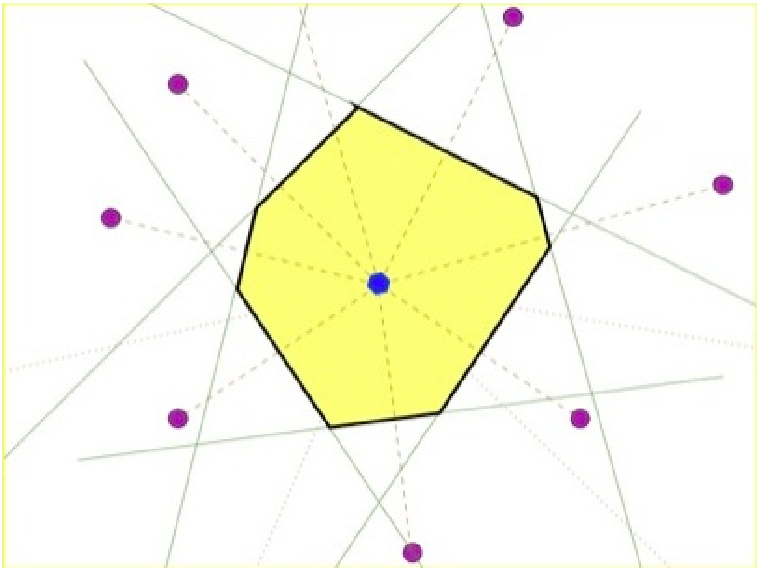
J. BELLISSARD, R. BENEDETTI, J.-M. GAMBAUDO, Commun. Math. Phys., **261**, (2006), 1-41.

The Voronoi Tilings

- Let $\mathcal{L}$ be a UD-set. If $x \in \mathcal{L}$ its *Voronoi cell* is defined by

$$V(x) = \{y \in \mathbb{R}^d \,;\, |y - x| < |y - x'| \, \forall x' \in \mathcal{L}\,,\, x' \neq x\}$$

$V(x)$ is open. Its closure $\overline{V(x)}$ is called the *Voronoi tile* of x



Proposition: *If $\mathcal{L} \in \mathrm{Del}_{r_0,r_1}$ the Voronoi tile of any $x \in \mathcal{L}$ is a convex polytope containing the ball $\overline{\mathrm{B}}(x;r_0)$ and contained in the ball $\overline{\mathrm{B}}(x;r_1)$*

The Voronoi Tilings

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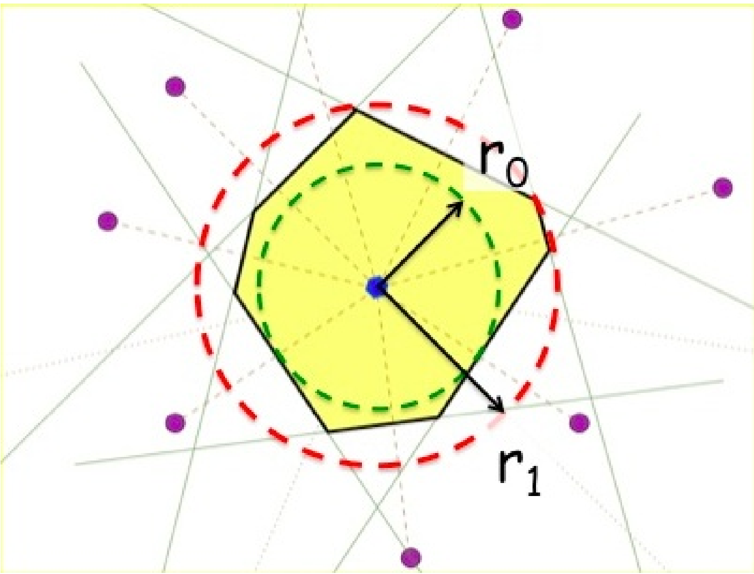

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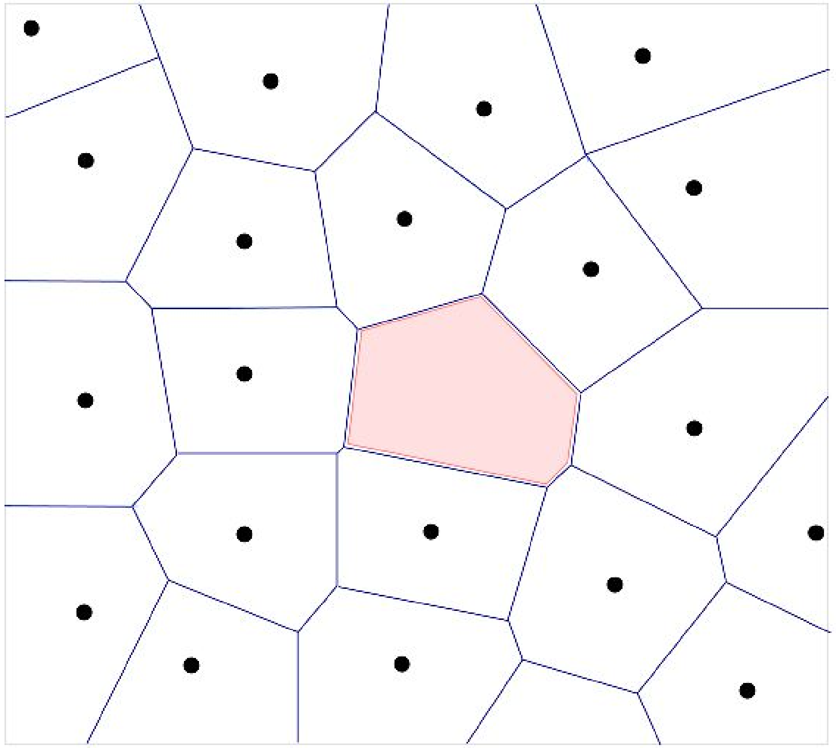
The Voronoi Tilings



Proposition: *the Voronoi tiles of a Delone set touch face-to-face*

$\mathcal{L}$ is FLC if and only if its Voronoi tiling has finitely many tiles modulo translations.

UD-sets versus Tilings

- A *tile* is a compact subset of $\mathbb{R}^d$ homeomorphic to a Euclidean ball. A *tiling* is a countable family of tiles with disjoint interiors, covering $\mathbb{R}^d$. A *prototile* is an equivalence class of tiles modulo *translations (or isometries)*.

- Given a tiling T of $\mathbb{R}^d$, endow each tile t with a *base point* x_t in its interior, such that if there is $a \in \mathbb{R}^d$ with $t' = t + a$ then $x_{t'} - x_t = a$. Then the set $\mathcal{L}_T$ of such base points is discrete.

- *With such a construction, the language of* UD-set *can be translated into the language of tilings* *(Kellendonk '97)*

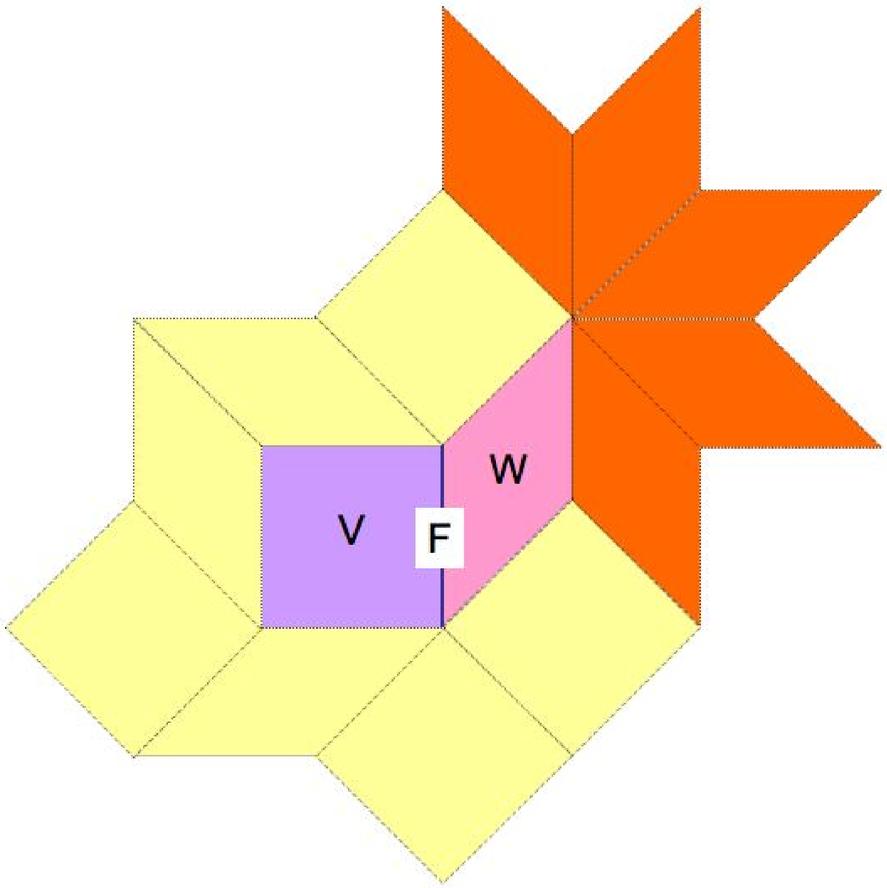
The Anderson-Putnam Complex

- Let T be a tiling which is *repetitive & FLC*. It will also be endowed with a set $\mathcal{L}_T$ of base points.

- T will be assumed to be *aperiodic*, namely there is $a \in \mathbb{R}^d$ such that $T + a = T$ if and only if $a = 0$.

- Given a tile t, its *collar* C_t is the set of all tiles touching it. The pair $\hat{t} = (t, C_t)$ is called a *collared tile*. $\mathbb{R}^d$ acts on the set of collared tiles. A *collared prototile* is an equivalence class of collared tile. The prototile associated with it is called its *geometrical support*.

The Anderson-Putnam Complex



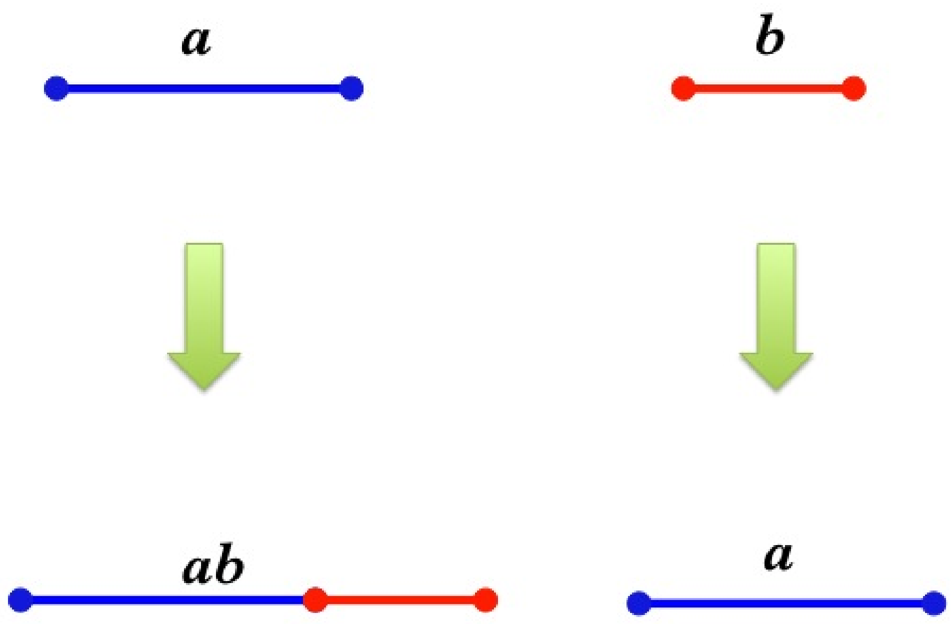
Two touching collared tiles in the octagonal lattice

The Anderson-Putnam Complex

- Let $\mathcal{C}$ denote the set of collared prototiles. By FLC, it is *finite*. Let then $\tilde{X}$ be the disjoint union of elements in $\mathcal{C}$. Two point $x, y \in \tilde{X}$, belonging to two the geometrical support of two collared prototiles $\tilde{t}, \tilde{t}'$, are *equivalent* ($x \sim y$), if there are two representing collared tiles $\hat{t}, \hat{t}'$ in the tiling, such that the points in the tiling representing x and y *coincide*.

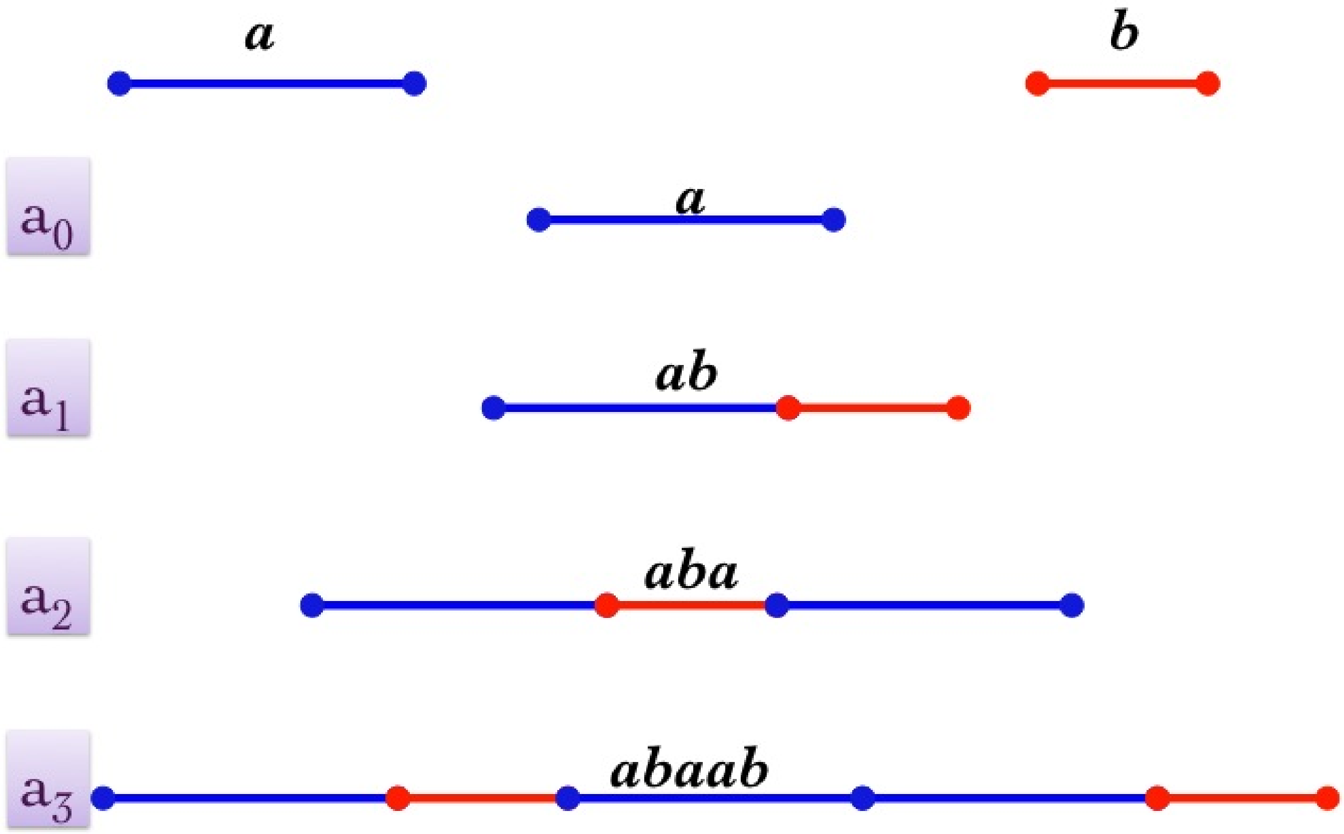
- The quotient space $X = X/\sim$ is a compact space called the *Anderson-Putnam complex*.

The AP-Complex for the Fibonacci Tiling



The Fibonacci Substitution

The AP-Complex for the Fibonacci Tiling



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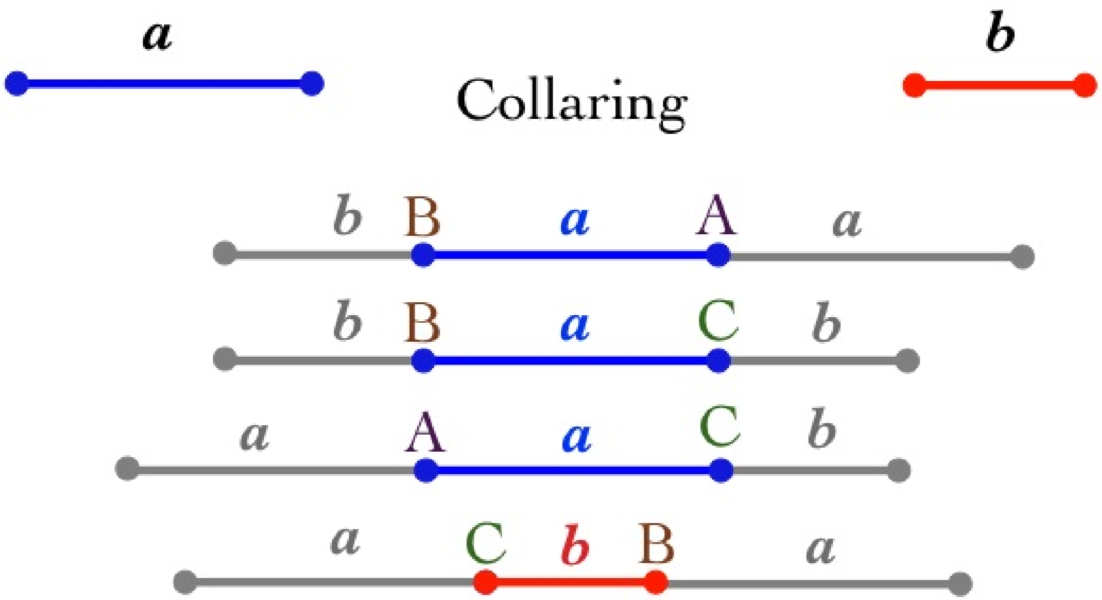
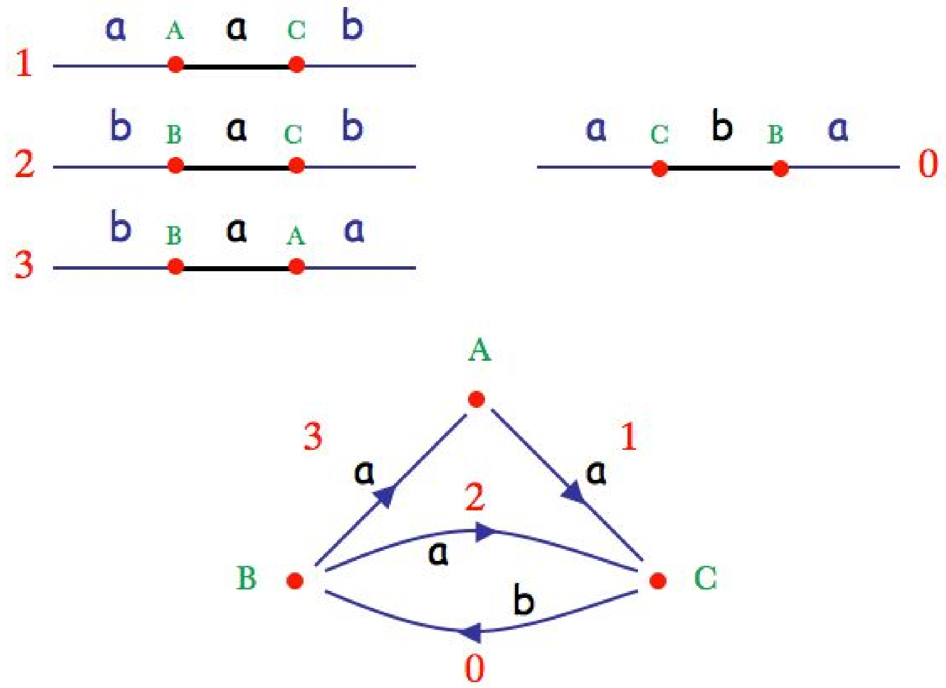


The AP-Complex for the Fibonacci Tiling



More generally, if the tiles of T have the structure of a *CW-complex*, so has X. In particular if T has *polyhedral tiles* its *Anderson-Putnam complex has the structure of a CW-complex.*

The Anderson-Putnam Complex

- **Theorem:** *(Anderson-Putnam '98, Bellissard-Benedetti-Gambaudo '01-'06)*
  Let T be an aperiodic, repetitive, FLC tiling with base points and polyhedral tiles. Then
  - *Any Anderson-Putnam complex has a natural structure of smooth, branched, oriented, flat, Riemannian manifold (BOF)*
  - *There is a sequence $(X_n)_{n\in\mathbb{N}}$ of Anderson-Putnam complexes together with maps $f_n : X_n \to X_{n-1}$ such that $Df_n = \mathbf{1}$ with the property that*

$$\mathrm{Hull}(T) \simeq \varprojlim (X_n, f_n)$$

  - *This homeomorphism conjugates the action of $\mathbb{R}^d$ on $\mathrm{Hull}(T)$ with the limit of the parallel transport by constant vector fields.*

It is time for coffee !