The Topology of Tiling Spaces

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Content

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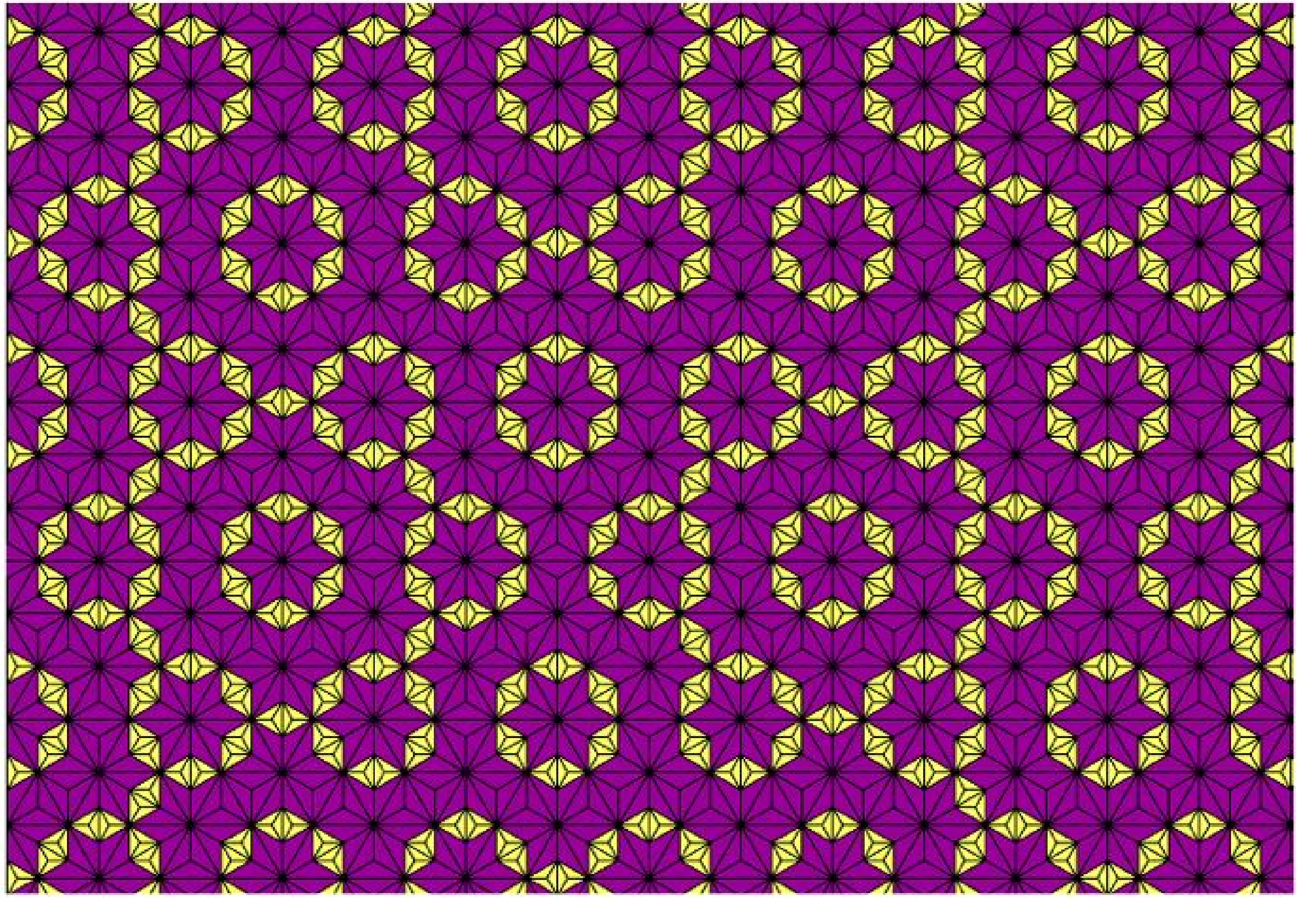
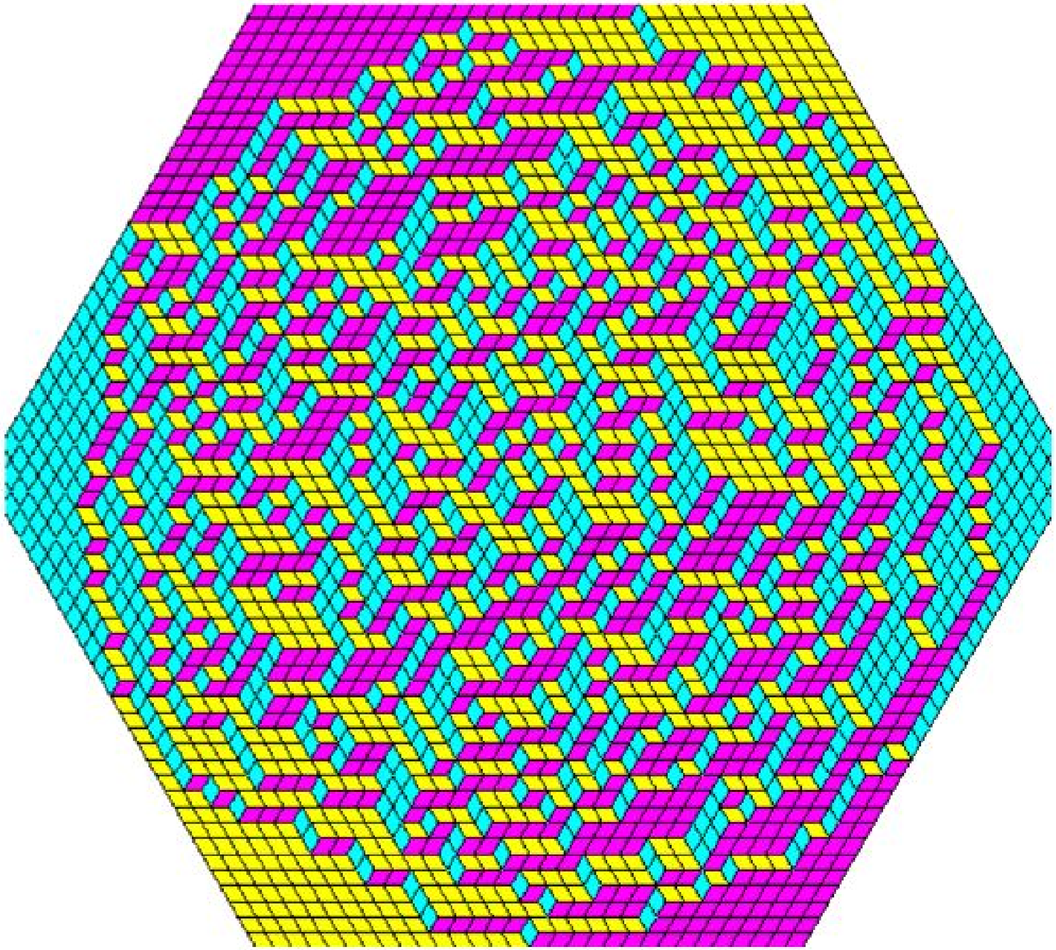
I - Tilings, Tilings,...



- A triangle tiling -



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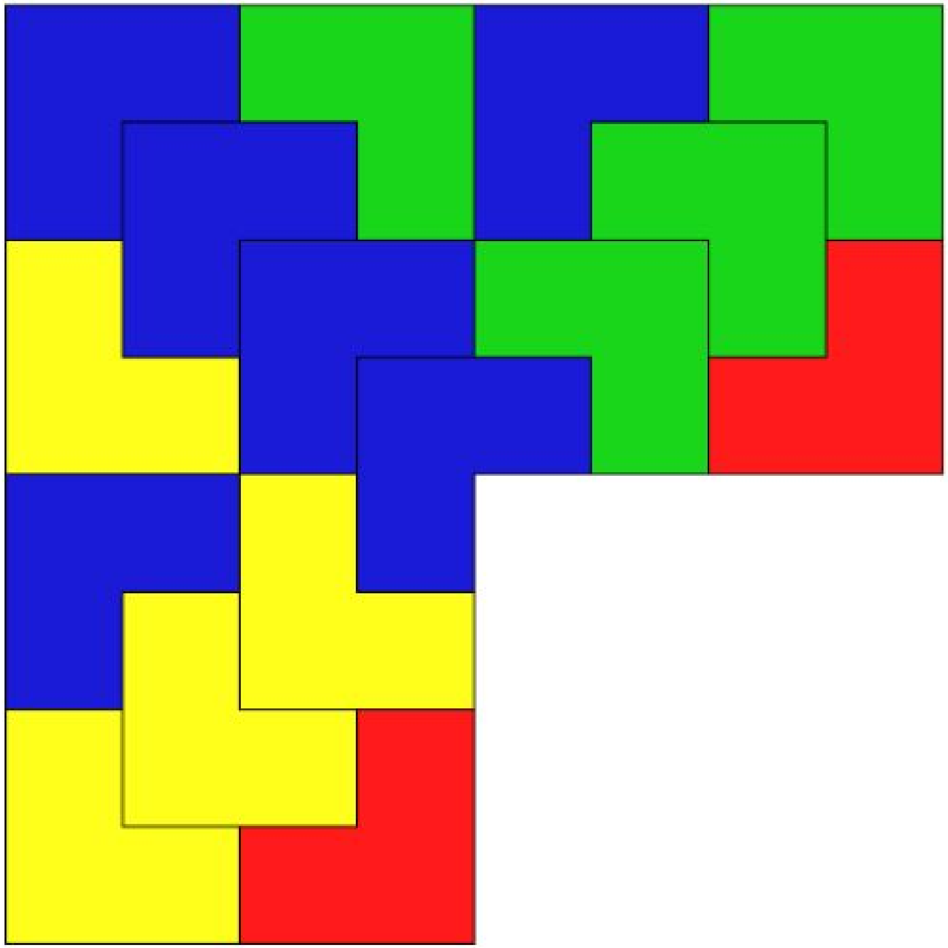
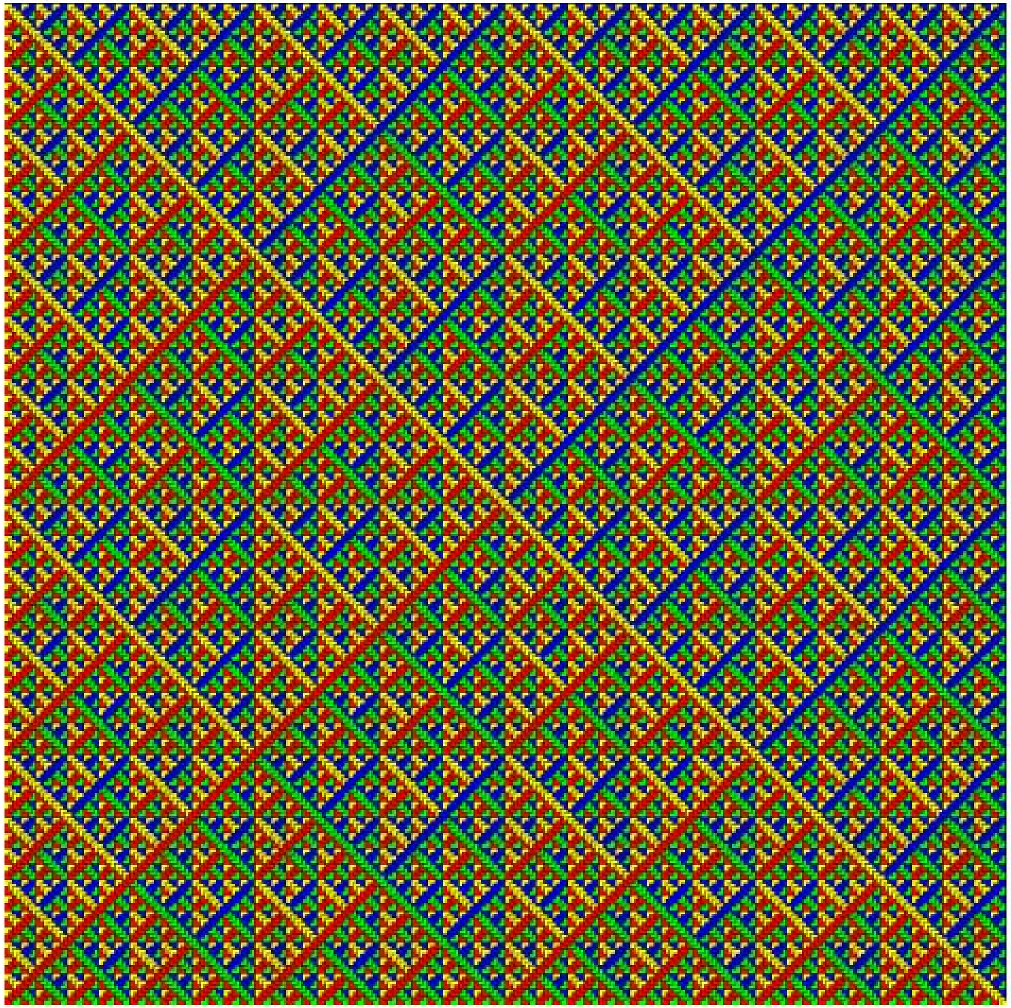
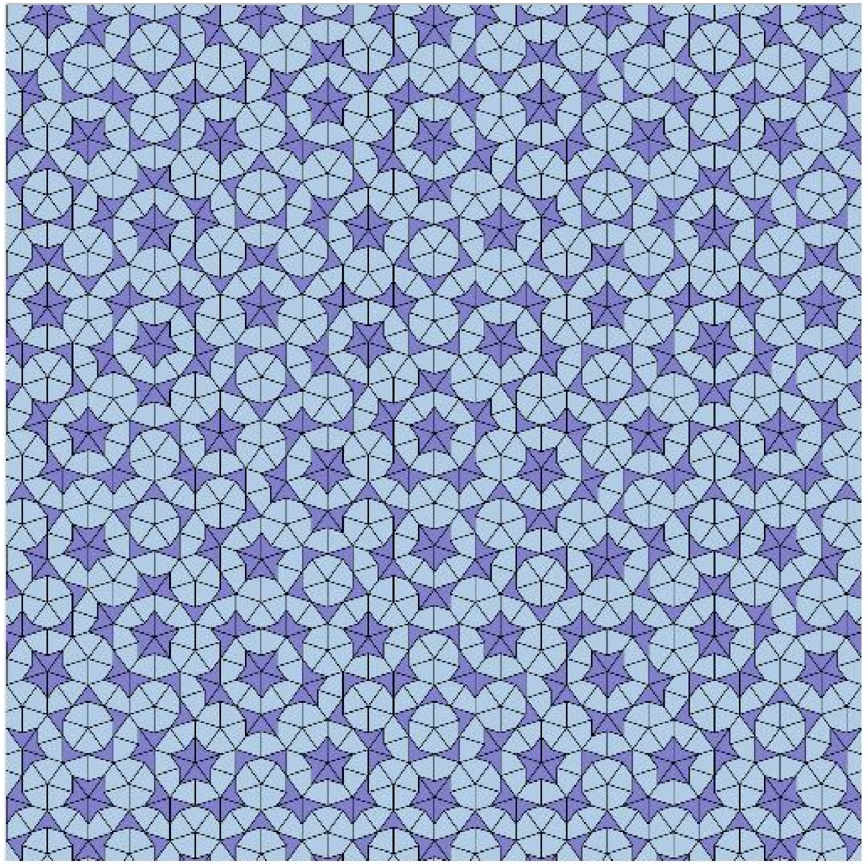


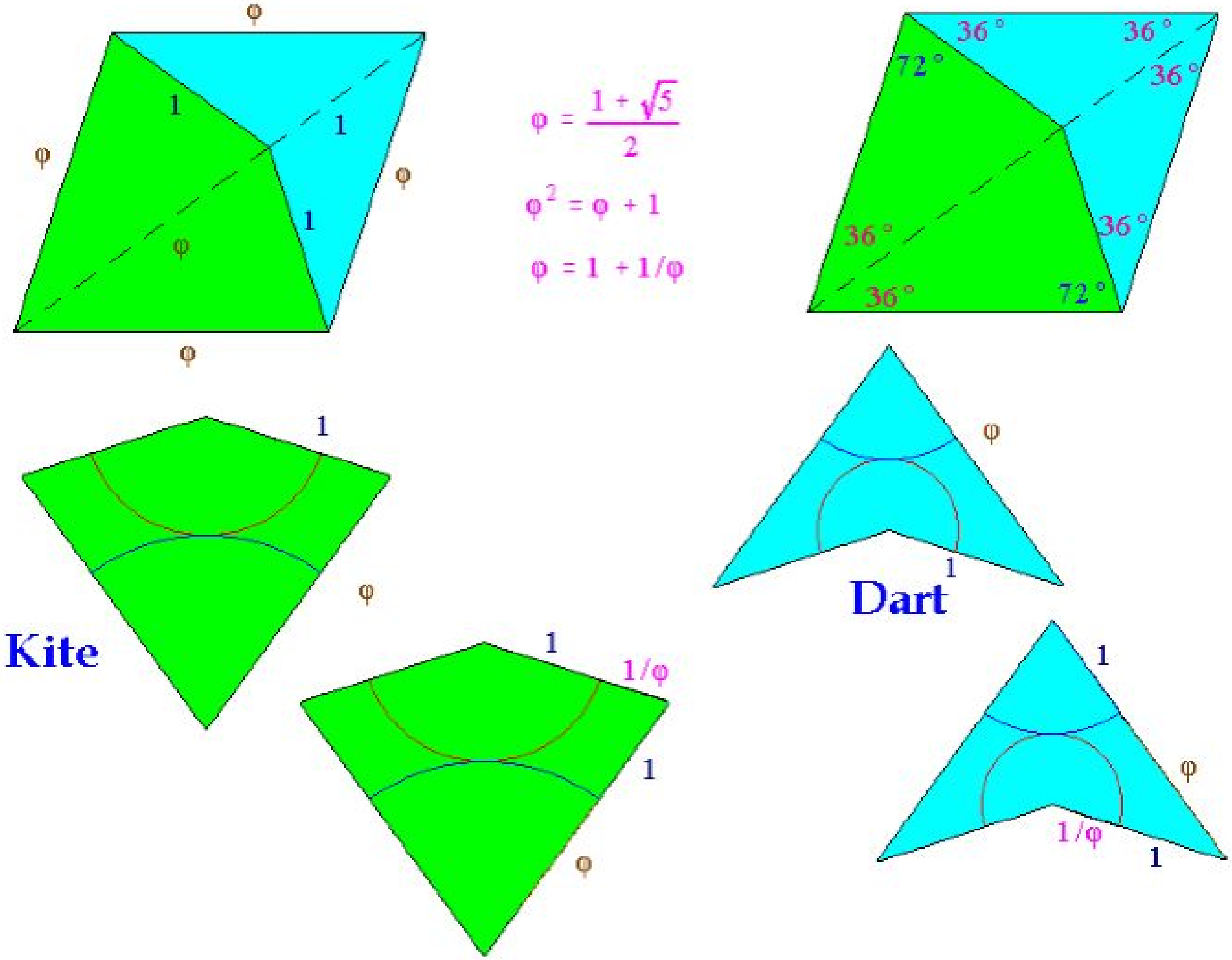
- Building the chair tiling -



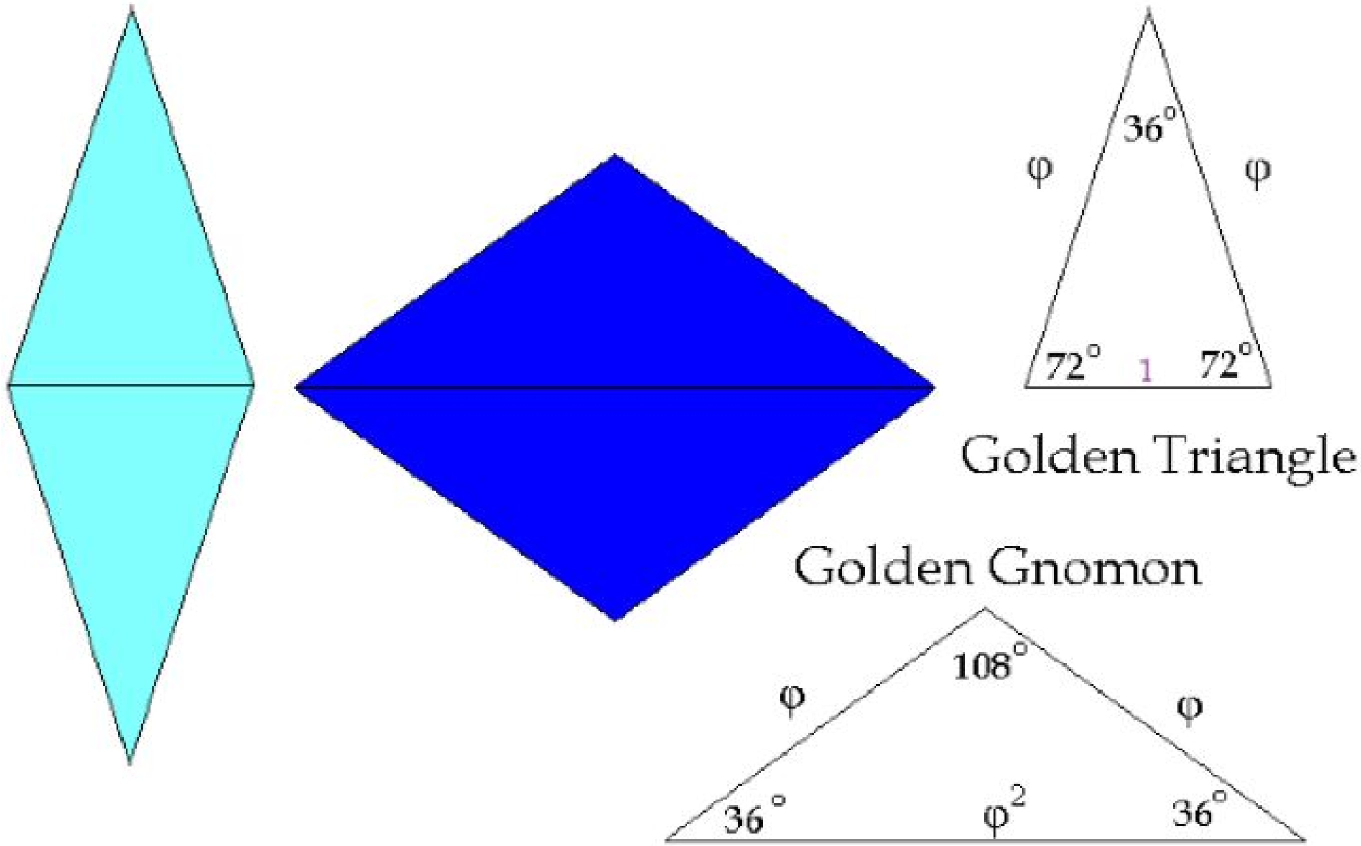
- The chair tiling -



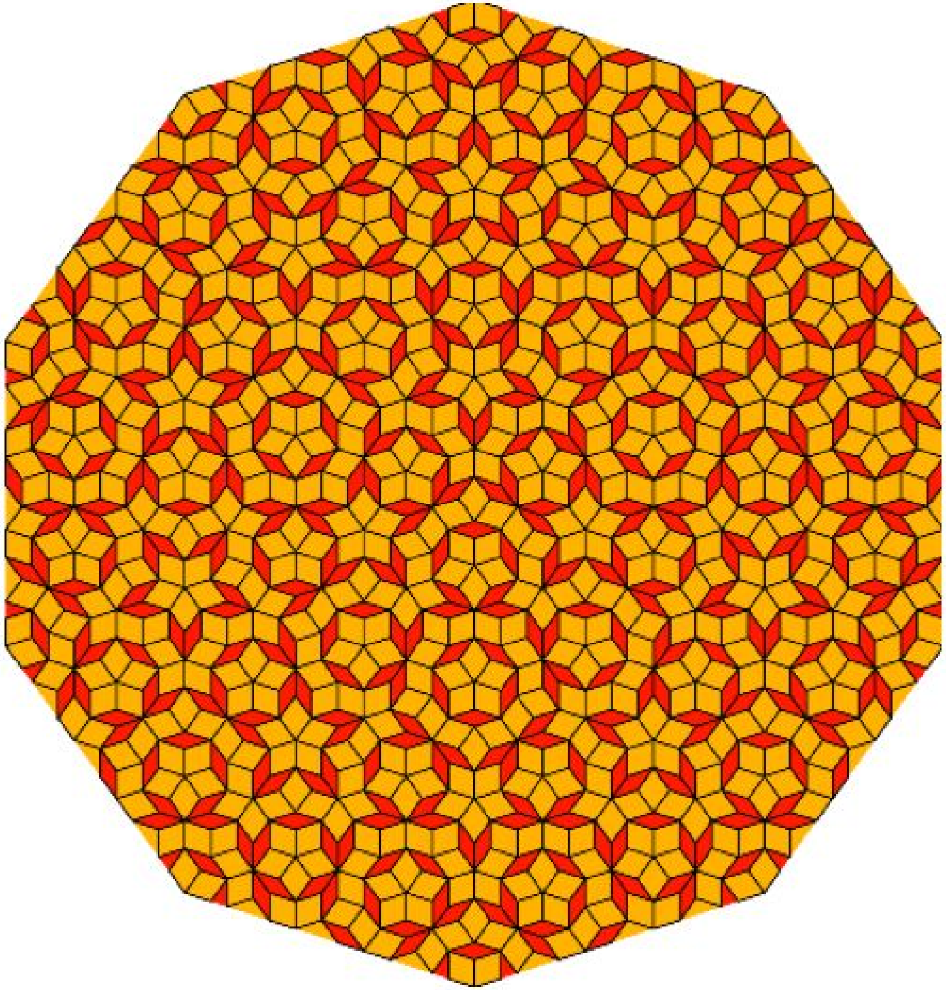
- The Penrose tiling -



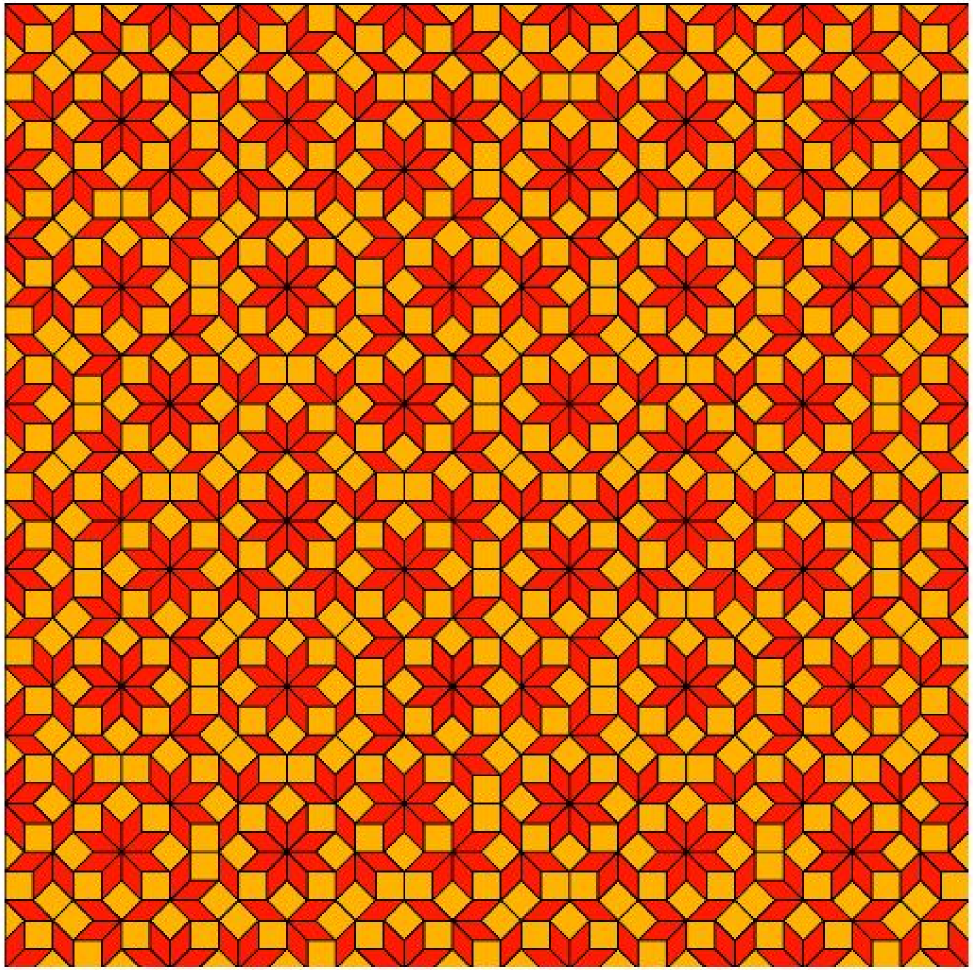
- Kites and Darts -



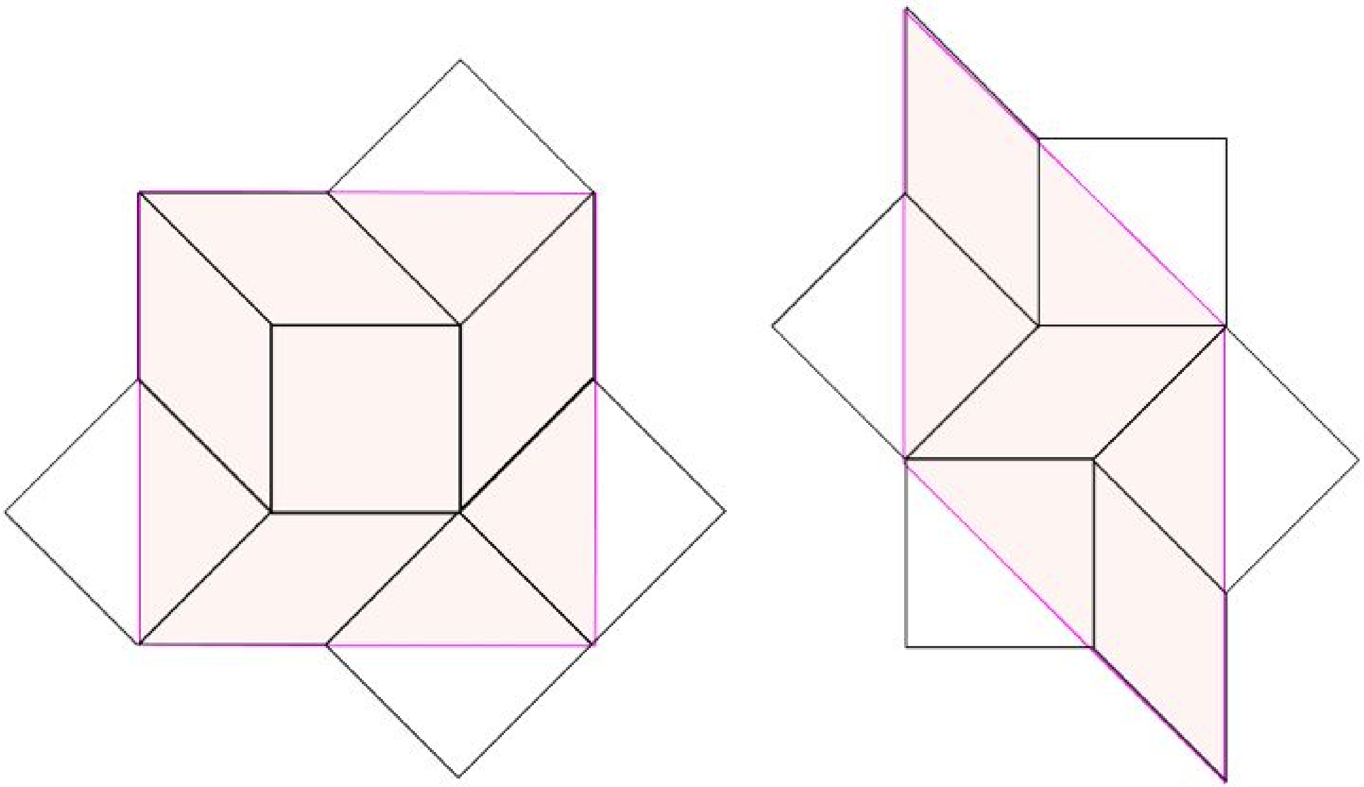
- Rhombi in Penrose’s tiling -



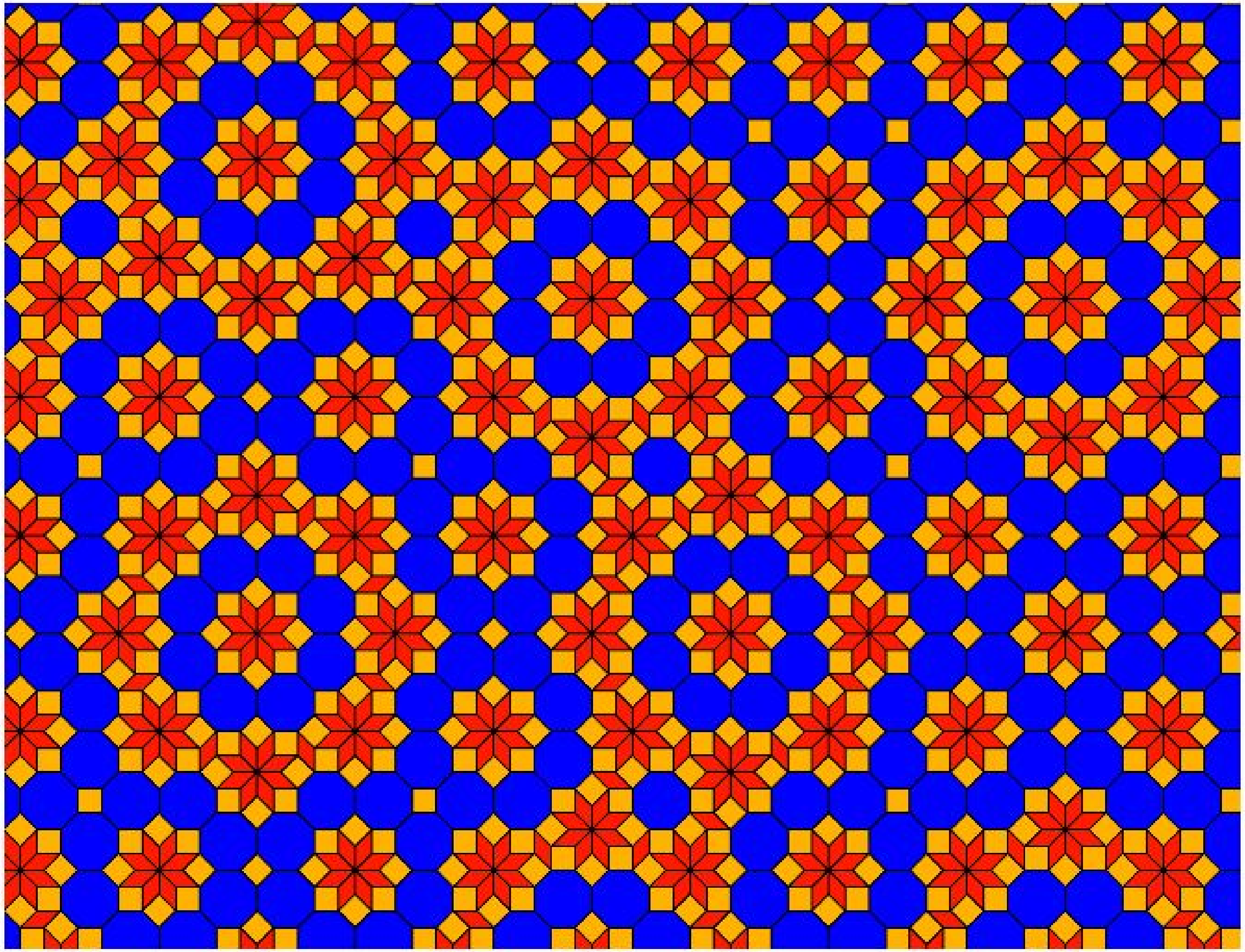
- The Penrose tiling -



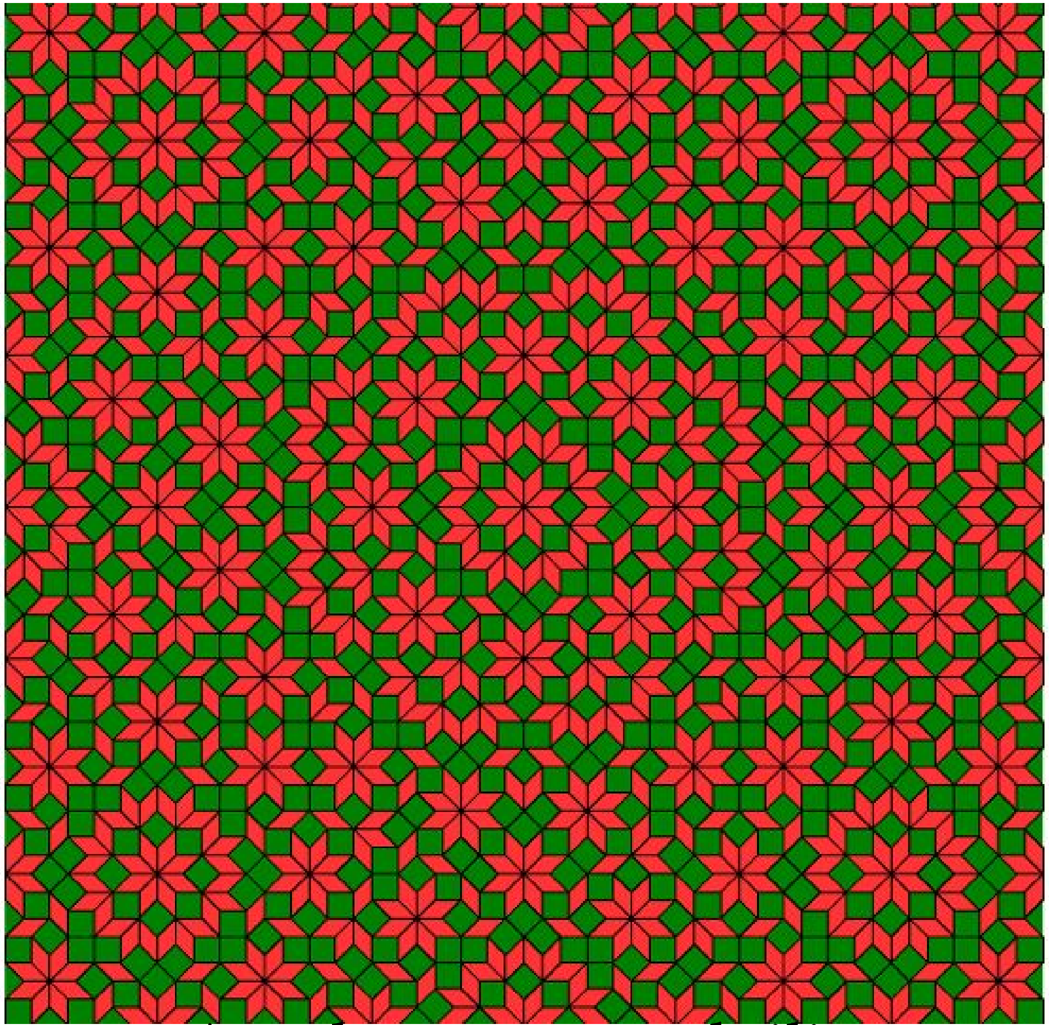
- The octagonal tiling -



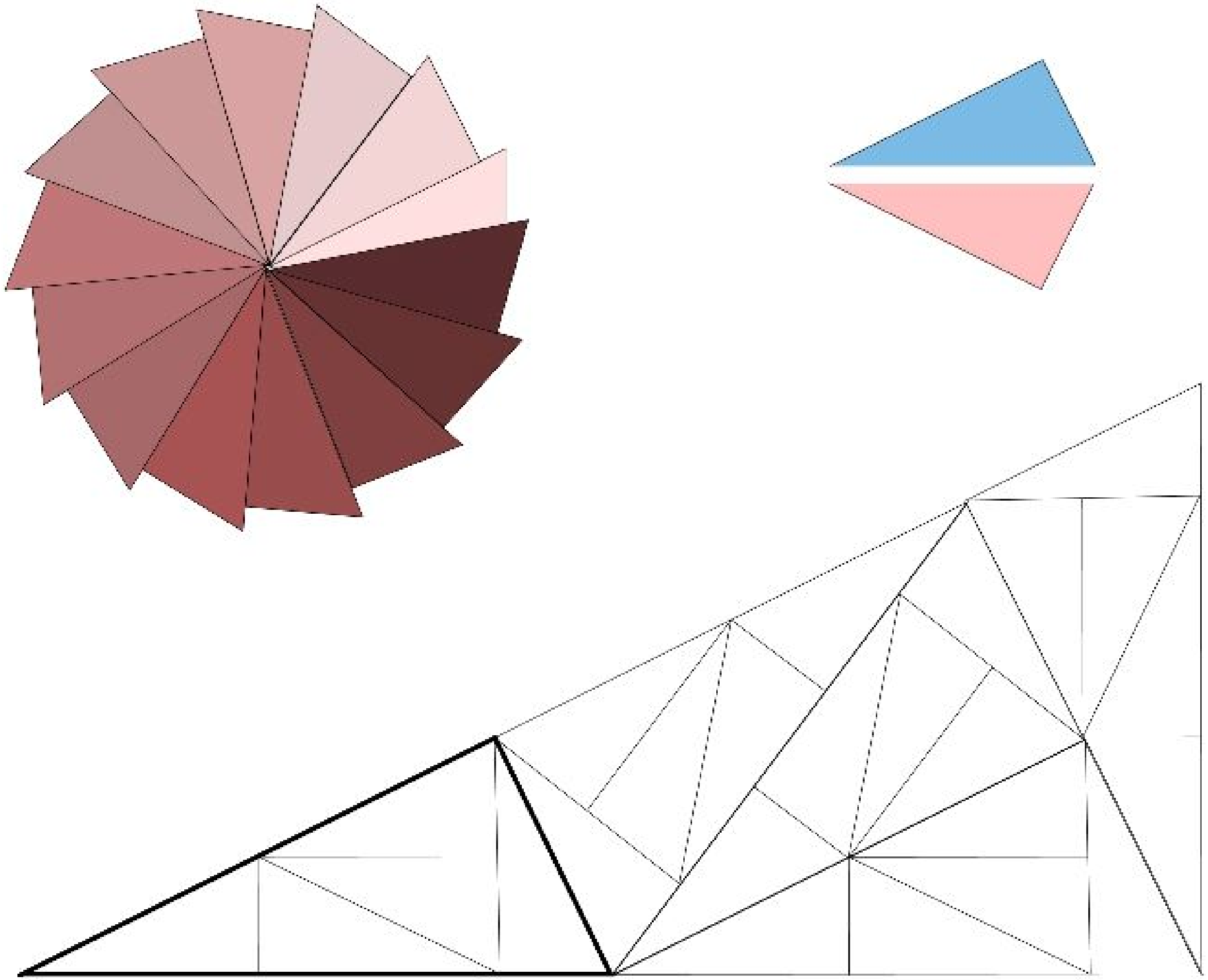
- Octagonal tiling: inflation rules -



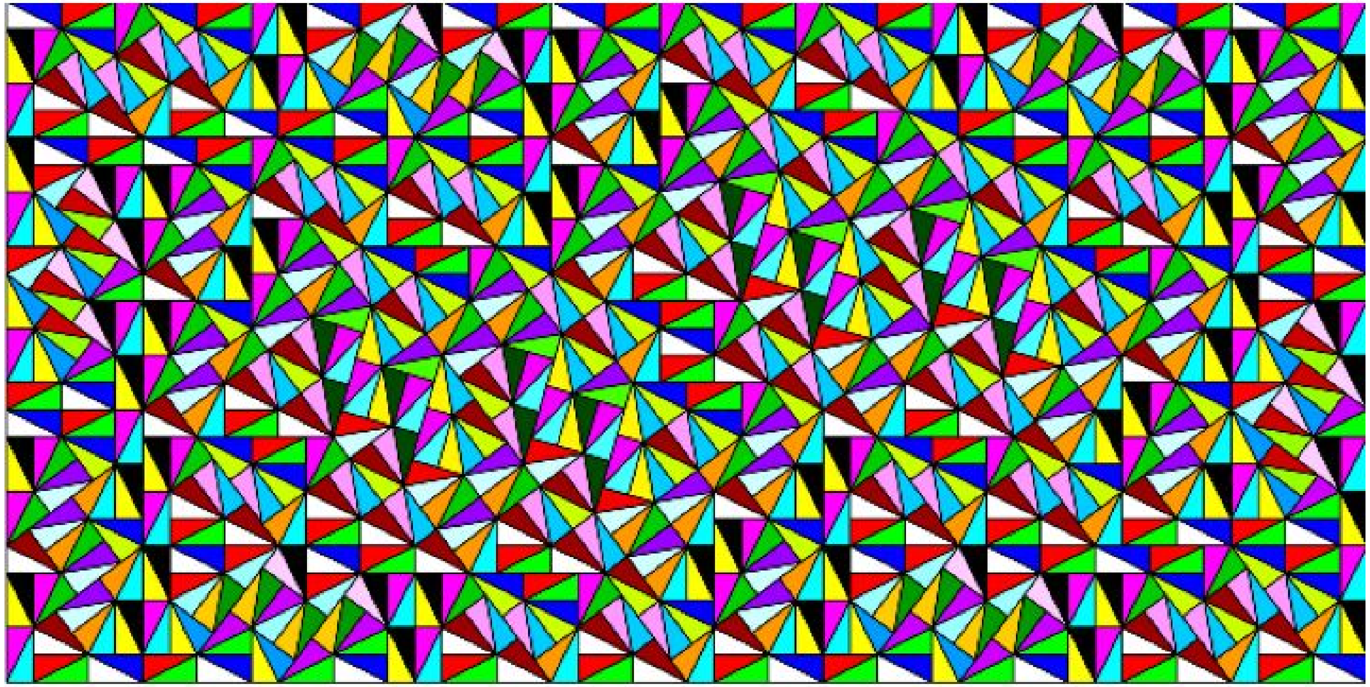
- Another octagonal tiling -



- Another octagonal tiling -



- Building the Pinwheel Tiling -



- The Pinwheel Tiling -

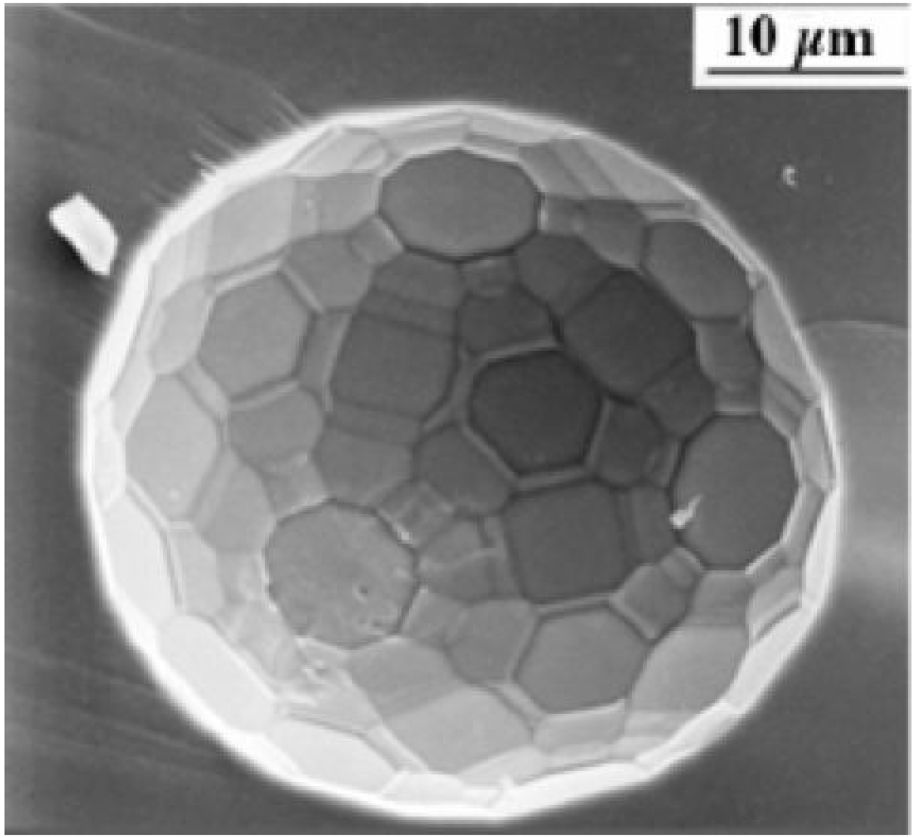
Aperiodic Materials

1. *Periodic Crystals* in d-dimensions:
   translation and crystal symmetries.
   Translation group $\mathcal{T} \simeq \mathbb{Z}^d$.

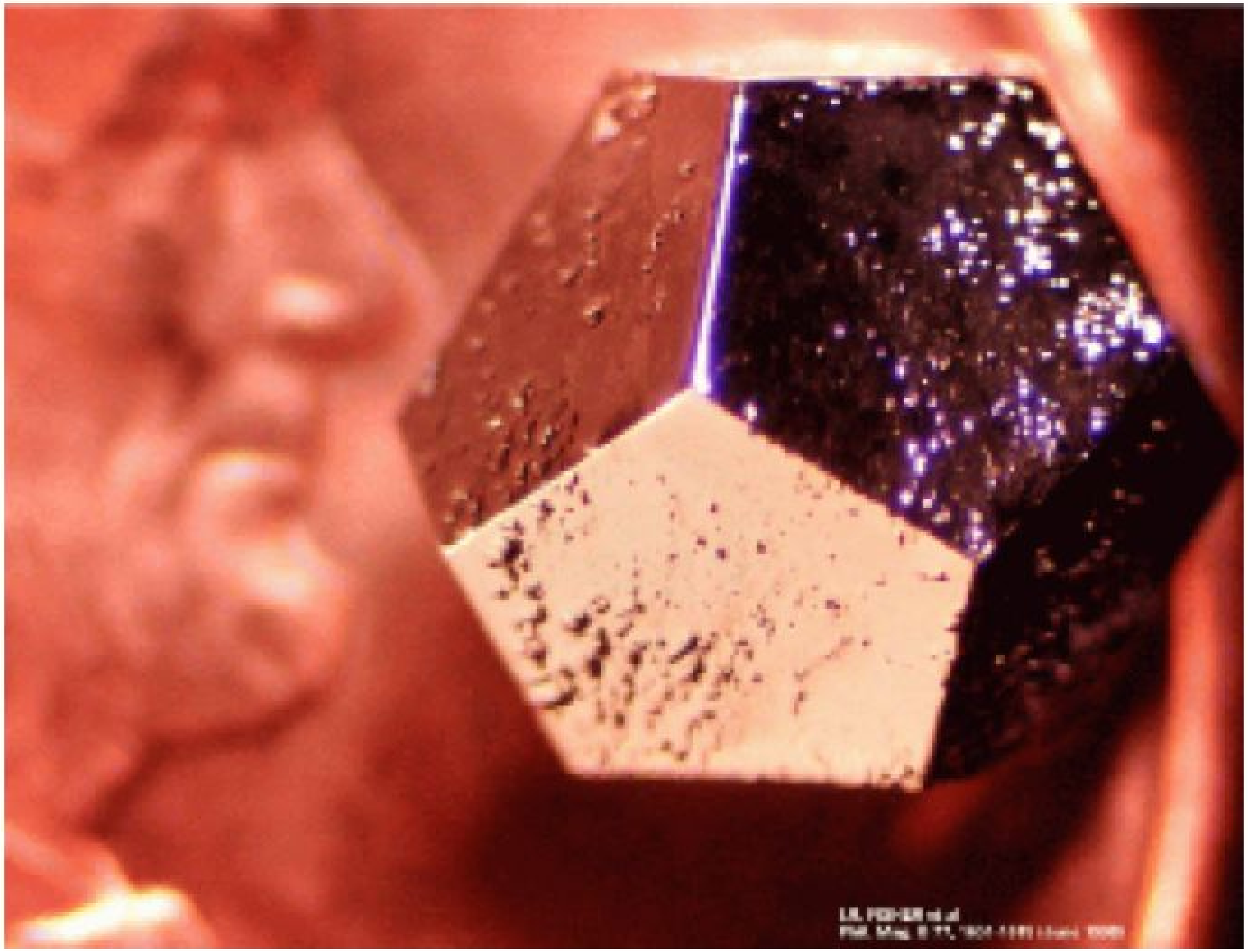
2. *Periodic Crystals in a Uniform Magnetic Field*;
   magnetic oscillations, Shubnikov-de Haas, de Haas-van Alfen.
   The magnetic field breaks the translation invariance to give some quasiperiodicity.

3. *Quasicrystals*: no translation symmetry, but icosahedral symmetry. Ex.:
   (a) $\mathbf{Al_{62.5}Cu_{25}Fe_{12.5}}$;
   (b) $\mathbf{Al_{70}Pd_{22}Mn_{8}}$;
   (c) $\mathbf{Al_{70}Pd_{22}Re_{8}}$;

4. *Disordered Media*: random atomic positions
   (a) Normal metals (with defects or impurities);
   (b) Alloys
   (c) Doped semiconductors (**Si**, **AsGa**, ...);



- The icosahedral quasicrystal *AlPdMn* -



- The icosahedral quasicrystal *HoMgZn*-

II - The Hull as a Dynamical System

Point Sets

A subset $\mathcal{L} \subset \mathbb{R}^d$ may be:

1. *Discrete*.
2. *Uniformly discrete*: $\exists r > 0$ s.t. each ball of radius r contains at most one point of $\mathcal{L}$.
3. *Relatively dense*: $\exists R > 0$ s.t. each ball of radius R contains at least one points of $\mathcal{L}$.
4. A *Delone* set: $\mathcal{L}$ is uniformly discrete and relatively dense.
5. *Finite Local Complexity (FLC)*: $\mathcal{L} - \mathcal{L}$ is discrete and closed.
6. *Meyer* set: $\mathcal{L}$ and $\mathcal{L} - \mathcal{L}$ are Delone.

Point Sets and Point Measures

$\mathfrak{M}(\mathbb{R}^d)$ is the set of Radon measures on $\mathbb{R}^d$ namely the dual space to $C_c(\mathbb{R}^d)$ (continuous functions with compact support), endowed with the weak* topology.

For $\mathcal{L}$ a *uniformly discrete* point set in $\mathbb{R}^d$:

$$\nu := \nu^{\mathcal{L}} = \sum_{y \in \mathcal{L}} \delta(x - y) \quad \in \mathfrak{M}(\mathbb{R}^d) .$$

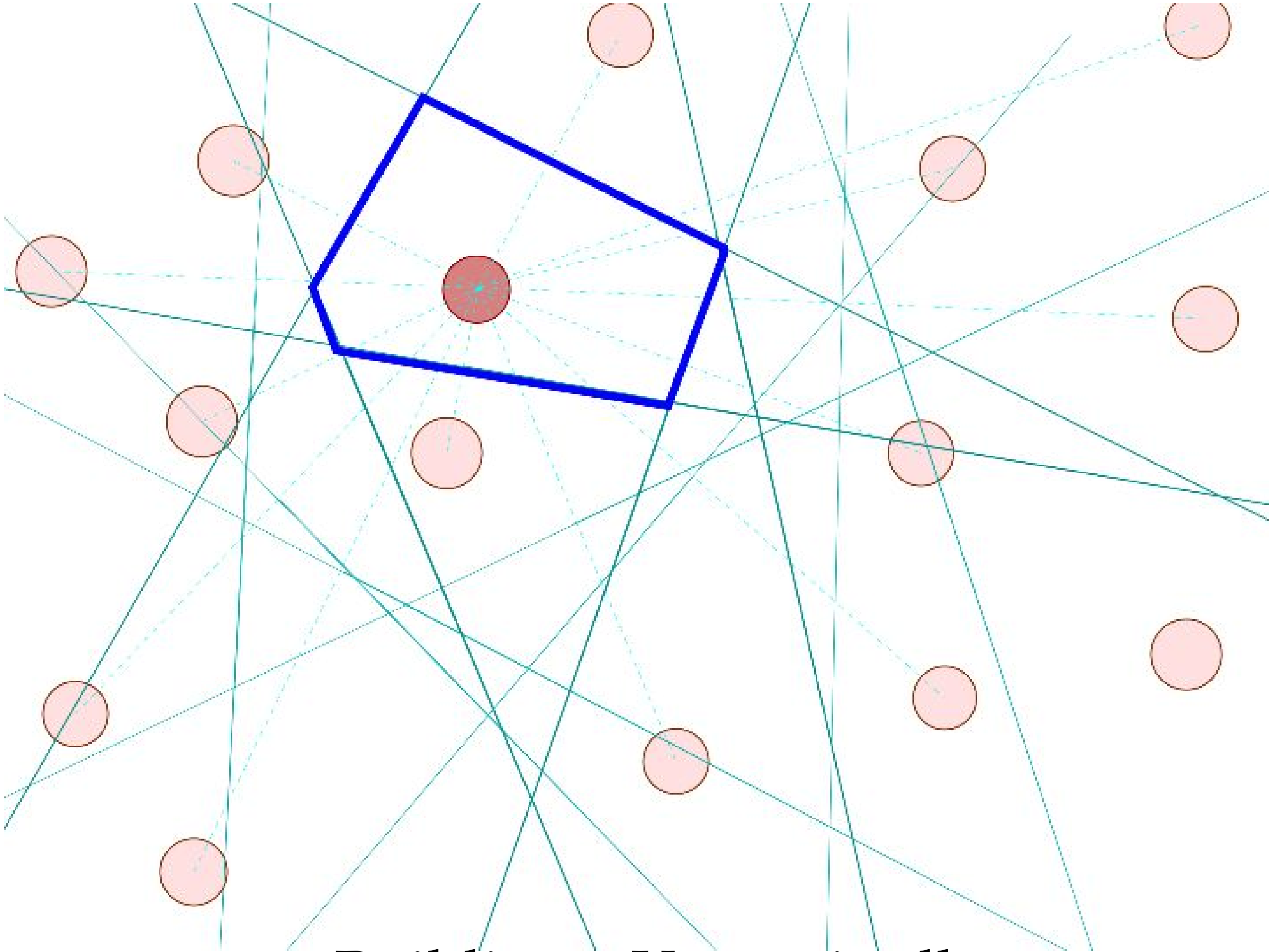
Point Sets and Tilings

Given a tiling with finitely many tiles *(modulo translations)*, a Delone set is obtained by defining a point in the interior of each *(translation equivalence class of)* tile.

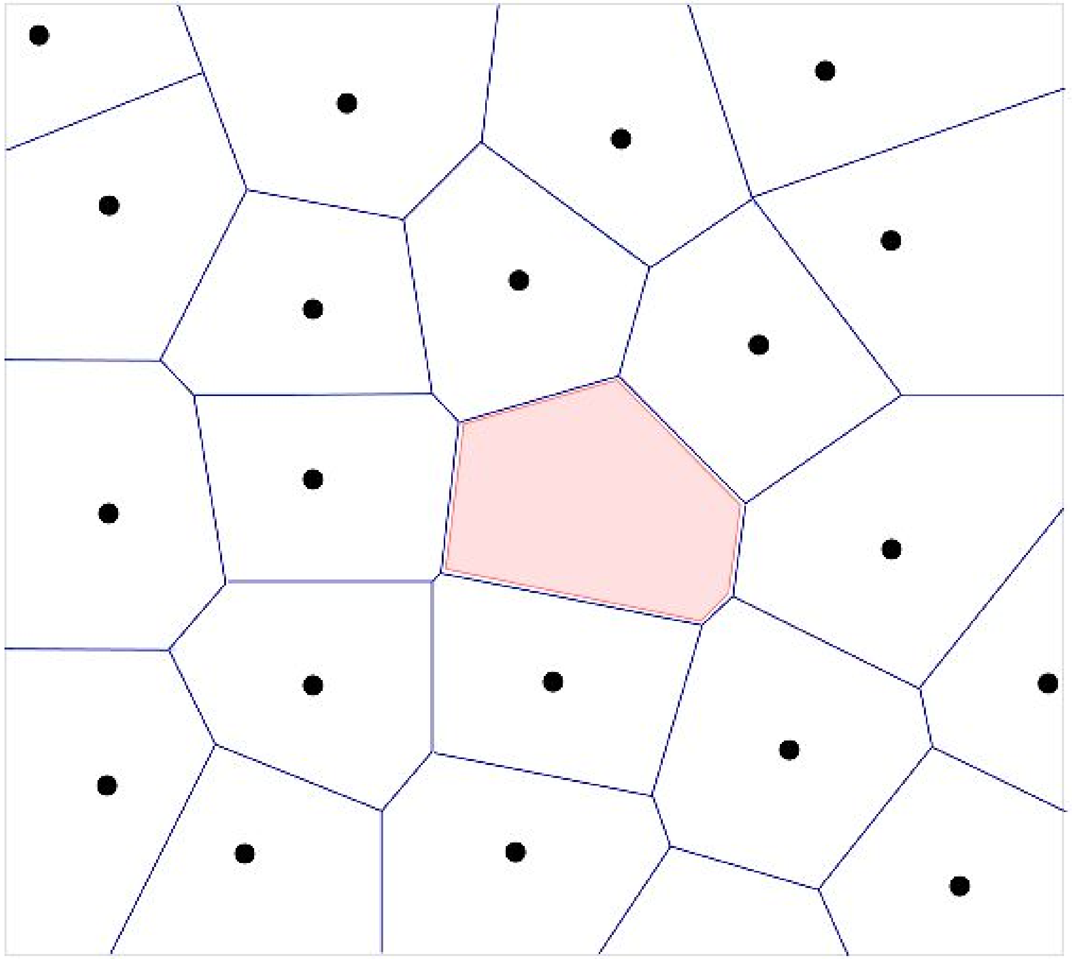
Conversely, given a Delone set, a tiling is built through the *Voronoi cells*

$$V(x) = \{a \in \mathbb{R}^d \, ; \, |a - x| < |a - y| \, , \forall y \mathcal{L} \setminus \{x\}\}$$

1. $V(x)$ is an *open convex polyhedron* containing $B(x;r)$ and contained into $\overline{B(x;R)}$.
2. Two Voronoi cells touch face-to-face.
3. If $\mathcal{L}$ is *FLC*, then the Voronoi tiling has finitely many tiles modulo translations.



- Building a Voronoi cell-



- A Delone set and its Voronoi Tiling-

The Hull

A point measure is $\mu \in \mathfrak{M}(\mathbb{R}^d)$ such that $\mu(B) \in \mathbb{N}$ for any ball $B \subset \mathbb{R}^d$. Its support is

1. *Discrete*.
2. *r-Uniformly discrete*: iff $\forall B$ ball of radius r, $\mu(B) \leq 1$.
3. *R-Relatively dense*: iff for each ball B of radius R, $\mu(B) \geq 1$.

$\mathbb{R}^d$ acts on $\mathfrak{M}(\mathbb{R}^d)$ by translation.

Theorem 1 *The set of r-uniformly discrete point measures is compact and $\mathbb{R}^d$-invariant.*
Its subset of R-relatively dense measures is compact and $\mathbb{R}^d$-invariant.

Definition 1 *Given $\mathcal{L}$ a uniformly discrete subset of $\mathbb{R}^d$, the Hull of $\mathcal{L}$ is the closure in $\mathfrak{M}(\mathbb{R}^d)$ of the $\mathbb{R}^d$-orbit of $\nu^{\mathcal{L}}$.*

Hence the Hull is a *compact metrizable space* on which $\mathbb{R}^d$ *acts by homeomorphisms*.

Properties of the Hull

If $\mathcal{L} \subset \mathbb{R}^d$ is r-uniformly discrete with Hull Ω then using compactness

1. each point $\omega \in \Omega$ *is an r-uniformly discrete* point measure with support $\mathcal{L}_\omega$.
2. if $\mathcal{L}$ is (r, R)*-Delone*, so are all $\mathcal{L}_\omega$'s.
3. if, in addition, $\mathcal{L}$ is *FLC*, so are all the $\mathcal{L}_\omega$'s.
   Moreover then $\mathcal{L} - \mathcal{L} = \mathcal{L}_\omega - \mathcal{L}_\omega \ \forall \omega \in \Omega$.

Definition 2 *The transversal of the Hull Ω of a uniformly discrete set is the set of $\omega \in \Omega$ such that $0 \in \mathcal{L}_\omega$.*

Theorem 2 *If $\mathcal{L}$ is FLC, then its transversal is completely discontinuous.*

Local Isomorphism Classes and Tiling Space

A *patch* is a finite subset of $\mathcal{L}$ of the form

$$p = (\mathcal{L} - x) \cap \overline{B(0, r_1)} \qquad x \in \mathcal{L},\ r_1 \geq 0$$

Given $\mathcal{L}$ a repetitive, FLC, Delone set let $\mathcal{W}$ be its set of finite patches: it is called the *the $\mathcal{L}$-dictionary*.

A Delone set (or a Tiling) $\mathcal{L}'$ is *locally isomorphic* to $\mathcal{L}$ if it has the same dictionary. The *Tiling Space* of $\mathcal{L}$ is the set of *Local Isomorphism Classes* of $\mathcal{L}$.

Theorem 3 *The Tiling Space of $\mathcal{L}$ coincides with its Hull.*

Minimality

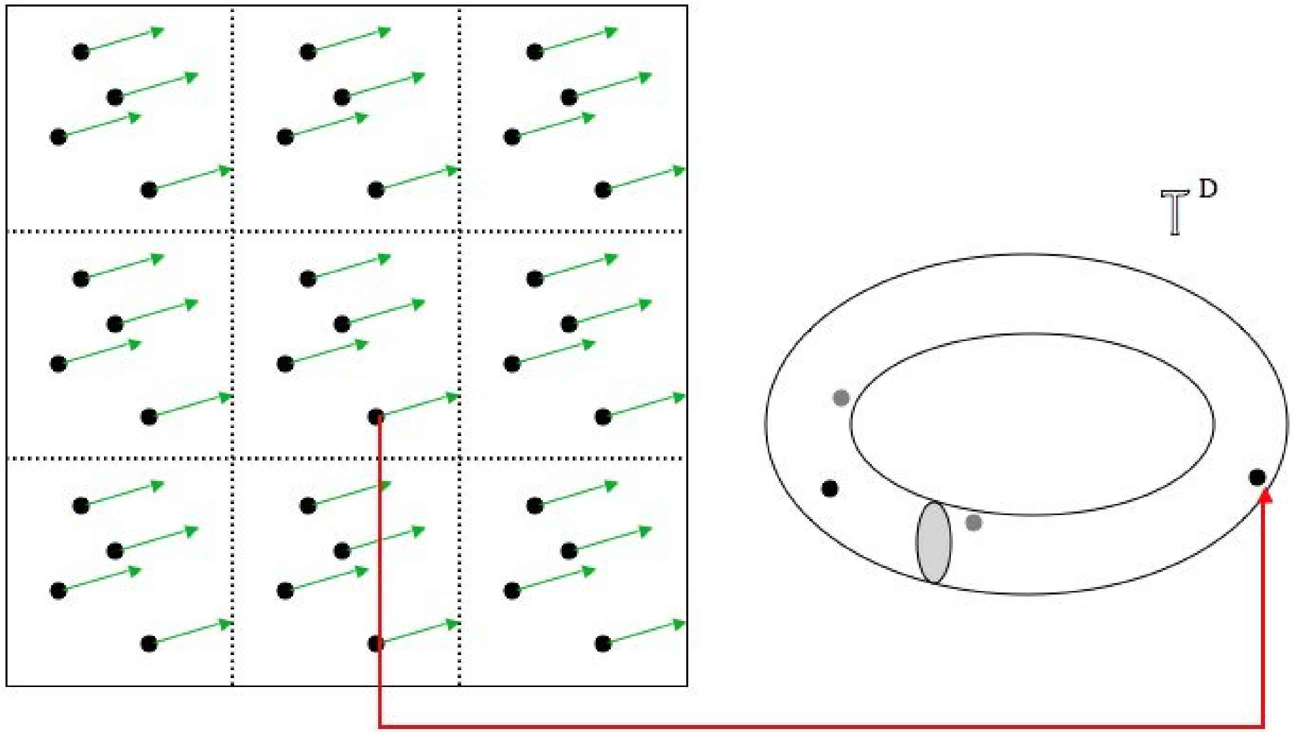
$\mathcal{L}$ is *repetitive* if for any finite patch p there is $R > 0$ such that each ball of radius R contains an ϵ-approximant of a translated of p.

Theorem 4 *$\mathbb{R}^d$ acts minimaly on Ω if and only if $\mathcal{L}$ is repetitive.*

Examples

1. *Crystals :* $\Omega = \mathbb{R}^d/\mathcal{T} \simeq \mathbb{T}^d$ with the quotient action of $\mathbb{R}^d$ on itself. (Here $\mathcal{T}$ is the translation group leaving the lattice invariant. $\mathcal{T}$ is isomorphic to $\mathbb{Z}^D$.)
   The transversal is a finite set (number of point per unit cell).

2. *Impurities in* **Si** *:* let $\mathcal{L}$ be the lattices sites for *Si* atoms (it is a Bravais lattice). Let $\mathfrak{A}$ be a finite set (alphabet) indexing the types of impurities.
   The transversal is $X = \mathfrak{A}^{\mathbb{Z}^d}$ with $\mathbb{Z}^d$-action given by shifts.
   The Hull Ω is the mapping torus of X.



- The Hull of a Periodic Lattice -

Quasicrystals

Use the *cut-and-project* construction:

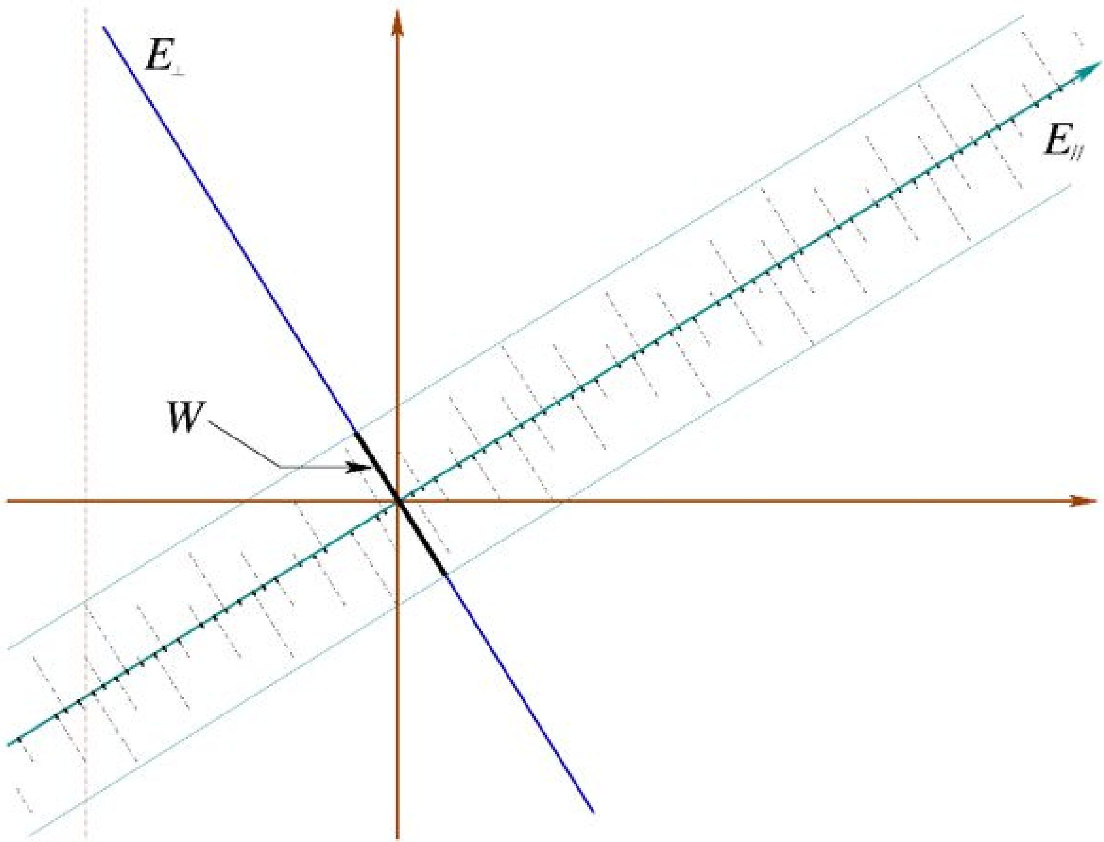
$$\mathbb{R}^d \simeq \mathcal{E}_{||} \xleftarrow{\pi_{||}} \mathbb{R}^n \xrightarrow{\pi_\perp} \mathcal{E}_\perp \simeq \mathbb{R}^{n-d}$$

$$\mathcal{L} \xleftarrow{\pi_{||}} \tilde{\mathcal{L}} \xrightarrow{\pi_\perp} W \quad ,$$

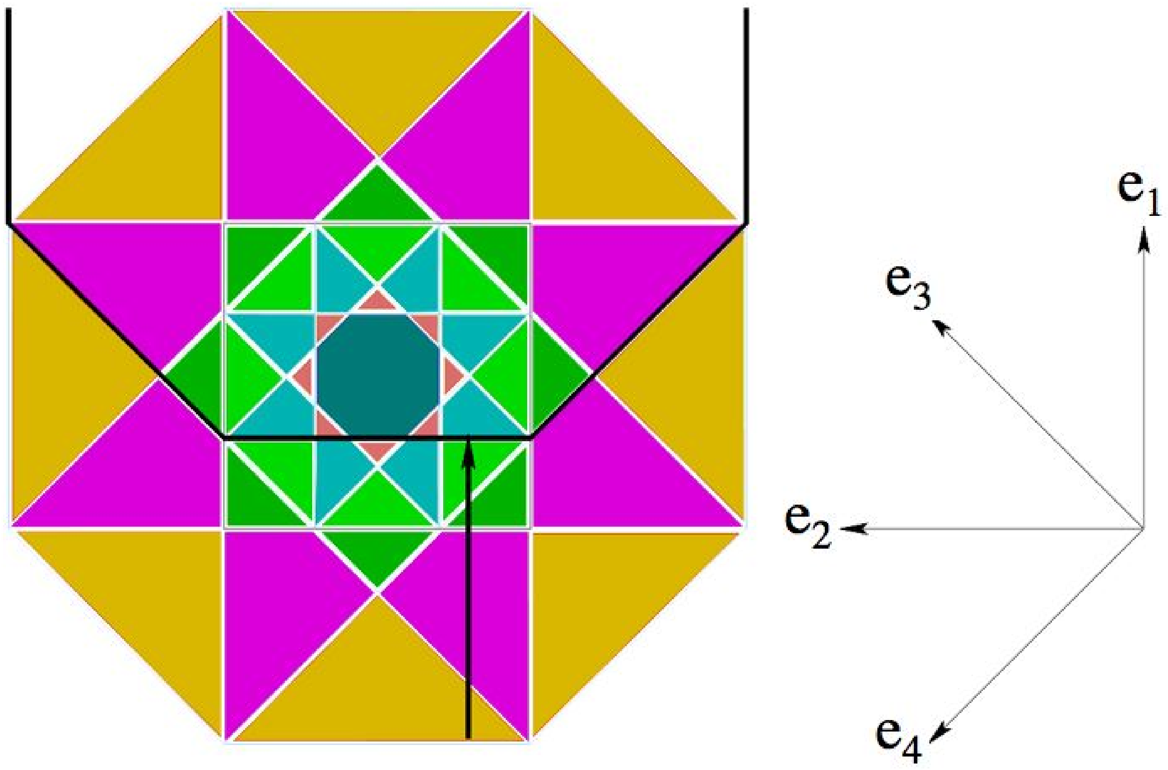
Here

1. $\tilde{\mathcal{L}}$ is a *lattice* in $\mathbb{R}^n$,
2. the *window* W is a compact polytope.
3. $\mathcal{L}$ is the *quasilattice* in $\mathcal{E}_{||}$ defined as

$$\mathcal{L} \;=\; \{\pi_{||}(m) \in \mathcal{E}_{||}\,;\, m \in \tilde{\mathcal{L}},\, \pi_\perp(m) \in W\}$$



– The cut–and–project construction –



- The transversal of the Octagonal Tiling is completely disconnected -

III - Branched Oriented Flat Riemannian Manifolds

Laminations and Boxes

A *lamination* is a foliated manifold with C^∞-structure along the leaves but only finite C^0-structure transversally. The *Hull of a Delone set is a lamination* with $\mathbb{R}^d$-orbits as leaves.

If $\mathcal{L}$ is a *FLC, repetitive, Delone* set, with Hull Ω a *box* is the homeomorphic image of

$$\phi : (\omega, x) \in F \times U \mapsto \mathrm{T}^{-x}\omega \in \Omega$$

if F is a clopen subset of the transversal, $U \subset \mathbb{R}^d$ is open and T denotes the $\mathbb{R}^d$-action on Ω.

A *quasi-partition* is a family $(B_i)_{i=1}^n$ of boxes such that $\bigcup_i \overline{B_i} = \Omega$ and $B_i \cap B_j = \emptyset$.

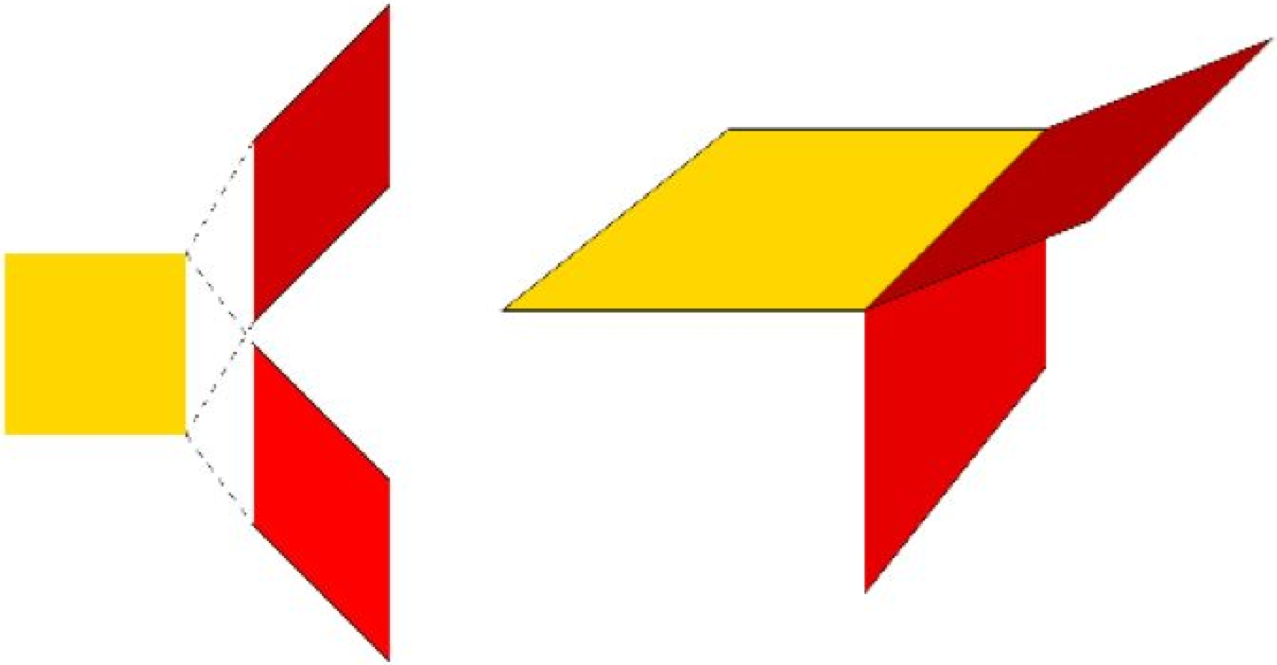
Theorem 5 *The Hull of a FLC, repetitive, Delone set admits a finite quasi-partition. It is always possible to choose these boxes as $\phi(F \times U)$ with U a d-rectangle.*

Branched Oriented Flat Manifolds

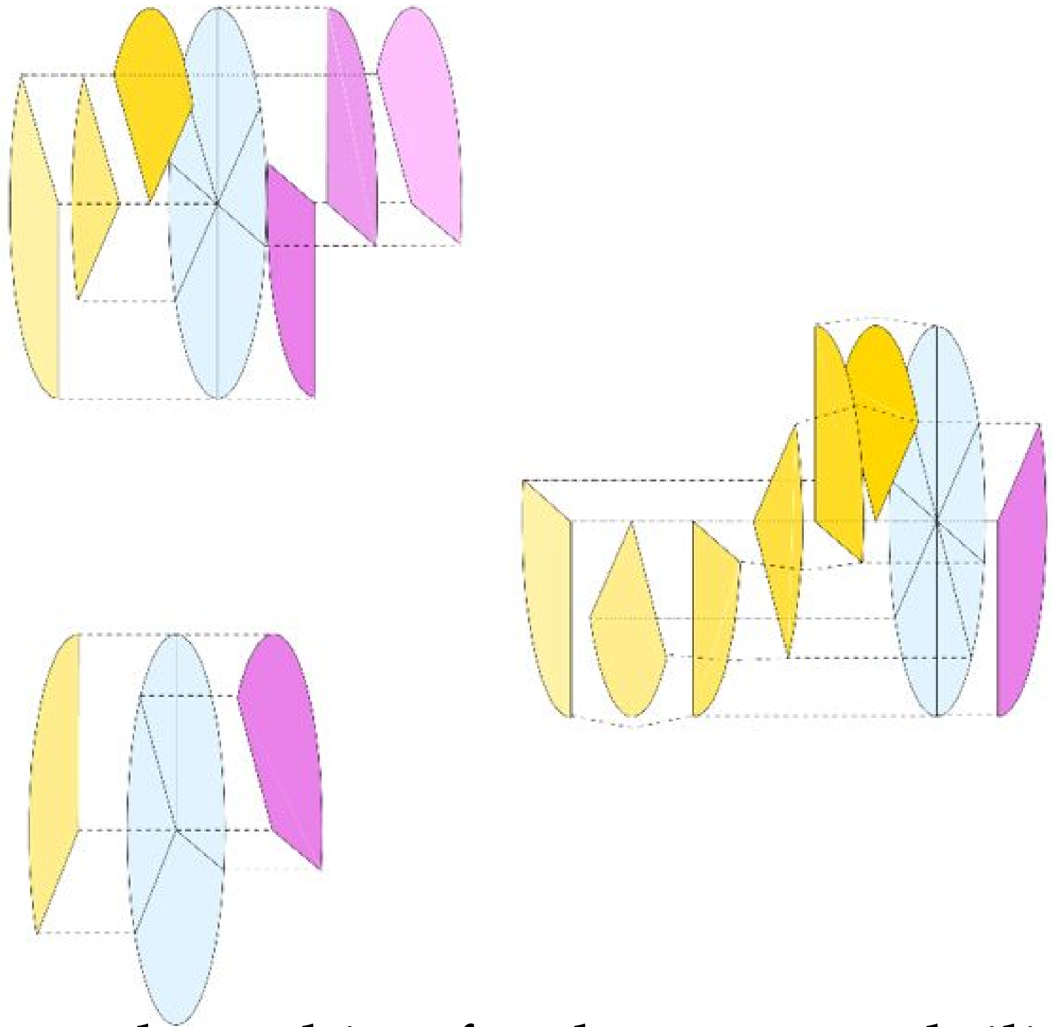
Flattening a box decomposition along the transverse direction leads to a *Branched Oriented Flat manifold*. Such manifolds can be built from the tiling itself as follows

Step 1:

1. X is the disjoint union of all *prototiles*;
2. glue prototiles T_1 and T_2 along a face $F_1 \subset T_1$ and $F_2 \subset T_2$ if F_2 is a translated of F_1 and if there are $x_1, x_2 \in \mathbb{R}^d$ such that $x_i + T_i$ are tiles of $\mathcal{T}$ with $(x_1 + T_1) \cap (x_2 + T_2) = x_1 + F_1 = x_2 + F_2$;
3. after identification of faces, X becomes a *branched oriented flat manifold* (BOF) B_0.



- Branching -



- Vertex branching for the octagonal tiling -

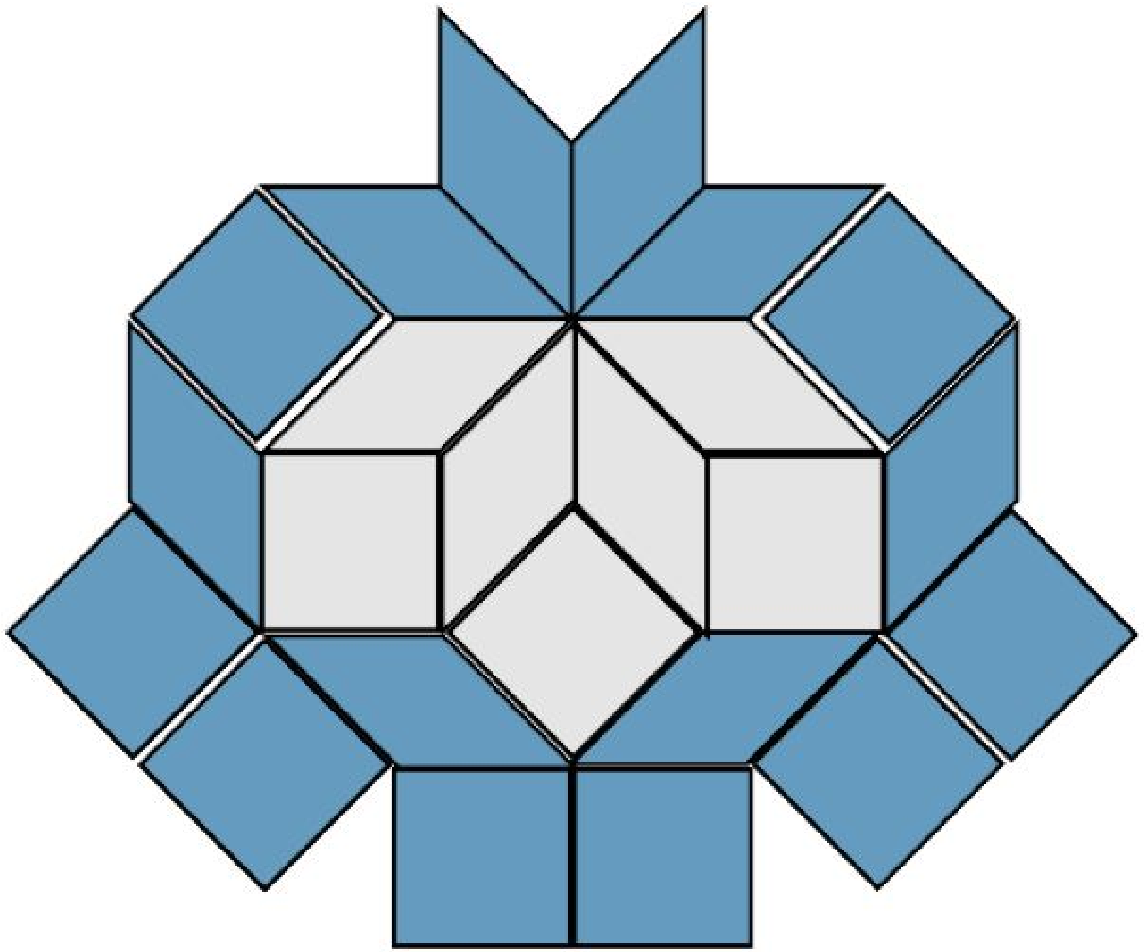
Step 2:

1. Having defined the patch p_n for $n \geq 0$, let $\mathcal{L}_n$ be the subset of $\mathcal{L}$ of points centered at a translated of p_n. By repetitivity this is a FLC repetitive Delone set too. Its prototiles are tiled by tiles of $\mathcal{L}$ and define a finite family $\mathfrak{P}_n$ of patches.

2. Each patch in $\mathcal{T} \in \mathfrak{P}_n$ will be collared by the patches of $\mathfrak{P}_{n-1}$ touching it from outside along its frontier. Call such a patch *modulo translation* a *a collared patch* and $\mathfrak{P}_n^c$ their set.

3. Proceed then as in Step 1 by replacing prototiles by collared patches to get the BOF-manifold B_n.

4. Then choose p_{n+1} to be the collared patch in $\mathfrak{P}_n^c$ containing p_n.



- A collared patch -

Step 3:

1. Define a *BOF-submersion* $f_n : B_{n+1} \mapsto B_n$ by identifying patches of order n in B_{n+1} with the prototiles of B_n. Note that $Df_n = \mathbf{1}$.
2. Call Ω the *projective limit* of the sequence

$$\cdots \xrightarrow{f_{n+1}} B_{n+1} \xrightarrow{f_n} B_n \xrightarrow{f_{n-1}} \cdots$$

3. $X_1, \cdots X_d$ are the commuting constant vector fields on B_n generating local translations and giving rise to a $\mathbb{R}^d$ action τ on Ω.

Theorem 6 *The dynamical system*

$$(\Omega, \mathbb{R}^d, \tau) = \varprojlim (B_n, f_n)$$

obtained as inverse limit of branched oriented flat manifolds, is conjugate to the Hull of the Delone set of the tiling $\mathcal{T}$ by an homemorphism.

IV - Cohomology and *K*-Theory

Čech Cohomology of the Hull

Let $\mathcal{U}$ be an *open covering* of the Hull. If $U \in \mathcal{U}$, $\mathcal{F}(U)$ is the space of integer valued locally constant function on U.

For $n \in \mathbb{N}$, the n-chains are the element of $C^n(\mathcal{U})$, namely the *free abelian group* generated by the elements of $\mathcal{F}(U_0 \cap \cdots \cap U_n)$ when the U_i varies in $\mathcal{U}$. A differential is defined by

$$d : C^n(\mathcal{U}) \mapsto C^{n+1}(\mathcal{U})$$

$$df\left(\bigcap_{i=0}^{n+1} U_i\right) = \sum_{j=0}^{n} (-1)^j f\left(\bigcap_{i:i\neq j} U_i\right)$$

This defines a *complex* with cohomology $\check{H}^n(\mathcal{U}, \mathbb{Z})$. The Čech cohomology group of the Hull Ω is defined as

$$\check{H}^n(\Omega, \mathbb{Z}) = \lim_{\rightarrow \mathcal{U}} \check{H}^n(\mathcal{U}, \mathbb{Z})$$

with ordering given by *refinement* on the set of open covers.

Longitudinal (co)-Homology

J. Bellissard, R. Benedetti, J.-. Gambaudo, Commun. Math. Phys., **261**, (2006), 1-41.
J. Kaminker, I. Putnam, Michigan Math. J., **51**, (2003), 537-546.
M. Benameur, H. Oyono-Oyono, C. R. Math. Acad. Sci. Paris, **334**, (2002), 667-670.

The Homology groups are defined by the inverse limit

$$H_*(\Omega, \mathbb{R}^d) = \lim_{\leftarrow} (H_*(B_n, \mathbb{R}), f_n^*)$$

Theorem 7 (JB, Benedetti, Gambaudo) *The homology group $H_d(\Omega, \mathbb{R}^d)$ admits a canonical positive cone induced by the orientation of $\mathbb{R}^d$, isomorphic to the affine set of positive $\mathbb{R}^d$-invariant measures on Ω.*

The cohomology groups are defined by the direct limit

$$H^*(\Omega, \mathbb{R}^d) = \varinjlim (H^*(B_n, \mathbb{R}), f_n^*)$$

The following result is known as the *Gap labeling Theorem* and was proved simultaneously by Kaminker-Putnam, Benameur & Oyono-Oyono, JB-Bendetti-Gambaudo. It is an extension of the *Connes index theorem* for foliations

Theorem 8 *If $\mathbb{P}$ is an $\mathbb{R}^d$-invariant probability on Ω, then the pairing with $H^d(\Omega, \mathbb{R}^d)$ satisfies*

$$\langle \mathbb{P} | H^d(\Omega, \mathbb{R}^d) \rangle = \int_\Xi d\mathbb{P}_{\text{tr}}\ C(\Xi, \mathbb{Z})$$

where Ξ is the transversal, $\mathbb{P}_{\text{tr}}$ is the probability on Ξ induced by $\mathbb{P}$ and $C(\Xi, \mathbb{Z})$ is the space of integer valued continuous functions on Ξ.

Pattern-Equivariant Cohomology

J. Kellendonk, J. Phys. **A36**, (2003), 5765-5772.
J. Kellendonk, I. Putnam, Math. Ann. **334**, (2006), 693-711.
L. Sadun, *Pattern-Equivariant Cohomology with Integer Cœfficients* (2007)

Let $\mathcal{L}$ be an FLC, repetitive Delone set in $\mathbb{R}^d$. A function $f : \mathbb{R}^d \mapsto X$ is *$\mathcal{L}$-pattern-equivariant* if there is $r > 0$ such that $f(x) = f(y)$ whenever $B(0;r) \cap (\mathcal{L} - x) = B(0;r) \cap (\mathcal{L} - y)$.

The Voronoi tiling of $\mathcal{L}$ can be seen as a *chain complex*, with tiles being the d-cells, and their k-faces being the k-cells.

A *k-cochain* with integer cœfficients is then a linear map α defined on the free abelian group of k-chains with values in $\mathbb{Z}$.

Let $C^k_{\mathcal{P}}(\mathcal{L})$ be the abelian group of $\mathcal{L}$-pattern equivariant k-cochains. The usual coboundary operator *(de Rham differential)*

$$d_n : C^n_{\mathcal{P}}(\mathcal{L}) \mapsto C^{n+1}_{\mathcal{P}}(\mathcal{L})$$

defines the *$\mathcal{L}$-pattern equivariant cohomology* denoted by

$$H^k_{\mathcal{P}}(\mathcal{L}, \mathbb{Z}) \; = \; \mathrm{Ker}\, d_n / \mathrm{Im}\, d_{n-1}$$

The PV-Cohomology

J. BELLISSARD, J.SAVINIEN, *arXiv*: 0705.2483, (2007).

Each cell of the *Voronoi complex* is punctured. The set $\mathcal{L}_s$ of such punctures defines the *simplicial transversal* Ξ_s. An equivalent class, modulo translation, of n-cell σ defines a compact subset $\Xi_s(\sigma)$. χ_σ denotes the characteristic function of $\Xi_s(\sigma)$.

If σ is such a cell and τ belongs to its boundary, then there is a unique vector $x_{\sigma\tau}$ joining the puncture of τ to the one of σ. Correspondingly the translation $\mathrm{T}^{x_{\sigma\tau}}$ in the Hull sends $\Xi_s(\tau)$ into a part of $\Xi_s(\tau)$, defining the translation operator

$$\theta_{\sigma\tau} = \chi_\sigma \mathrm{T}^{x_{\sigma\tau}} \chi_\tau$$

where χ_σ denotes the characterictic function of $\Xi_s(\sigma)$.

A *PV-n-cochain* will be a group homomorphism from the group of (oriented) n-chains on the BOF manifold B_0 into the group $C(\Xi_s, \mathbb{Z})$. The Pimsner differential is defined by

$$df(\sigma) = \sum_{\tau \in \partial\sigma} [\sigma : \tau] f(\tau) \circ \theta_{\sigma\tau}$$

Here $[\sigma : \tau]$ denoted the *incidence number* of τ relative to σ. The associate cohomology is $H^n_P(B_0, C(\Xi_s, \mathbb{Z}))$.

Cohomology and K-theory

The main topological property of the Hull (or tiling psace) is summarized in the following

Theorem 9 *(i) The various cohomologies, Čech, longitudinal, pattern-equivariant and PV, are isomorphic.*
(ii) There is a spectral sequence converging to the K-group of the Hull with page 2 given by the cohomology of the Hull.
(iii) In dimension $d \leq 3$ the K-group coincides with the cohomology.

Conclusion

1. *Tilings* can be equivalently be represented by *Delone sets* or *point measures*.

2. The *Hull* allows to give tilings the structure of a *dynamical system* with a transversal.

3. This dynamical system can be seen as a *lamination* or, equivalently, as the *inverse limit* of *Branched Oriented Flat Riemannian Manifolds*.

4. The Čech cohomology is equivalent to the longitudinal one, obtained by inverse limit, to the pattern-equivariant one or to the Pimsner cohomology are equivalent *Cohomology* of the Hull. The *K-group* of the Hull can be computed through a spectral sequence with the cohomology in page 2.

5. In maximum degree, the *Homology* gives the family of *invariant measures* and the *Gap Labelling Theorem*.