RIEMANNIAN GEOMETRY

on

METRIC CANTOR SETS

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Main References

J. Pearson, J. Bellissard,
Noncommutative Riemannian Geometry and Diffusion on Ultrametric Cantor Sets,
`arXiv: 0802.1336v1 [math.OA]`, Feb. 2008

A. Connes,
Noncommutative Geometry,
Academic Press, 1994.

G. Michon,
Les Cantors réguliers,
C. R. Acad. Sci. Paris Sér. I Math., (19), **300**, (1985) 673-675.

K. Falconer,
Fractal Geometry: Mathematical Foundations and Applications,
John Wiley and Sons 1990.

Content

1. Michon's Trees
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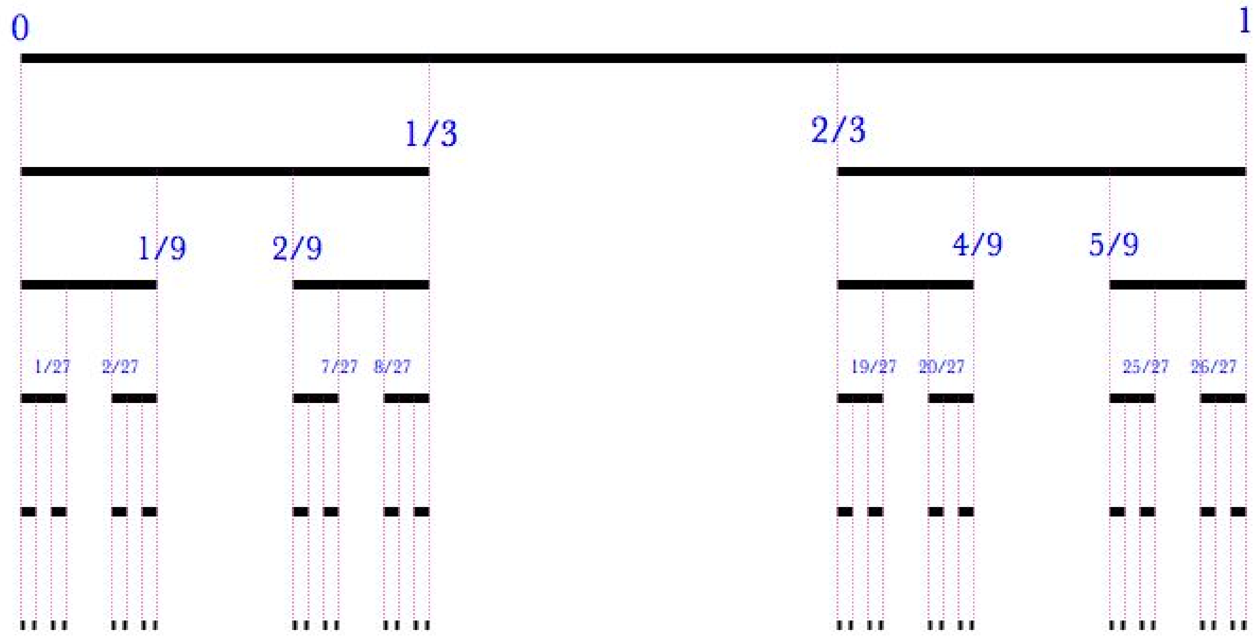
I - Michon's Trees

G. Michon, "Les Cantors réguliers", *C. R. Acad. Sci. Paris Sér. I Math.*, (19), **300**, (1985) 673-675.

I.1)- Cantor sets



The triadic Cantor set

Definition *A Cantor set is a compact, completely disconnected set without isolated points*

Theorem *Any Cantor set is homeomorphic to* $\{0,1\}^{\mathbb{N}}$.

L. BROUWER, "On the structure of perfect sets of points", *Proc. Akad. Amsterdam*, **12**, (1910), 785-794.

Hence without extra structure there is only one Cantor set.

I.2)- Metrics

Definition *Let X be a set. A metric d on X is a map $d : X \times X \mapsto \mathbb{R}_+$ such that, for all $x, y, z \in X$*
(i) $d(x, y) = 0$ *if and only if* $x = y$*,*
(ii) $d(x, y) = d(y, x)$*,*
(iii) $d(x, y) \leq d(x, z) + d(z, y)$*.*

Definition *A metric d on a set X is an ultrametric if it satisfies*

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

for all family x, y, z of points of C.

Given (C, d) a metric space, for $\epsilon > 0$ let $\stackrel{\epsilon}{\sim}$ be the equivalence relation defined by

$$x \stackrel{\epsilon}{\sim} y \quad \Leftrightarrow \quad \exists x_0 = x, x_1, \cdots, x_{n-1}, x_n = y \quad d(x_{k-1}, x_k) < \epsilon$$

Theorem *Let (C, d) be a metric Cantor set. Then there is a sequence $\epsilon_1 > \epsilon_2 > \cdots \epsilon_n > \cdots \geq 0$ converging to 0, such that $\stackrel{\epsilon}{\sim} = \stackrel{\epsilon_n}{\sim}$ whenever $\epsilon_n \geq \epsilon > \epsilon_{n+1}$.*

For each $\epsilon > 0$ there is a finite number of equivalence classes and each of them is close and open.

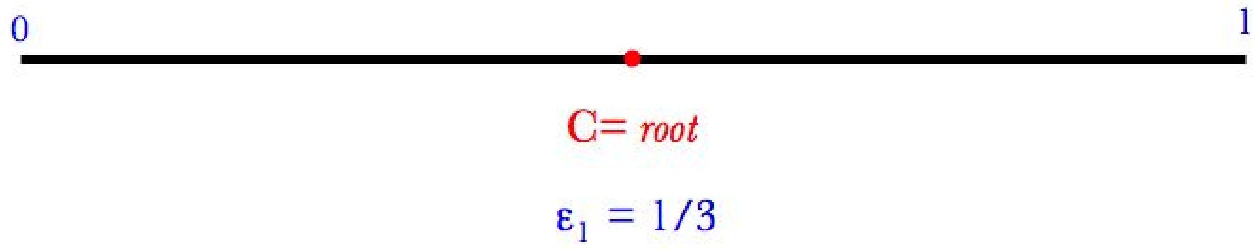
Moreover, the sequence $[x]_{\epsilon_n}$ of clopen sets converges to $\{x\}$ as $n \to \infty$.

I.3)- Michon's graph

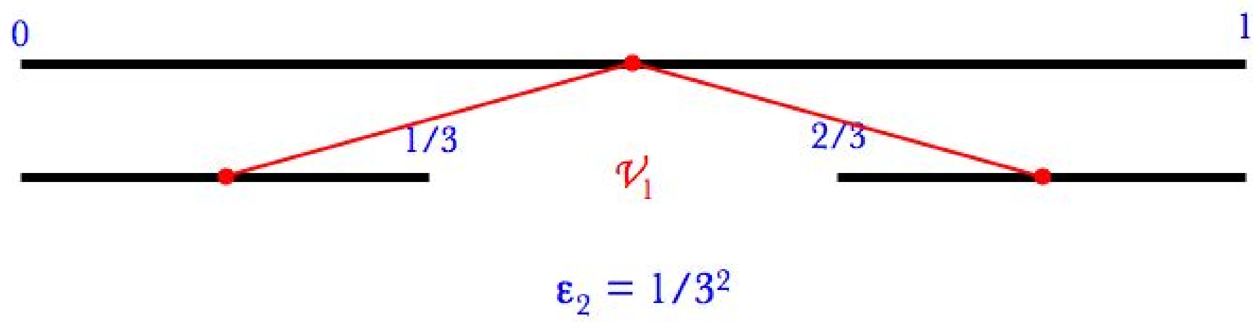
Set

- $\mathcal{V}_0 = \{C\}$ (called the *root*),
- for $n \geq 1$, $\mathcal{V}_n = \{[x]_{\epsilon_n}; x \in C\}$,
- $\mathcal{V}$ is the disjoint union of the $\mathcal{V}_n$'s,
- $\mathcal{E} = \{(v, v') \in \mathcal{V} \times \mathcal{V} \; ; \; \exists n \in \mathbb{N} \, , \, v \in \mathcal{V}_n, v' \in \mathcal{V}_{n+1} \, , \, v' \subset v\}$,
- $\delta(v) = \mathrm{diam}\{v\}$.

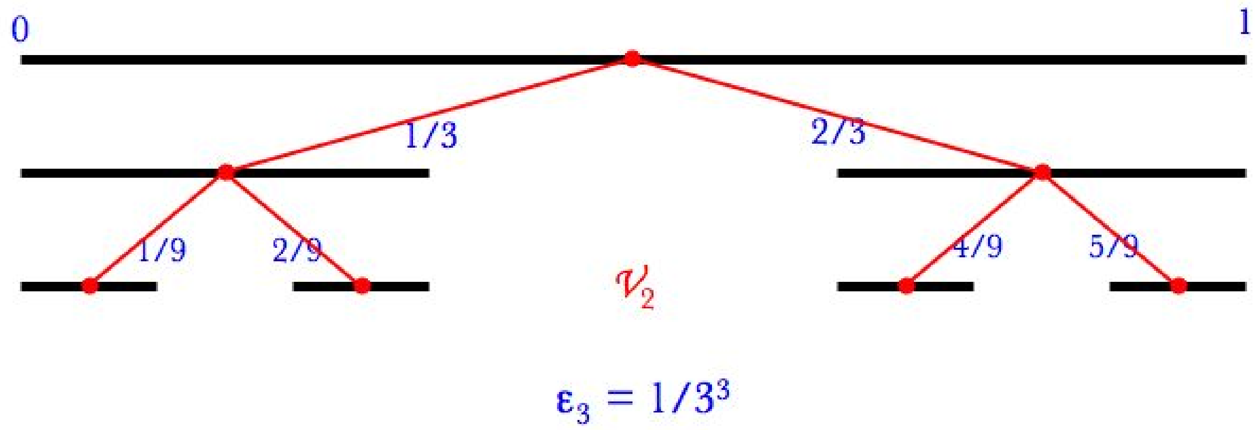
The family $\mathcal{T} = (C, \mathcal{V}, \mathcal{E}, \delta)$ defines a weighted rooted tree, with root C, set of vertices $\mathcal{V}$, set of edges $\mathcal{E}$ and weight δ



The Michon tree for the triadic Cantor set



The Michon tree for the triadic Cantor set



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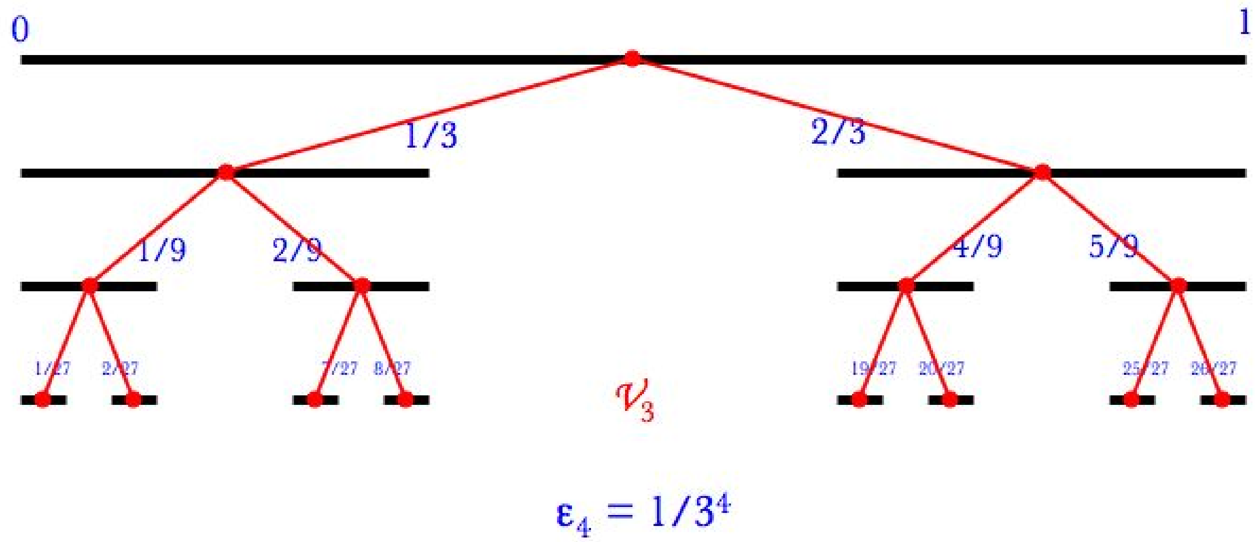

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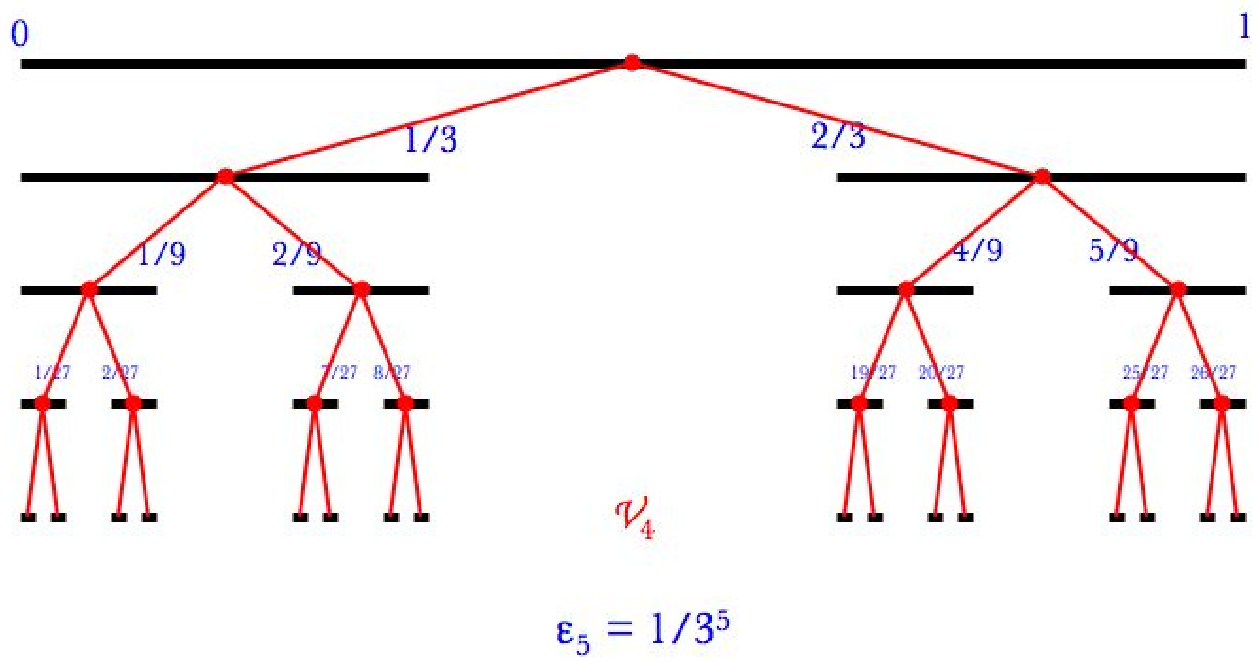

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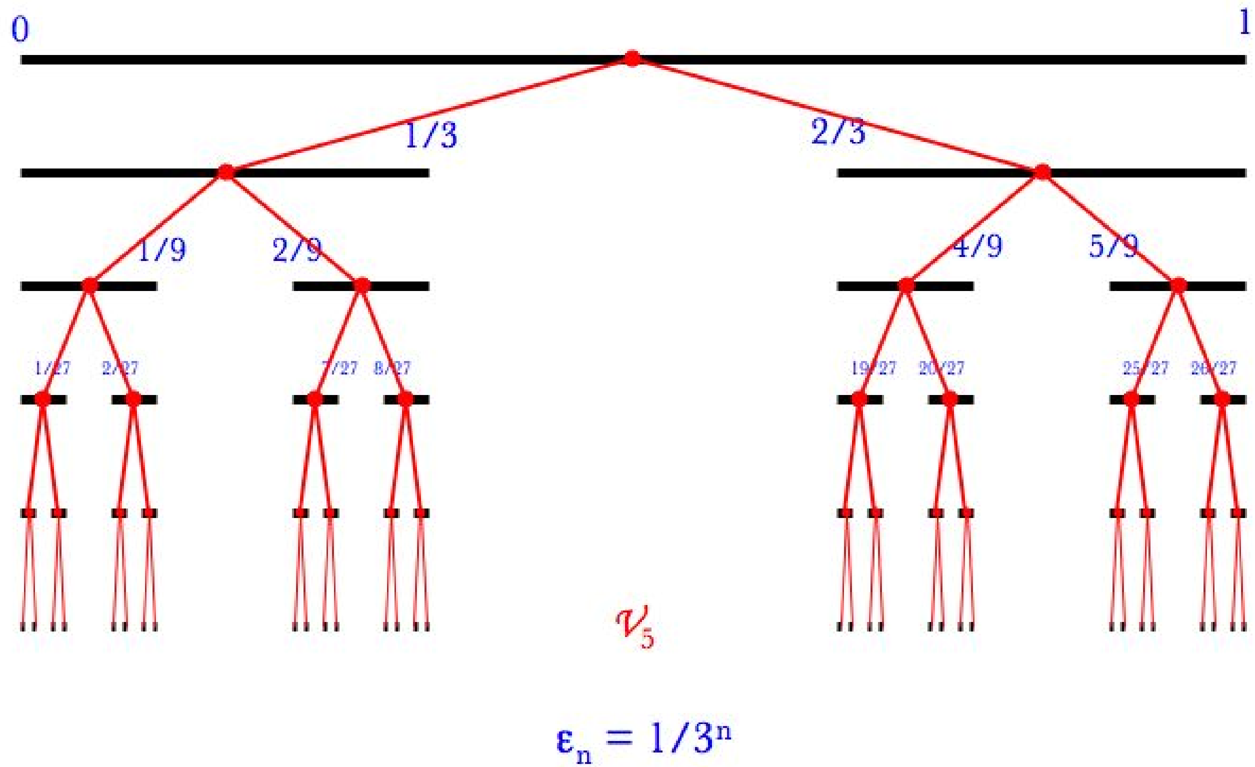
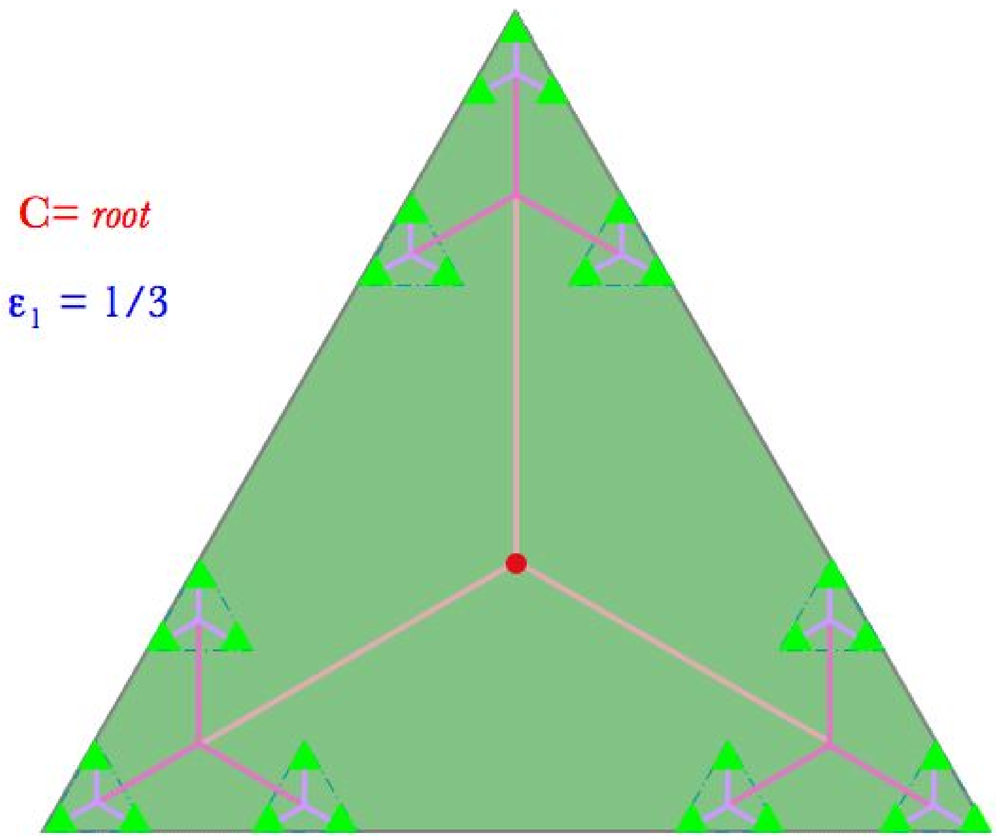
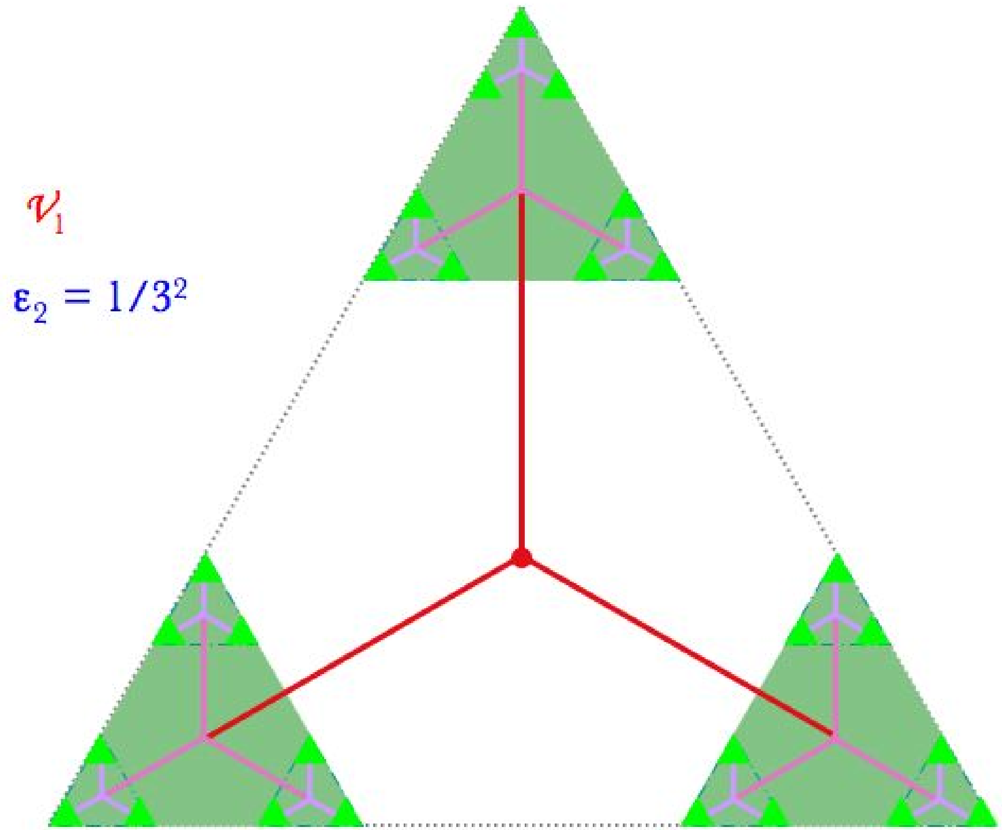


The Michon tree for the triadic Cantor set



The Michon tree for the triadic ring $\mathbb{Z}(3)$



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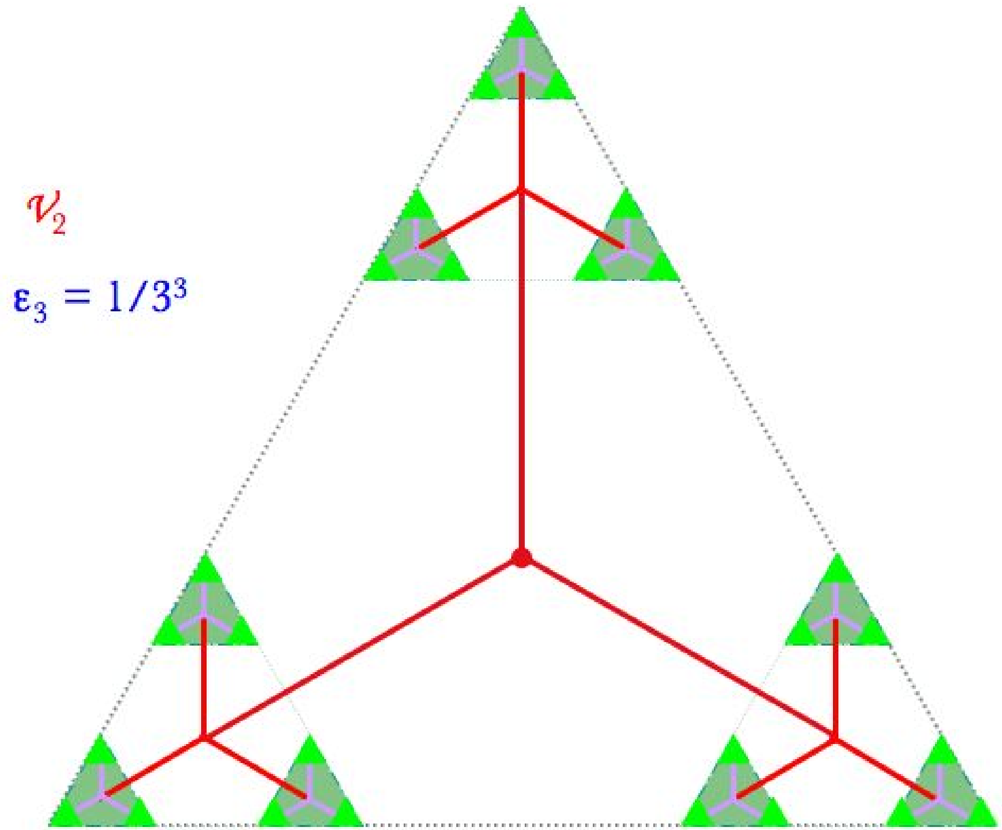
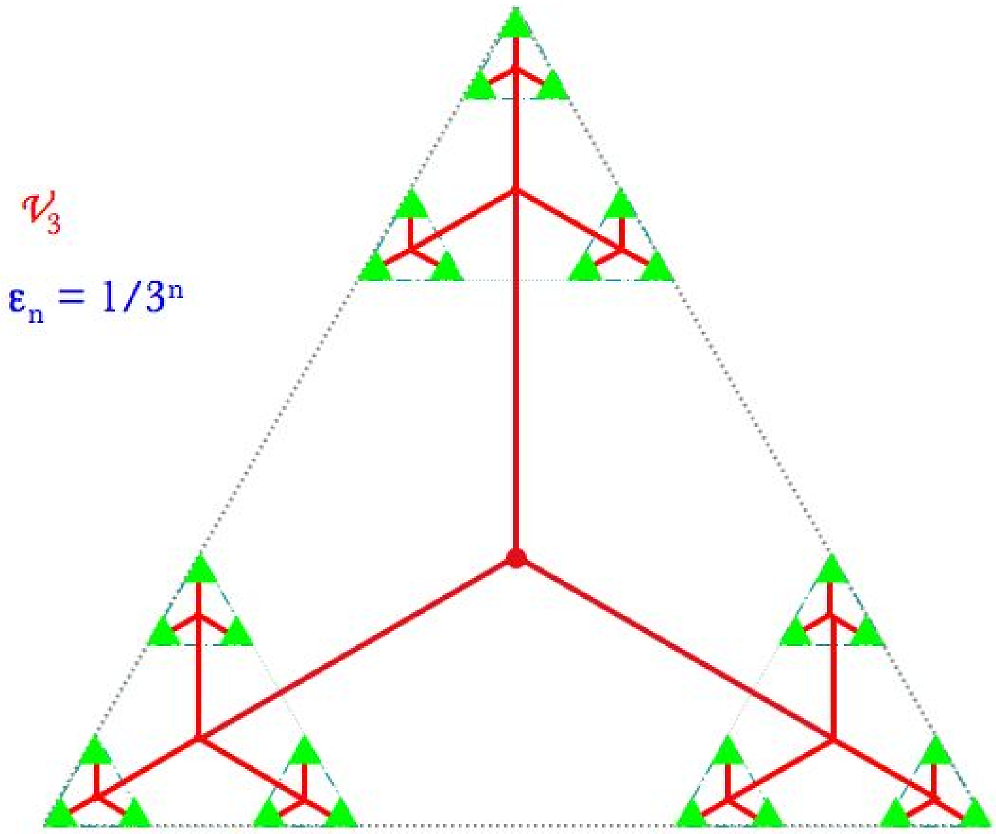


The Michon tree for the triadic ring $\mathbb{Z}(3)$



The Michon tree for the triadic ring $\mathbb{Z}(3)$

I.4)- The boundary of a tree

Let $\mathcal{T} = (0, \mathcal{V}, \mathcal{E})$ be a rooted tree. It will be called *Cantorian* if

- *Each vertex admits one descendant with more than one child*
- *Each vertex has only a finite number of children.*

Then $\partial\mathcal{T}$ is the set of infinite path starting form the root. If $v \in \mathcal{V}$ then $[v]$ will denote the set of such paths passing through v

Theorem *The family $\{[v]\,;\, v \in \mathcal{V}\}$ is the basis of a topology making $\partial\mathcal{T}$ a Cantor set.*

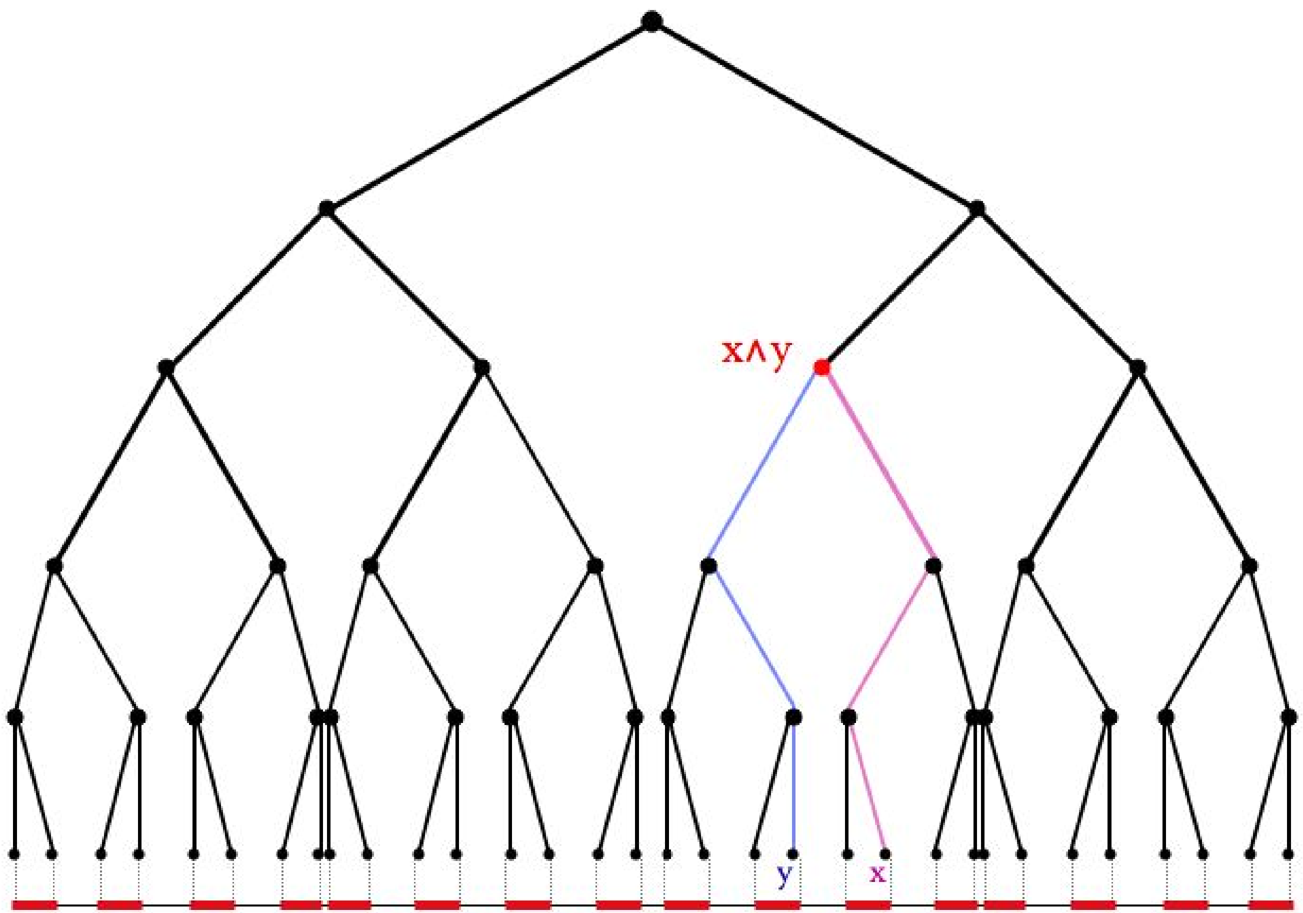
A *weight* on $\mathcal{T}$ is a map $\delta : \mathcal{V} \mapsto \mathbb{R}_+$ such that

- If $w \in \mathcal{V}$ is a child of v then $\delta(v) \geq \delta(w)$,
- If $v \in \mathcal{V}$ has only one child w then $\delta(v) = \delta(w)$, otherwise $\delta(v) > \delta(w)$,
- If v_n is the decreasing sequence of vertices along an infinite path $x \in \partial\mathcal{T}$ then $\lim_{n\to\infty} \delta(v_n) = 0$.

Theorem *If $\mathcal{T}$ is a Cantorian rooted tree with a weight δ, then $\partial\mathcal{T}$ admits a canonical* ultrametric *d_δ defined by.*

$$d_\delta(x, y) = \delta([x \wedge y])$$

where $[x \wedge y]$ is the least common ancestor of x and y.



The least common ancestor of x and y

Theorem *Let $\mathcal{T}$ be a Cantorian rooted tree with weight δ. Then if $v \in \mathcal{V}$, $\delta(v)$ coincides with the diameter of $[v]$ for the canonical metric.*

Conversely, if $\mathcal{T}$ is the Michon tree of a metric Cantor set (C, d), with weight $\delta(v) = \mathrm{diam}(v)$, then there is a contracting homeomorphism from (C, d) onto $(\partial\mathcal{T}, d_\delta)$ and d_δ is the smallest ultrametric dominating d.

In particular, if d is an ultrametric, then $d = d_\delta$ and the homeomorphism is an isometry.

This gives a representation of all ultrametric Cantor sets together with a parametrization of the space of ultrametrics.

II - Spectral Triples

A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

II.1)- Spectral Triples

A *spectral triple* is a family $(\mathcal{H}, \mathcal{A}, D)$, such that

- $\mathcal{H}$ is a Hilbert space
- $\mathcal{A}$ is a $*$-algebra invariant by holomorphic functional calculus, with a representation π into $\mathcal{H}$ by bounded operators
- D is a self-adjoint operator on $\mathcal{H}$ with *compact resolvent* such that $[D, \pi(f)] \in \mathcal{B}(\mathcal{H})$ is a bounded operator for all $f \in \mathcal{A}$.
- $(\mathcal{H}, \mathcal{A}, D)$ is called *even* if there is $G \in \mathcal{B}(\mathcal{H})$ such that
  - $G = G^* = G^{-1}$
  - $[G, \pi(f)] = 0$ for $f \in \mathcal{A}$
  - $GD = -DG$

II.2)- The spectral triple of an ultrametric Cantor set

Let $\mathcal{T} = (C, \mathcal{V}, \mathcal{E}, \delta)$ be the *reduced* Michon tree associated with an *ultrametric Cantor set* (C, d). Then

- $\mathcal{H} = \ell^2(\mathcal{V}) \otimes \mathbb{C}^2$: any $\psi \in \mathcal{H}$ will be seen as a sequence $(\psi_v)_{v \in \mathcal{V}}$ with $\psi_v \in \mathbb{C}^2$
- $G,\ D$ are defined by

$$(D\psi)_v = \frac{1}{\delta(v)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi_v \qquad (G\psi)_v = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \psi_v$$

  so that they anticommute.
- $\mathcal{A} = C_{\text{Lip}}(C)$ is the space of Lipshitz continuous functions on (C, d)

II.3)- Choices

The tree $\mathcal{T}$ is *reduced*, meaning that only the vertices with more than one child are considered.

A *choice* will be a function $\tau : \mathcal{V} \mapsto C \times C$ such that if $\tau(v) = (x, y)$ then

- $x, y \in [v]$
- $d(x, y) = \delta(v) = \mathrm{diam}([v])$

Let $\mathrm{Ch}(v)$ be the set of children of v. Consequently, the set $\Upsilon(C)$ of choices is given by

$$\Upsilon(C) = \prod_{v \in \mathcal{V}} \Upsilon_v \qquad \Upsilon_v = \bigsqcup_{w \neq w' \in \mathrm{Ch}(v)} [w] \times [w']$$

The set $\mathcal{V}$ of vertices can be seen as a coarse-grained approximation of the Cantor set C.

Similarly, the set Υ_v can be seen as a coarse-grained approximation the unit tangent vectors at v.

Within this interpretation, the set $\Upsilon(C)$ can be seen as the unit sphere bundle inside the tangent bundle.

II.4)- Representations of $\mathcal{A}$

Let $\tau \in \Upsilon(C)$ be a choice. If $v \in \mathcal{V}$ write $\tau(v) = (\tau_+(v), \tau_-(v))$. Then π_τ is the representation of $C_{\text{Lip}}(C)$ into $\mathcal{H}$ defined by

$$(\pi_\tau(f)\psi)_v = \begin{bmatrix} f(\tau_+(v)) & 0 \\ 0 & f(\tau_-(v)) \end{bmatrix} \psi_v \qquad f \in C_{\text{Lip}}(C)$$

Theorem *The distance d on C can be recovered from the following Connes formula*

$$d(x,y) = \sup\left\{ \left| f(x) - f(y) \right| \; ; \; \sup_{\tau \in \Upsilon(C)} \|[D, \pi_\tau(f)]\| \leq 1 \right\}$$

Remark: the commutator $[D, \pi_\tau(f)]$ is given by

$$([D, \pi_\tau(f)]\psi)_v = \frac{f(\tau_+(v)) - f(\tau_-(v))}{d_\delta(\tau_+(v), \tau_-(v))} \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \psi_v$$

In particular $\sup_\tau \|[D, \pi_\tau(f)]\|$ is the Lipshitz norm of f

$$\|f\|_{\text{Lip}} = \sup_{x \neq y \in C} \left| \frac{f(x) - f(y)}{d_\delta(x, y)} \right|$$

III - ζ-function and Metric Measure

A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley and Sons 1990.

G.H. Hardy & M. Riesz, *The General Theory of Dirichlet's Series*, Cambridge University Press (1915).

III.1)- ζ-function

The *ζ-function* of the Dirac operator is defined by

$$\zeta(s) \;=\; \operatorname{Tr}\left(\frac{1}{|D|^s}\right) \qquad\qquad s \in \mathbb{C}$$

The *abscissa of convergence* is a positive real number $s_0 > 0$ so that the series defined by the trace above converges for $\Re(s) > s_0$.

Theorem *Let (C, d) be an ultrametric Cantor set. The abscissa of convergence of the ζ-function of the corresponding Dirac operator coincides with the* upper box dimension *of (C, d).*

- The *upper box dimension* of a compact metric space (X, d) is defined by

$$\overline{\dim}_B(C) = \limsup_{\delta \downarrow 0} \frac{\log N_\delta(C)}{-\log \delta}$$

where $N_\delta(X)$ is the least number of sets of diameter at most δ that cover X.

- Thanks to the definition of the Dirac operator

$$\zeta(s) = 2 \sum_{v \in \mathcal{V}} \delta(v)^s$$

- There are examples of metric Cantor sets with *infinite upper box dimension*. This is the case for the transversal of tilings with positive entropy.

III.2)- Dixmier Trace & Metric Measure

If the abscissa of convergence is finite, then a *probability measure* μ on (C, d) can be defined as follows (if the limit exists)

$$\mu(f) = \lim_{s \downarrow s_0} \frac{\mathrm{Tr}\,(|D|^{-s} \pi_\tau(f))}{\mathrm{Tr}\,(|D|^{-s})} \qquad f \in \mathcal{C}_{\mathrm{Lip}}(C)$$

This limit coincides with the *normalized Dixmier trace*

$$\frac{\mathrm{Tr}_{\mathrm{Dix}}(|D|^{-s_0} \pi_\tau(f))}{\mathrm{Tr}_{\mathrm{Dix}}(|D|^{-s_0})}$$

Theorem *The definition of the* Metric Measure μ *is independent of the choice* τ*.*

- If ζ admits an *isolated simple pole at $s = s_0$*, then $|D|^{-1}$ belongs to the *Mačaev ideal $\mathcal{L}^{s_0+}(\mathcal{H})$*. Therefore the measure μ is well defined.

- There is a large class of Cantor sets (such as *Iterated Function System*) for which the measure μ coincides with the *Hausdorff measure* associated with the upper box dimension.

- In particular μ is the *metric analog of the Lebesgue measure class* on a Riemannian manifold, in that the measure of a ball of radius r behaves like r^{s_0} for r small

$$\mu(B(x,r)) \overset{r\downarrow 0}{\sim} r^{s_0}$$

- μ is the analog of the *volume form* on a Riemannian manifold.

As a consequence μ defines a *canonical probability measure* ν on the space of choices Υ as follows

$$\nu = \bigotimes_{v \in \mathcal{V}} \nu_v \qquad \nu_v = \frac{1}{Z_v} \sum_{w \neq w' \in \mathrm{Ch}(v)} \mu \otimes \mu|_{[w]\times[w]}$$

where Z_v is a normalization constant given by

$$Z_v = \sum_{w \neq w' \in \mathrm{Ch}(v)} \mu([w])\mu([w'])$$

IV - The Laplace-Beltrami Operator

M. Fukushima, *Dirichlet Forms and Markov Processes*, North-Holland (1980).

J. Pearson, J. Bellissard,
Noncommutative Riemannian Geometry and Diffusion on Ultrametric Cantor Sets,
arXiv: 0802.1336v1 [math.OA], Feb. 2008

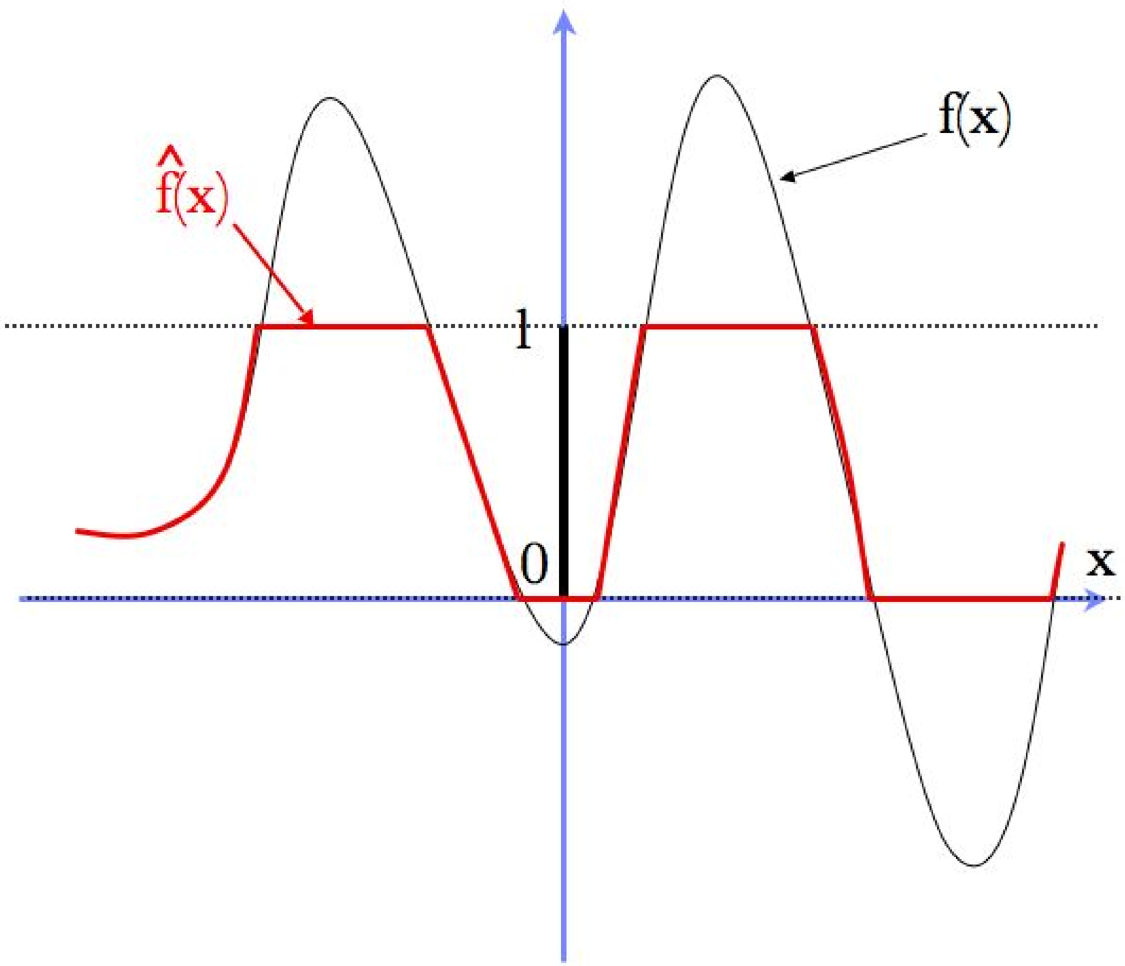
IV.1)- Dirichlet Forms

Let (X, μ) be a probability space space. For f a *real valued* measurable function on X, let $\hat{f}$ be the function obtained as

$$\hat{f}(x) = \begin{cases} 1 & \text{if} \quad f(x) \geq 1 \\ f(x) & \text{if} \quad 0 \leq f(x) \leq 1 \\ 0 & \text{if} \quad f(x) \leq 0 \end{cases}$$

A Dirichlet form Q on X is a *positive definite sesquilinear form* $Q : L^2(X, \mu) \times L^2(X, \mu) \mapsto \mathbb{C}$ such that

- Q is densely defined with domain $\mathcal{D} \subset L^2(X, \mu)$
- Q is closed
- Q is *Markovian*, namely if $f \in \mathcal{D}$, then $Q(\hat{f}, \hat{f}) \leq Q(f, f)$



Markovian cut-off of a real valued function

The simplest typical example of Dirichlet form is related to the Laplacian Δ_Ω on a bounded domain $\Omega \subset \mathbb{R}^D$

$$Q_\Omega(f, g) = \int_\Omega d^D x \ \overline{\nabla f(x)} \cdot \nabla g(x)$$

with domain $\mathcal{D} = C_0^1(\Omega)$ the space of continuously differentiable functions on Ω vanishing on the boundary.

This form is closeable in $L^2(\Omega)$ and its closure defines a Dirichlet form.

Any closed positive sesquilinear form Q on a Hilbert space, defines canonically a *positive self-adjoint operator* $-\Delta_Q$ satisfying

$$\langle f| - \Delta_Q \, g\rangle = Q(f, g)$$

In particular $\Phi_t = \exp(t\Delta_Q)$ (defined for $t \in \mathbb{R}_+$) is a strongly continuous *contraction* semigroup.

If Q is a Dirichlet form on X, then the contraction semigroup $\Phi = (\Phi_t)_{t\geq 0}$ is a *Markov semigroup*.

A *Markov semi-group* Φ on $L^2(X,\mu)$ is a family $(\Phi_t)_{t\in[0,+\infty)}$ where

- For each $t \geq 0$, Φ_t is a *contraction* from $L^2(X,\mu)$ into itself
- *(Markov property)* $\Phi_t \circ \Phi_s = \Phi_{t+s}$
- *(Strong continuity)* the map $t \in [0,+\infty) \mapsto \Phi_t$ is strongly continuous
- $\forall t \geq 0$, Φ_t is *positivity preserving* : $f \geq 0 \;\Rightarrow\; \Phi_t(f) \geq 0$
- Φ_t is *normalized*, namely $\Phi_t(1) = 1$.

Theorem (Fukushima) *A contraction semi-group on $L^2(X,\mu)$ is a Markov semi-group if and only if its generator is defined by a Dirichlet form.*

IV.2)- The Laplace-Beltrami Form

Let M be a *Riemannian manifold* of dimension D. The *Laplace-Beltrami operator* is associated with the Dirichlet form

$$Q_{\mathrm{M}}(f,g) = \sum_{i,j=1}^{D} \int_M d^{\mathrm{D}}x \, \sqrt{\det(g(x))}\, g_{ij}(x)\, \overline{\partial_i f(x)}\, \partial_j g(x)$$

where g is the metric. Equivalently (in local coordinates)

$$Q_{\mathrm{M}}(f,g) = \int_M d^{\mathrm{D}}x \, \sqrt{\det(g(x))} \int_{S(x)} d\nu_x(u)\, \overline{u\cdot\nabla f(x)}\, u\cdot\nabla g(x)$$

where $S(x)$ represent the *unit sphere* in the tangent space whereas ν_x is the *normalized Haar measure* on $S(x)$.

Similarly, if (C, d) is an ultrametric Cantor set, the expression

$$[D, \pi_\tau(f)]$$

can be interpreted as a *directional derivative*, analogous to $u \cdot \nabla f$, since a choice τ has been interpreted as a unit tangent vector.

The Laplace Beltrami operator is defined by

$$Q_s(f, g) \;=\; \int_\Upsilon d\nu(\tau)\ \mathrm{Tr}\left\{\frac{1}{|D|^s}[D, \pi_\tau(f)]^*\ [D, \pi_\tau(g)]\right\}$$

for $f, g \in C_{\mathrm{Lip}}(C)$ and $s > 0$.

Let $\mathcal{D}$ be the linear subspace of $L^2(C, \mu)$ generated by the *characteristic functions* of the clopen sets $[v]$, $v \in \mathcal{V}$. Then

Theorem *For any $s \in \mathbb{R}$, the form Q_s defined on $\mathcal{D}$ is closeable on $L^2(C, \mu)$ and its closure is a Dirichlet form.*

The corresponding operator $-\Delta_s$ leaves $\mathcal{D}$ invariant, has a discrete spectrum.

For $s < s_0 + 2$, $-\Delta_s$ is unbounded with compact resolvent.

IV.3)- Jumps Process over Gaps

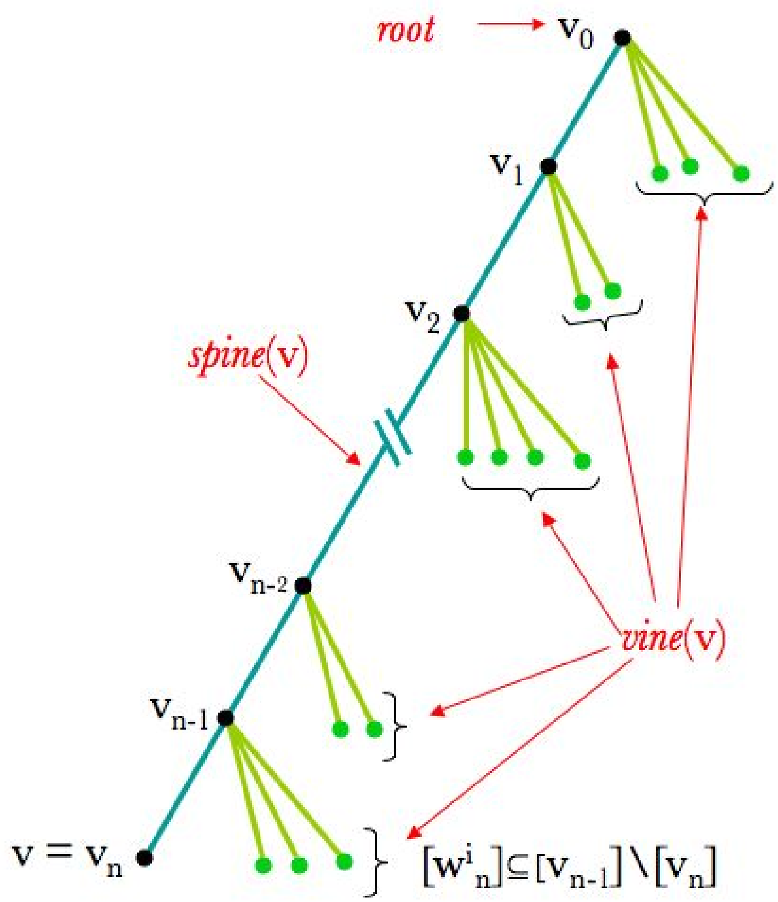
Δ_s generates a Markov semigroup, thus a stochastic process $(X_t)_{t\geq 0}$ where the X_t's takes on values in C.

Given $v \in \mathcal{V}$, its *spine* is the set of vertices located along the finite path joining the root to v. The *vine $\mathcal{V}(v)$ of v* is the set of vertices w, not in the spine, which are children of one vertex of the spine.

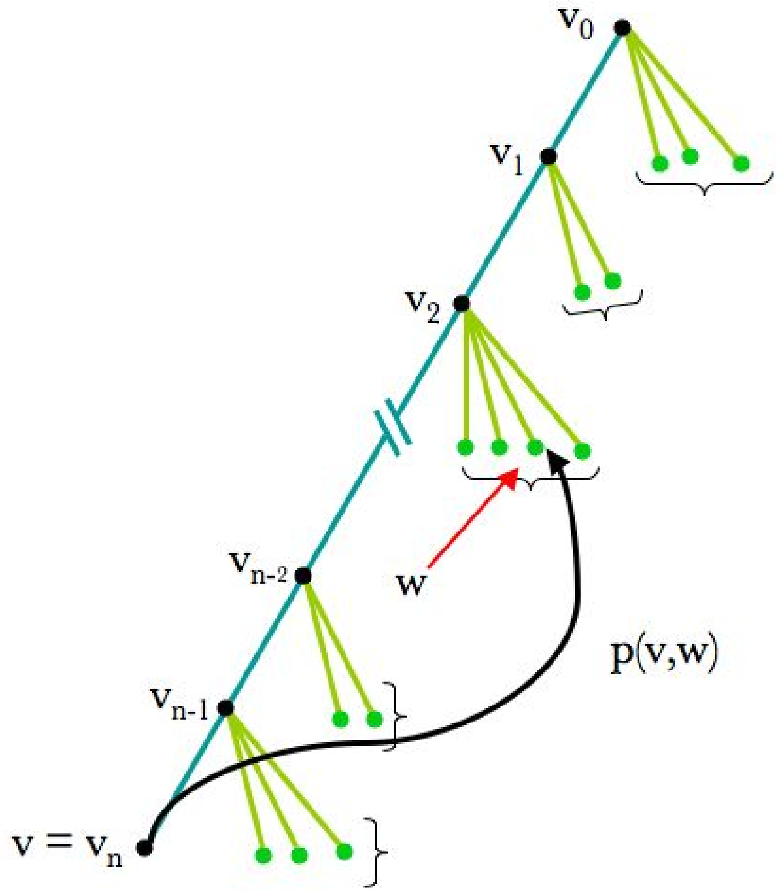
Then if χ_v is the characteristic function of $[v]$

$$\Delta_s \chi_v = \sum_{w \in \mathcal{V}(v)} p(v,w)(\chi_w - \chi_v)$$

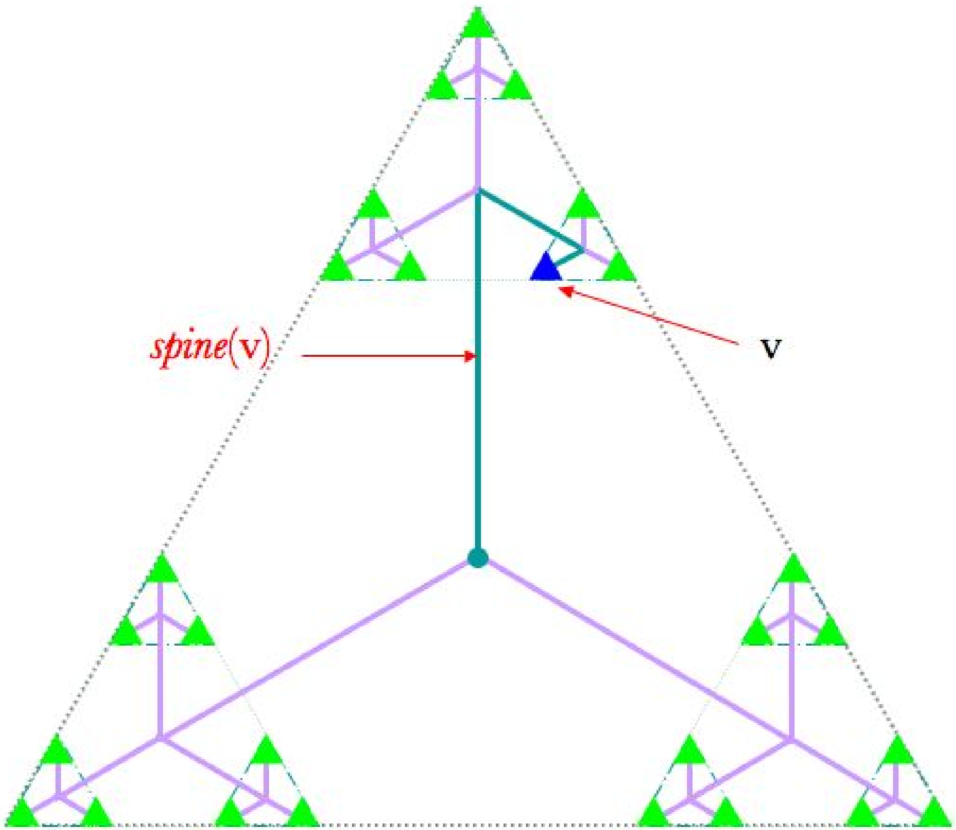
where $p(v,w) > 0$ represents the *probability for X_t to jump from v to w per unit time*.



The vine of a vertex v

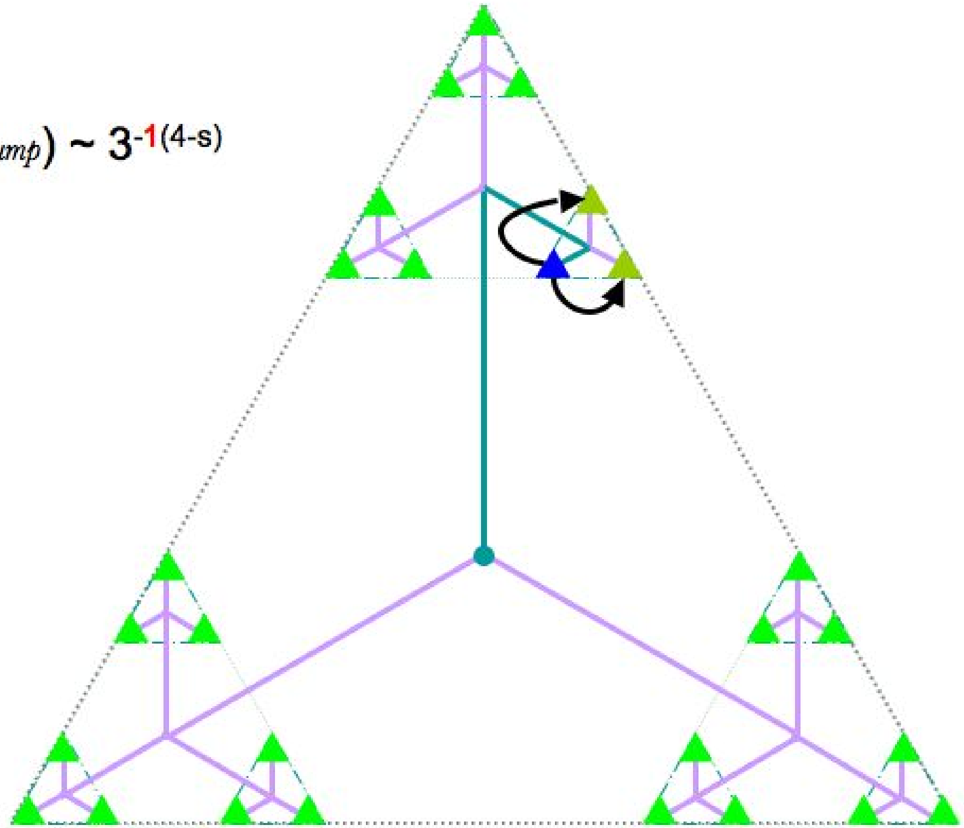


Jump process from v to w

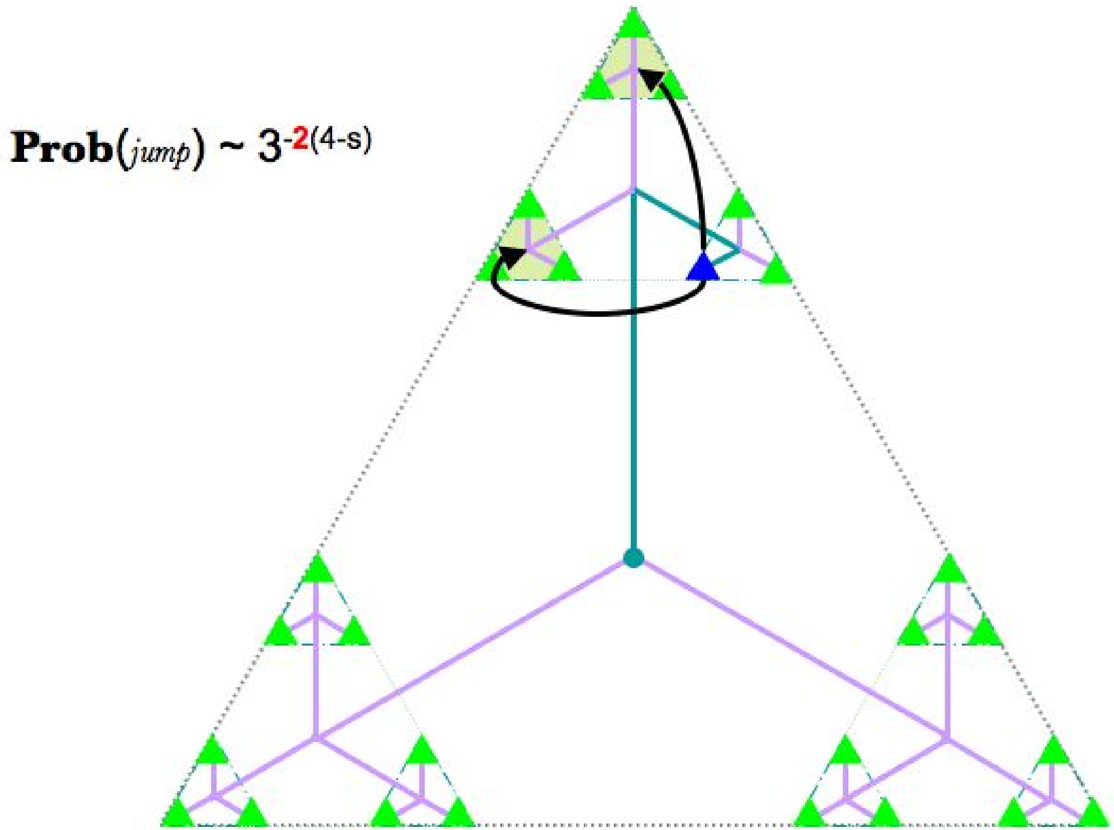


The tree for the triadic ring $\mathbb{Z}(3)$

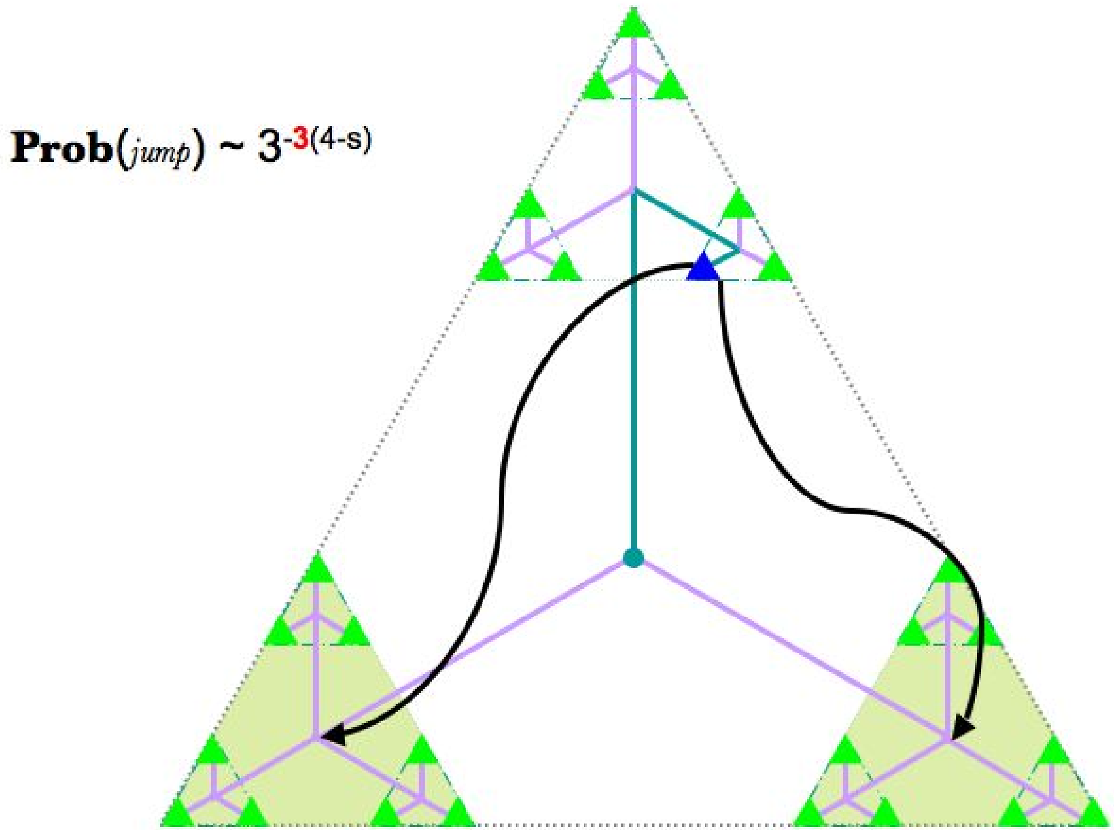
Prob(*jump*) ~ $3^{-1(4-s)}$



Jump process in $\mathbb{Z}(3)$



Jump process in $\mathbb{Z}(3)$



Jump process in $\mathbb{Z}(3)$

Remarks:

- Each vine defines a subspace of $L^2(C, \mu)$ of *finite dimension*
- Each such subspace is invariant by Δ_s: Δ_s has a *pure point spectrum*
- Δ_s can be seen as a 1*D-difference operator*, by restricting it to the set of vines of any infinite path $x \in \partial\mathcal{T}$
- The transition rate $p(v, w)$ can be explicitly written in terms of the $\mu([w])$'s and their diameters.

Concretely, if $\hat{w}$ denotes the *father* of w (which belongs to the spine)

$$p(v,w) \ = \ 2\delta(\hat{w})^{s-2} \ \frac{\mu([v])}{Z_{\hat{w}}}$$

where $Z_{\hat{w}}$ is the normalization constant for the measure $\nu_{\hat{w}}$ on the set of choices at $\hat{w}$, namely

$$Z_{\hat{w}} = \sum_{u \neq u' \in \mathrm{Ch}(\hat{w})} \mu([u])\mu([u'])$$

IV.4)- The Triadic Cantor Set

If C is the *triadic Cantor set*

- The eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ of Δ_s can be computed explicitly
- The eigenfunctions can also be computed explicitly
- The *density of state* $\mathcal{N}(\lambda) = \#\{n \in \mathbb{N}\,;\, \lambda_n \leq \lambda\}$ satisfies the Weyl asymptotics (where $k > 0$ is explicit)

$$\mathcal{N}(\lambda) \overset{\lambda\uparrow\infty}{=} 2\left(\frac{\lambda}{k}\right)^{s_0/2+s_0-s} (1 + o(1))$$

- If $s = s_0$ then $\mathcal{N}(\lambda) \sim \lambda^{s_0/2}$ suggesting that s_0 is the right dimension for the *noncommutative Riemannian manifold* (C, d).

More precisely, the eigenvalues are

$$\lambda_n = -2\left(1 + 3^{s_0+2-s} + \cdots + 3^{(s_0+2-s)(n-2)} + 2 \cdot 3^{(s_0+2-s)(n-1)}\right)$$

with $n \geq 1$ *and with multiplicity*

$$g_n = 2^{n-1}$$

In the triadic Cantor set a vertex v at level n of the hierarchy, can be labeled by a finite string 0110001 of 0's and 1's of length n.

The eigenfunctions are given by the Haar functions *defined by*

$$\varphi_\omega = \sum_{v \in \{0,1\}^n} (-1)^{\omega \cdot v} \chi_v$$

where $\omega \in \{0, 1\}^{\mathbb{N}}$ *and* $|\omega| \leq n$ *if* $|\omega|$ *denotes the maximum index* k *such that* $\omega_k = 1$.

In addition, the stochastic process has an *anomalous diffusion*

$$\mathbb{E}\{d(X_{t_0}, X_{t_0+t})^2\} \overset{t\downarrow 0}{=} D\,t\,\ln(1/t)\,(1 + o(1))$$

for some explicit positive D.

V - To conclude

V.1)- Results

- Ultrametric Cantor sets can be described as *Riemannian manifolds*, through Noncommutative Geometry.
- An analog of the *tangent unit sphere* is given by *choices*
- The *upper box dimension* plays the role of the dimension
- A *volume measure* is defined through the Dixmier trace
- A *Laplace-Beltrami operator* is defined with compact resolvent and Weyl asymptotics
- It generates a *jump process* playing the role of the *Brownian motion*.
- This process exhibits *anomalous diffusion*.

V. 2)- Quasi-isometric Embedding

- Every ultrametric Cantor set can be *isometrically embedded in a real Hilbert space*

- Sufficient conditions exist on the metric to allow for *quasi-isometric embeddings* into an finite dimensional Euclidean space (*embeddability*)

- **Prove or disprove:** *the atomic surface (transversal) of a quasicrystal, endowed with the combinatorial metric, is NOT embeddable*

(JB *work in progress)*

V.3)- Tilings & Aperiodic Solids

Open Problems:

- Construct a spectral triple for the groupoid of the transversal of an aperiodic repetitive tiling with finite local complexity

  (JB & J Savinien *work in progress)*

- Interpret the Laplace-Beltrami operator as the generator of atomic diffusion in quasicrystals (flip-flops or phason modes)