RIEMANNIAN GEOMETRY

on

METRIC CANTOR SETS

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Content

1. Michon's Trees
2. Spectral Triples
3. ζ-function and Metric Measure
4. The Laplace-Beltrami Operator
5. To conclude

I - Michon's Trees

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I.1)- Cantor sets

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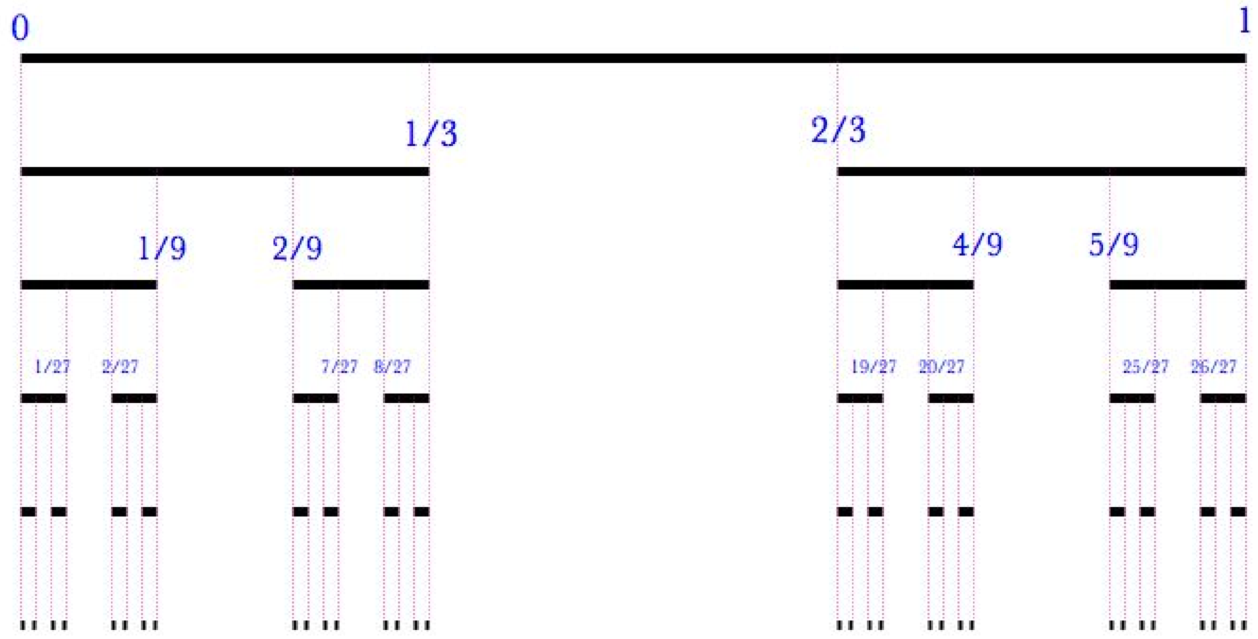

The triadic Cantor set

Definition *A Cantor set is a compact, completely disconnected set without isolated points*

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Hence without extra structure there is only one Cantor set.

I.2)- Metrics

Definition *Let X be a set. A metric d on X is a map $d : X \times X \mapsto \mathbb{R}_+$ such that, for all $x, y, z \in X$*

(i) $d(x, y) = 0$ *if and only if* $x = y$,

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Definition *A metric d on a set X is an ultrametric if it satisfies*

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

for all family x, y, z *of points of C.*

Given (C, d) a metric space, for $\epsilon > 0$ let $\overset{\epsilon}{\sim}$ be the equivalence relation defined by

$$x \overset{\epsilon}{\sim} y \quad \Leftrightarrow \quad \exists x_0 = x, x_1, \cdots, x_{n-1}, x_n = y \quad d(x_{k-1}, x_k) < \epsilon$$

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Moreover, the sequence $[x]_{\epsilon_n}$ of clopen sets converges to $\{x\}$ as $n \to \infty$.

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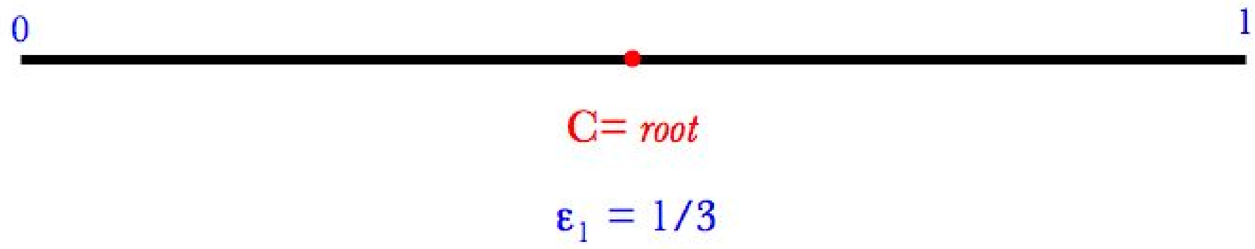
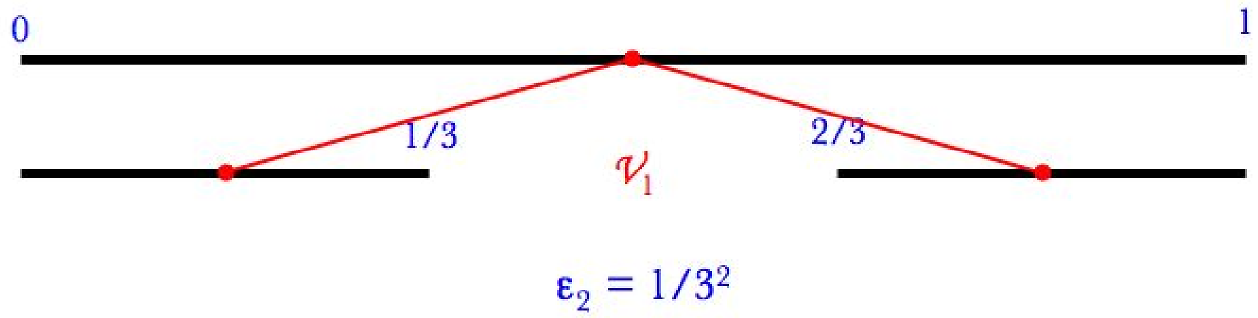
The family $\mathcal{T} = (C, \mathcal{V}, \mathcal{E}, \delta)$ defines a weighted rooted tree, with root C, set of vertices $\mathcal{V}$, set of edges $\mathcal{E}$ and weight δ



The Michon tree for the triadic Cantor set



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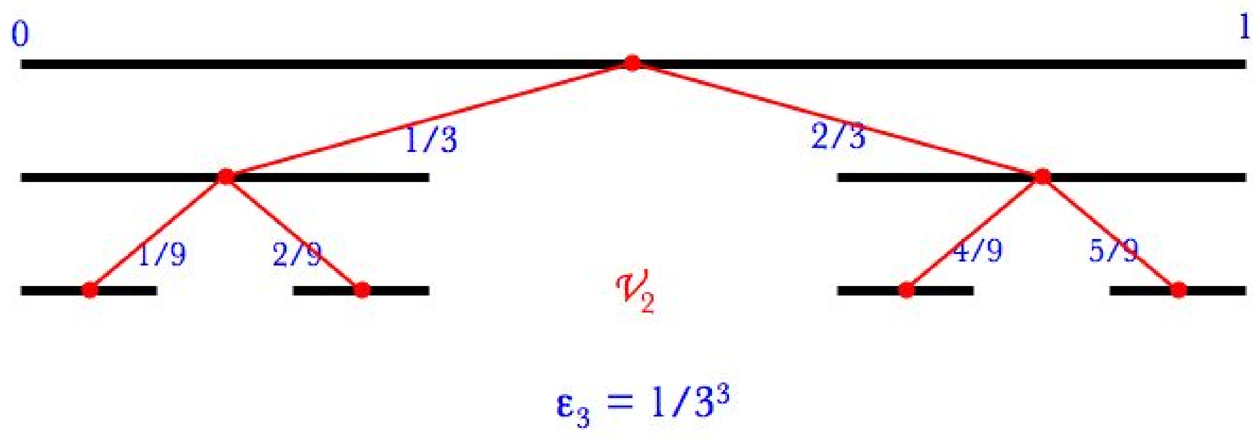

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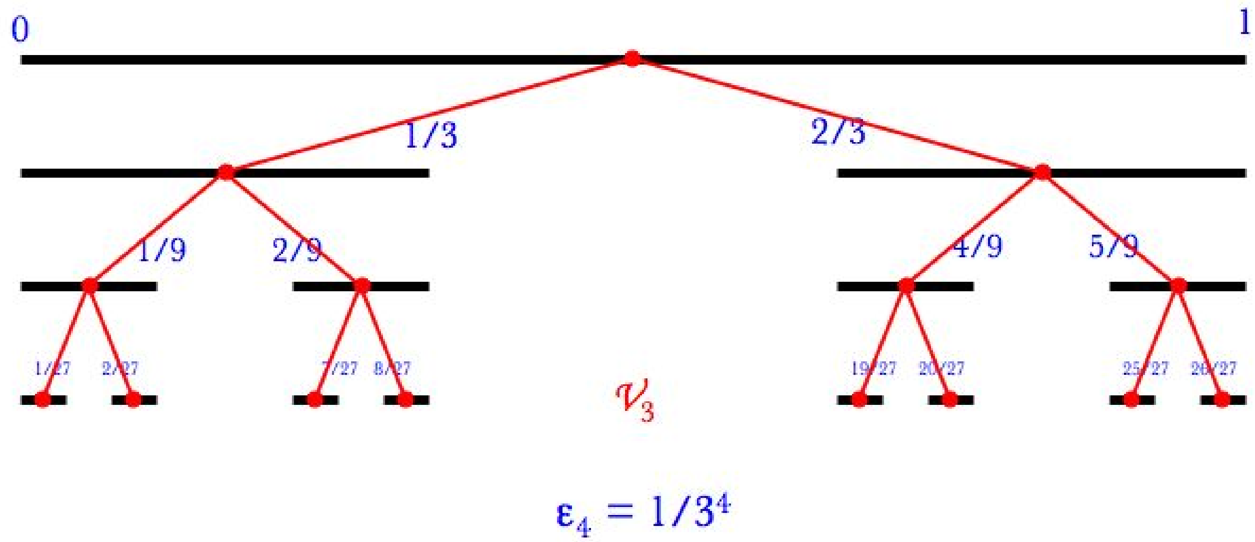

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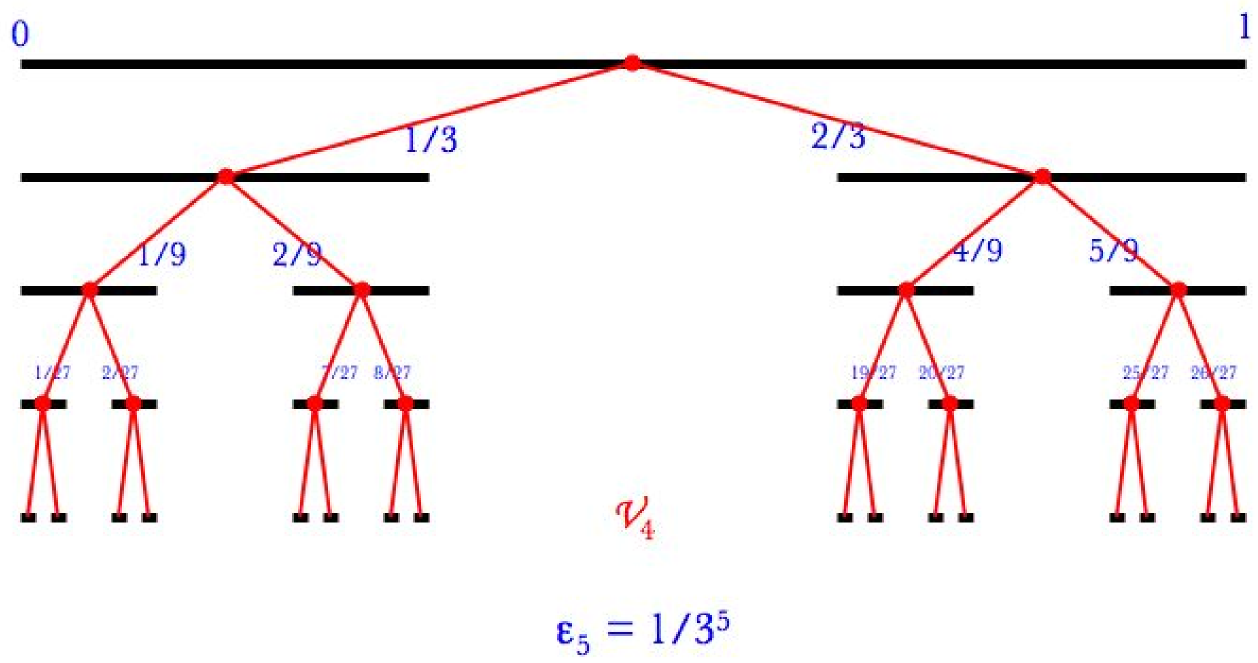

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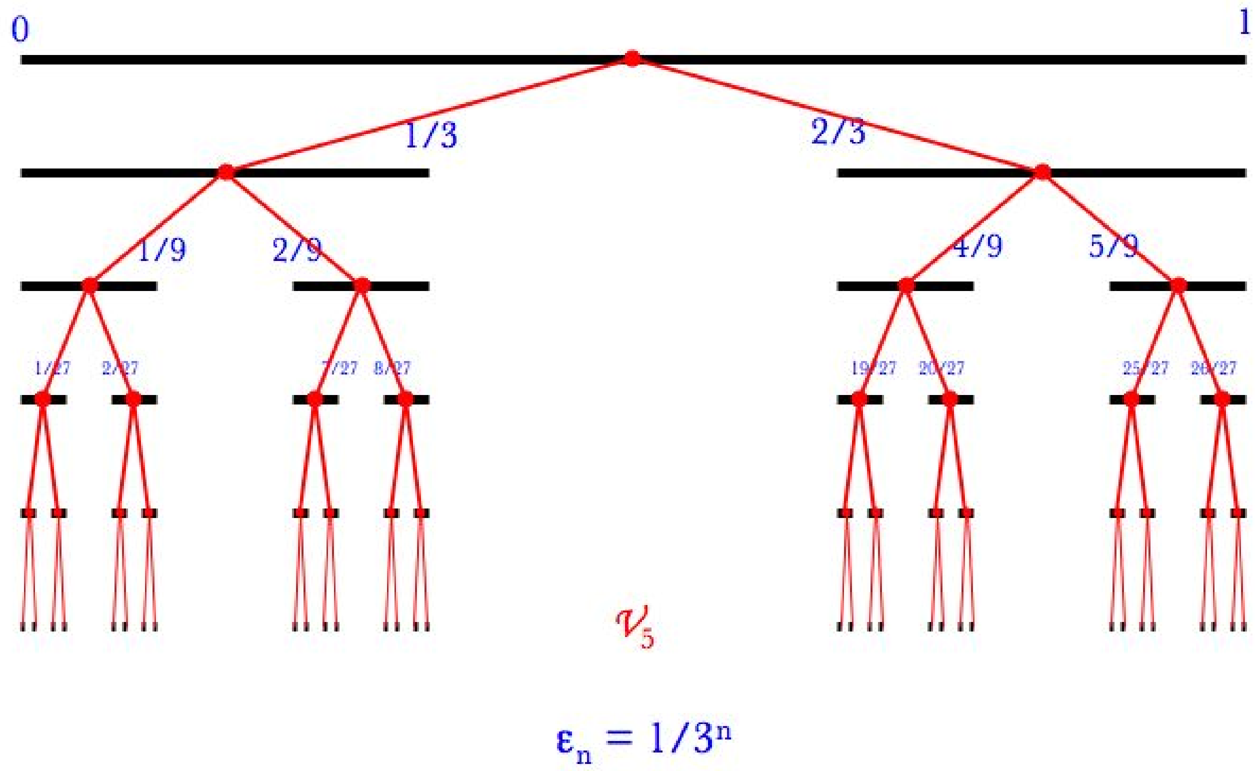

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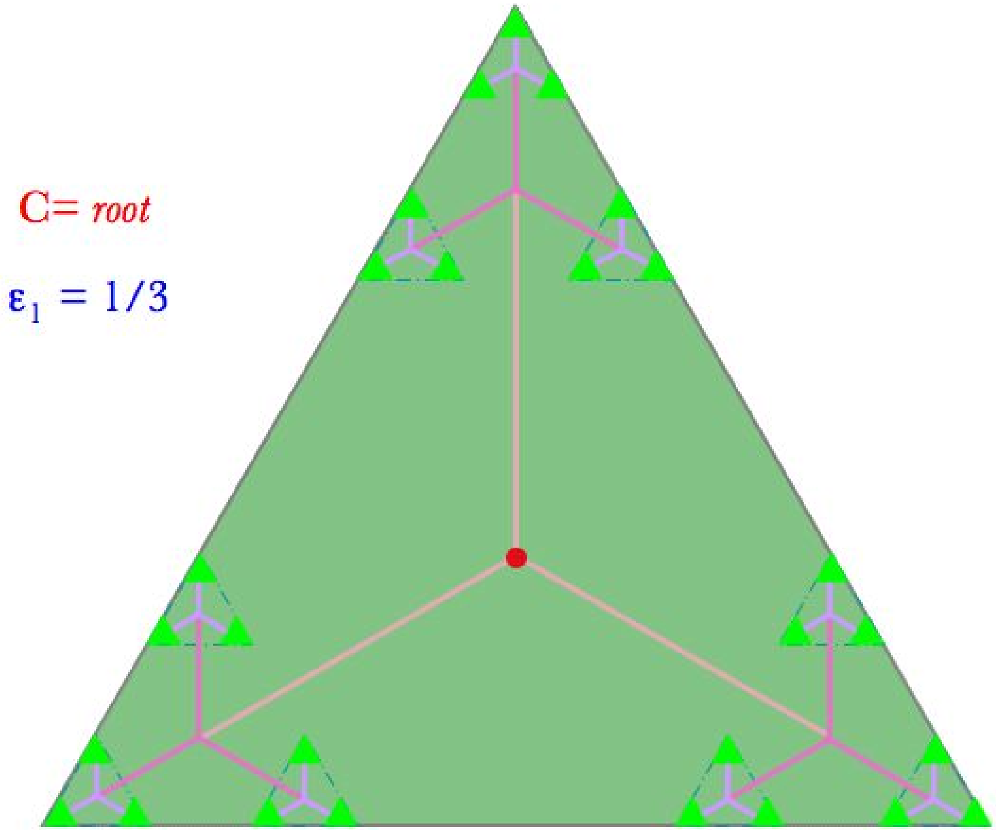

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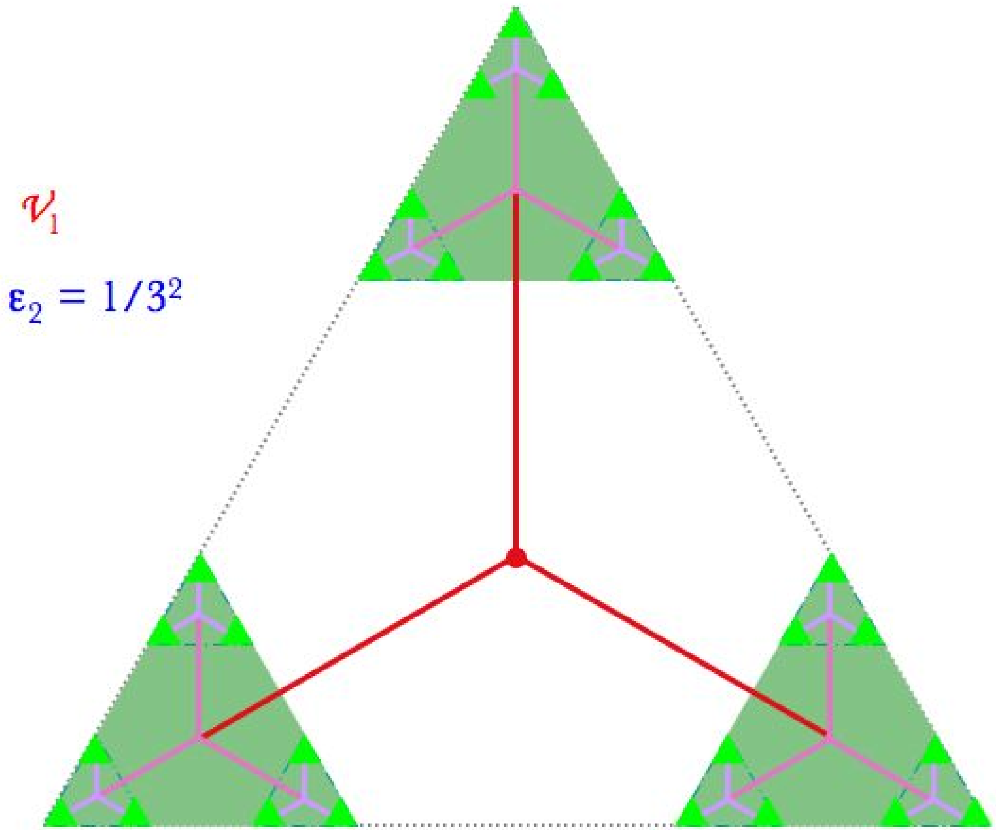

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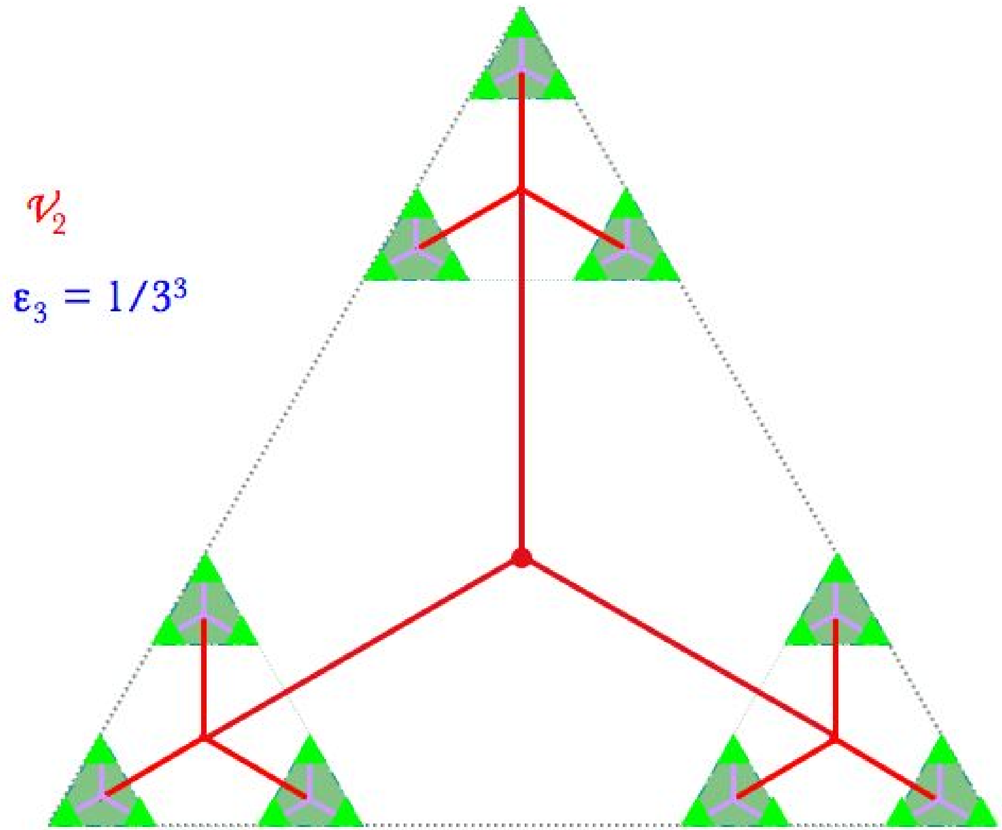

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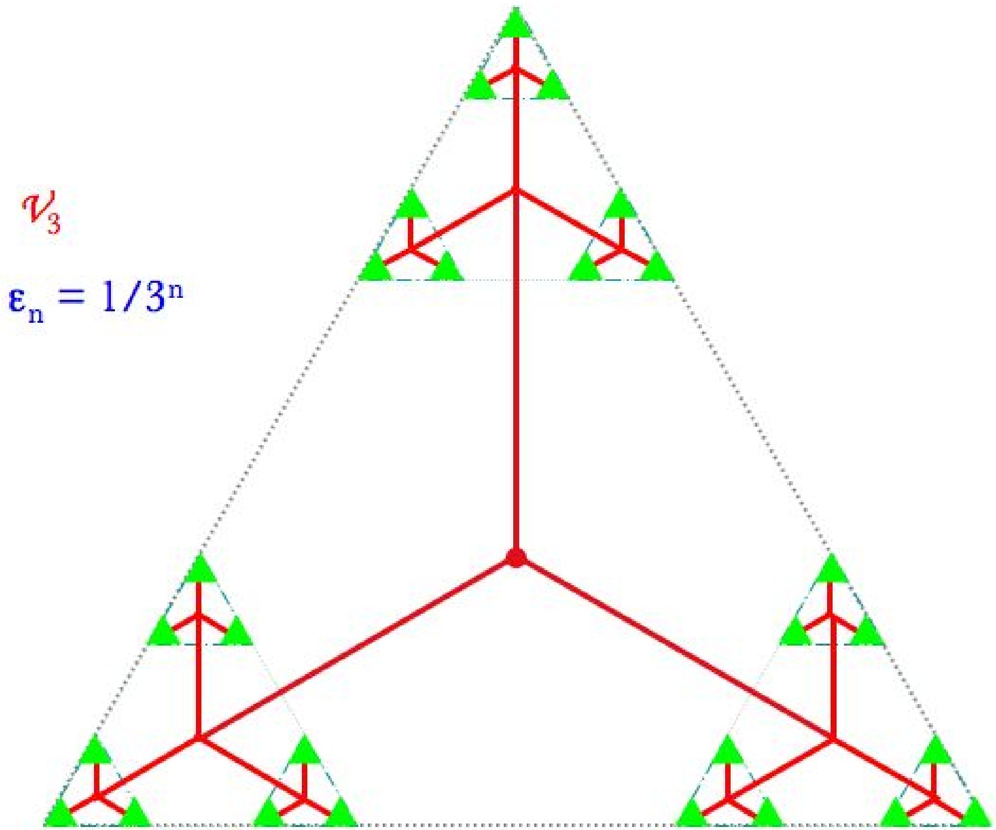

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Theorem *The family* $\{[v]\,;\, v \in \mathcal{V}\}$ *is the basis of a topology making* $\partial\mathcal{T}$ *a Cantor set.*

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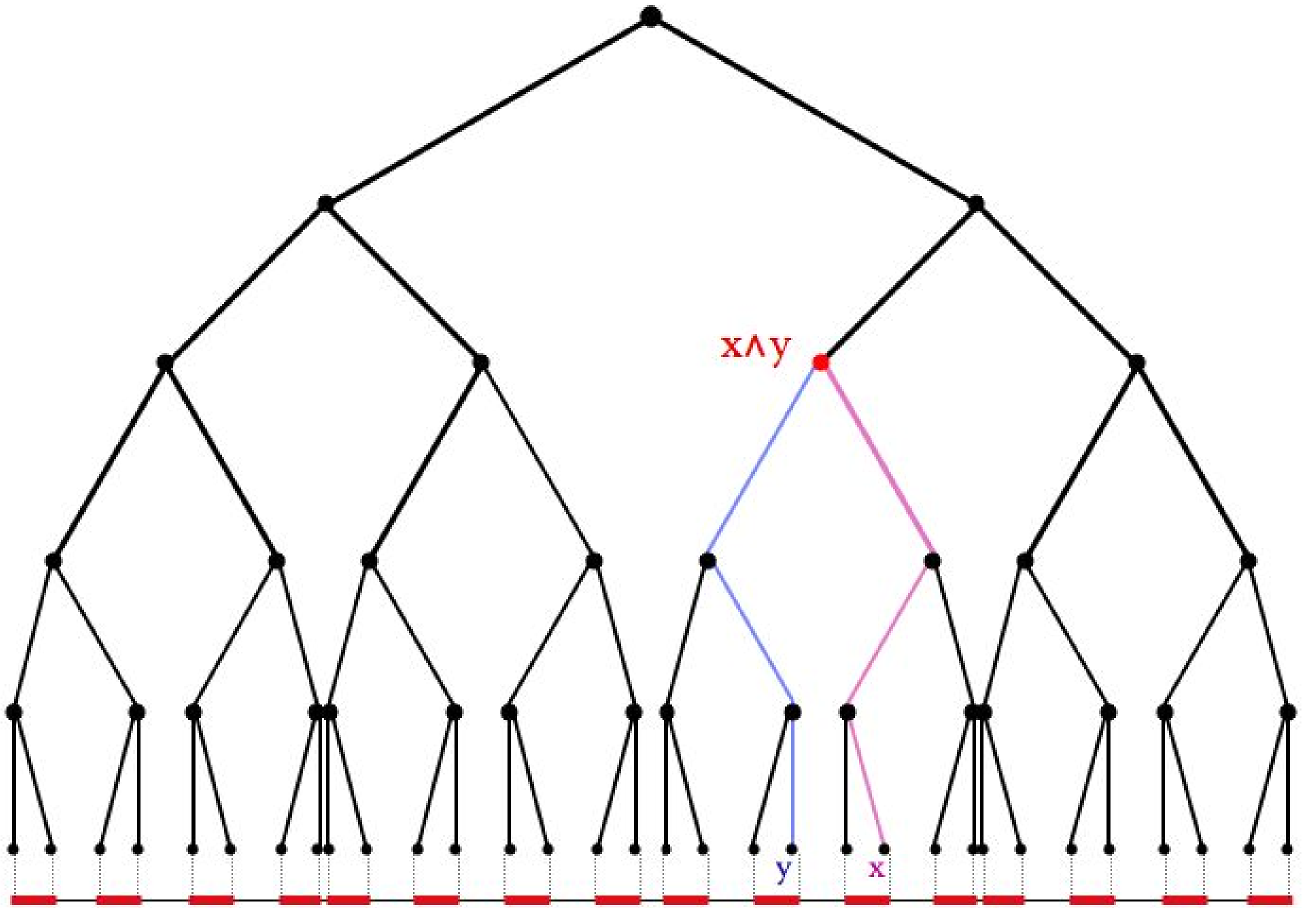
Theorem *If $\mathcal{T}$ is a Cantorian rooted tree with a weight δ, then $\partial\mathcal{T}$ admits a canonical* ultrametric *d_δ defined by.*

$$d_\delta(x, y) = \delta([x \wedge y])$$

where $[x \wedge y]$ is the least common ancestor of x and y.



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In particular, if d is an ultrametric, then $d = d_\delta$ and the homeomorphism is an isometry.

This gives a representation of all ultrametric Cantor sets together with a parametrization of the space of ultrametrics.

II - Spectral Triples

A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

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$$(D\psi)_v = \frac{1}{\delta(v)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi_v \qquad (G\psi)_v = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \psi_v$$

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Let $\mathrm{Ch}(v)$ be the set of children of v. Consequently, the set $\Upsilon(C)$ of choices is given by

$$\Upsilon(C) = \prod_{v \in \mathcal{V}} \Upsilon_v \qquad \Upsilon_v = \bigsqcup_{w \neq w' \in \mathrm{Ch}(v)} [w] \times [w']$$

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Within this interpretation, the set $\Upsilon(C)$ can be seen as the unit sphere bundle inside the tangent bundle.

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Let $\tau \in \Upsilon(C)$ be a choice. If $v \in \mathcal{V}$ write $\tau(v) = (\tau_+(v), \tau_-(v))$. Then π_τ is the representation of $C_{\mathrm{Lip}}(C)$ into $\mathcal{H}$ defined by

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$$(\pi_\tau(f)\psi)_v = \begin{bmatrix} f(\tau_+(v)) & 0 \\ 0 & f(\tau_-(v)) \end{bmatrix} \psi_v \qquad f \in C_{\text{Lip}}(C)$$

II.4)- Representations of $\mathcal{A}$

Let $\tau \in \Upsilon(C)$ be a choice. If $v \in \mathcal{V}$ write $\tau(v) = (\tau_+(v), \tau_-(v))$. Then π_τ is the representation of $C_{\text{Lip}}(C)$ into $\mathcal{H}$ defined by

$$(\pi_\tau(f)\psi)_v = \begin{bmatrix} f(\tau_+(v)) & 0 \\ 0 & f(\tau_-(v)) \end{bmatrix} \psi_v \qquad f \in C_{\text{Lip}}(C)$$

Theorem *The distance d on C can be recovered from the following Connes formula*

$$d(x,y) = \sup\left\{ \left|f(x) - f(y)\right| \; ; \; \sup_{\tau \in \Upsilon(C)} \|[D, \pi_\tau(f)]\| \leq 1 \right\}$$

Remark: the commutator $[D, \pi_\tau(f)]$ is given by

$$([D, \pi_\tau(f)]\psi)_v = \frac{f(\tau_+(v)) - f(\tau_-(v))}{d_\delta(\tau_+(v), \tau_-(v))} \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \psi_v$$

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In particular $\sup_\tau \|[D, \pi_\tau(f)]\|$ is the Lipshitz norm of f

$$\|f\|_{\text{Lip}} = \sup_{x \neq y \in C} \left| \frac{f(x) - f(y)}{d_\delta(x, y)} \right|$$

III - ζ-function and Metric Measure

A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley and Sons 1990.

G.H. Hardy & M. Riesz, *The General Theory of Dirichlet's Series*, Cambridge University Press (1915).

III.1)- ζ-function

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Theorem *Let* (C, d) *be an ultrametric Cantor set. The abscissa of convergence of the ζ-function of the corresponding Dirac operator coincides with the* upper box dimension *of* (C, d).

- The *upper box dimension* of a compact metric space (X, d) is defined by

$$\overline{\dim}_B(C) = \limsup_{\delta \downarrow 0} \frac{\log N_\delta(C)}{-\log \delta}$$

where $N_\delta(X)$ is the least number of sets of diameter at most δ that cover X.

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- There are examples of metric Cantor sets with *infinite upper box dimension*. This is the case for the transversal of tilings with positive entropy.

III.2)- Dixmier Trace & Metric Measure

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If the abscissa of convergence is finite, then a *probability measure* μ on (C, d) can be defined as follows (if the limit exists)

$$\mu(f) = \lim_{s \downarrow s_0} \frac{\mathrm{Tr}\left(|D|^{-s} \pi_\tau(f)\right)}{\mathrm{Tr}\left(|D|^{-s}\right)} \qquad f \in \mathcal{C}_{\mathrm{Lip}}(C)$$

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Theorem *The definition of the* Metric Measure μ *is independent of the choice* τ.

- If ζ admits an *isolated simple pole at* $s = s_0$, then $|D|^{-1}$ belongs to the *Mačaev ideal* $\mathcal{L}^{s_0+}(\mathcal{H})$. Therefore the measure μ is well defined.

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- In particular μ is the *metric analog of the Lebesgue measure class* on a Riemannian manifold, in that the measure of a ball of radius r behaves like r^{s_0} for r small

$$\mu(B(x,r)) \overset{r\downarrow 0}{\sim} r^{s_0}$$

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- μ is the analog of the *volume form* on a Riemannian manifold.

As a consequence μ defines a *canonical probability measure* ν on the space of choices Υ as follows

$$\nu = \bigotimes_{v \in \mathcal{V}} \nu_v \qquad \nu_v = \frac{1}{Z_v} \sum_{w \neq w' \in \mathrm{Ch}(v)} \mu \otimes \mu|_{[w] \times [w]}$$

where Z_v is a normalization constant given by

$$Z_v = \sum_{w \neq w' \in \mathrm{Ch}(v)} \mu([w])\mu([w'])$$

IV - The Laplace-Beltrami Operator

M. Fukushima, *Dirichlet Forms and Markov Processes*, North-Holland (1980).

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Noncommutative Riemannian Geometry and Diffusion on Ultrametric Cantor Sets,
J. Noncommutative Geometry, **3**, (2009), 447-480.

A. Julien, J. Savinien,
Transverse Laplacians for Substitution Tilings,
`arXiv:0908.1095`, August 2009.

IV.1)- Dirichlet Forms

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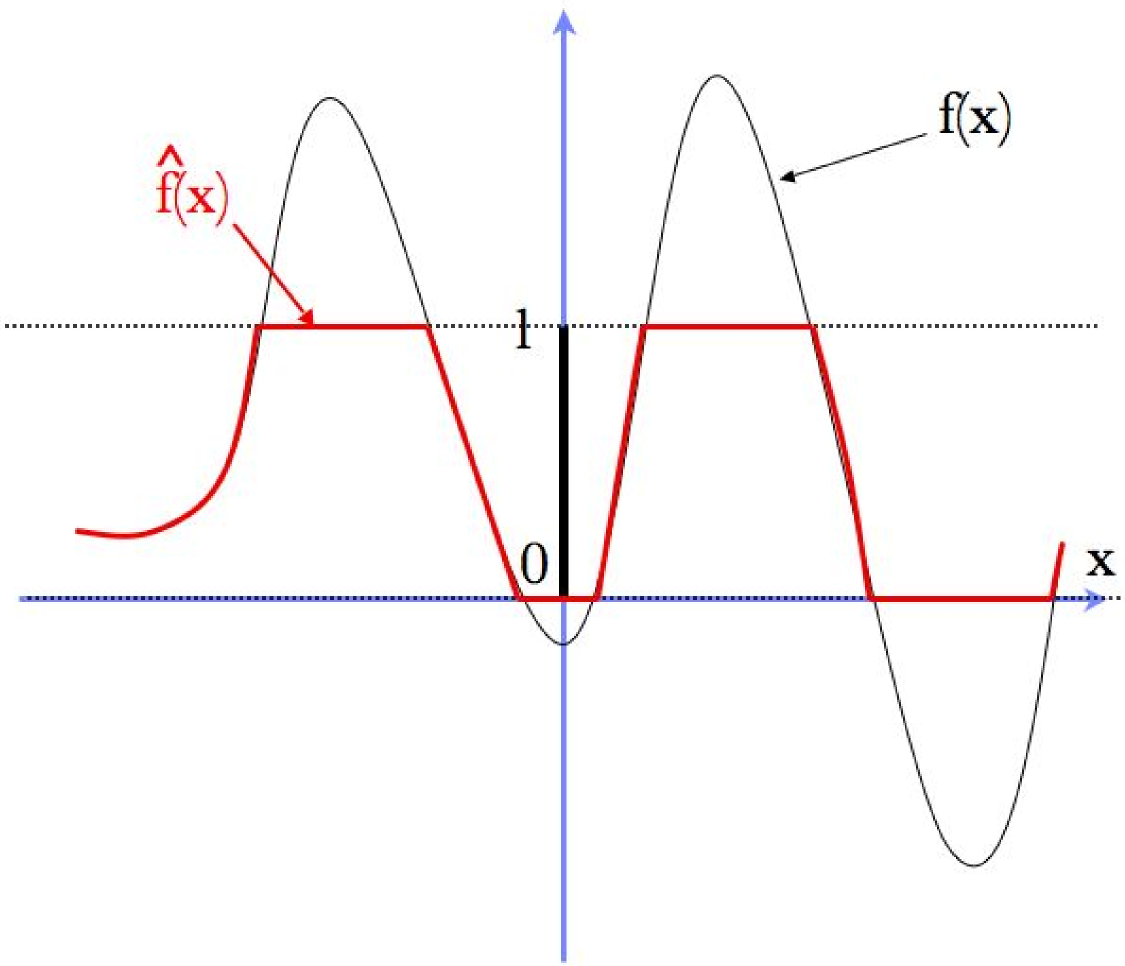
Let (X, μ) be a probability space space. For f a *real valued* measurable function on X, let $\hat{f}$ be the function obtained as

$$\hat{f}(x) = \begin{cases} 1 & \text{if} \quad f(x) \geq 1 \\ f(x) & \text{if} \quad 0 \leq f(x) \leq 1 \\ 0 & \text{if} \quad f(x) \leq 0 \end{cases}$$



Markovian cut-off of a real valued function

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- Q is densely defined with domain $\mathcal{D} \subset L^2(X, \mu)$
- Q is closed
- Q is *Markovian*, namely if $f \in \mathcal{D}$, then $Q(\hat{f}, \hat{f}) \leq Q(f, f)$

The simplest typical example of Dirichlet form is related to the Laplacian Δ_Ω on a bounded domain $\Omega \subset \mathbb{R}^D$

$$Q_\Omega(f, g) = \int_\Omega d^D x \ \overline{\nabla f(x)} \cdot \nabla g(x)$$

with domain $\mathcal{D} = C_0^1(\Omega)$ the space of continuously differentiable functions on Ω vanishing on the boundary.

This form is closeable in $L^2(\Omega)$ and its closure defines a Dirichlet form.

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If Q is a Dirichlet form on X, then the contraction semigroup $\Phi = (\Phi_t)_{t\geq 0}$ is a *Markov semigroup*.

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Theorem (Fukushima) *A contraction semi-group on $L^2(X,\mu)$ is a Markov semi-group if and only if its generator is defined by a Dirichlet form.*

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where g is the metric. Equivalently (in local coordinates)

$$Q_{\mathrm{M}}(f,g) \;=\; \int_M d^{\mathrm{D}}x\ \sqrt{\det(g(x))} \int_{S(x)} dv_x(u)\, \overline{u\cdot\nabla f(x)}\; u\cdot\nabla g(x)$$

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where $S(x)$ represent the *unit sphere* in the tangent space whereas ν_x is the *normalized Haar measure* on $S(x)$.

Similarly, if (C, d) is an ultrametric Cantor set, the expression

$$[D, \pi_\tau(f)]$$

can be interpreted as a *directional derivative*, analogous to $u \cdot \nabla f$, since a choice τ has been interpreted as a unit tangent vector.

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The *Laplace-Pearson operators* are defined, by analogy, by

$$Q_s(f, g) \;=\; \int_\Upsilon d\nu(\tau)\; \mathrm{Tr}\left\{ \frac{1}{|D|^s} [D, \pi_\tau(f)]^* \, [D, \pi_\tau(g)] \right\}$$

for $f, g \in C_{\mathrm{Lip}}(\mathrm{C})$ and $s > 0$.

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For $s < s_0 + 2$, $-\Delta_s$ is unbounded with compact resolvent.

IV.3)- Jumps Process over Gaps

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Then if χ_v is the characteristic function of $[v]$

$$\Delta_s \chi_v = \sum_{w \in \mathcal{V}(v)} p(v,w)(\chi_w - \chi_v)$$

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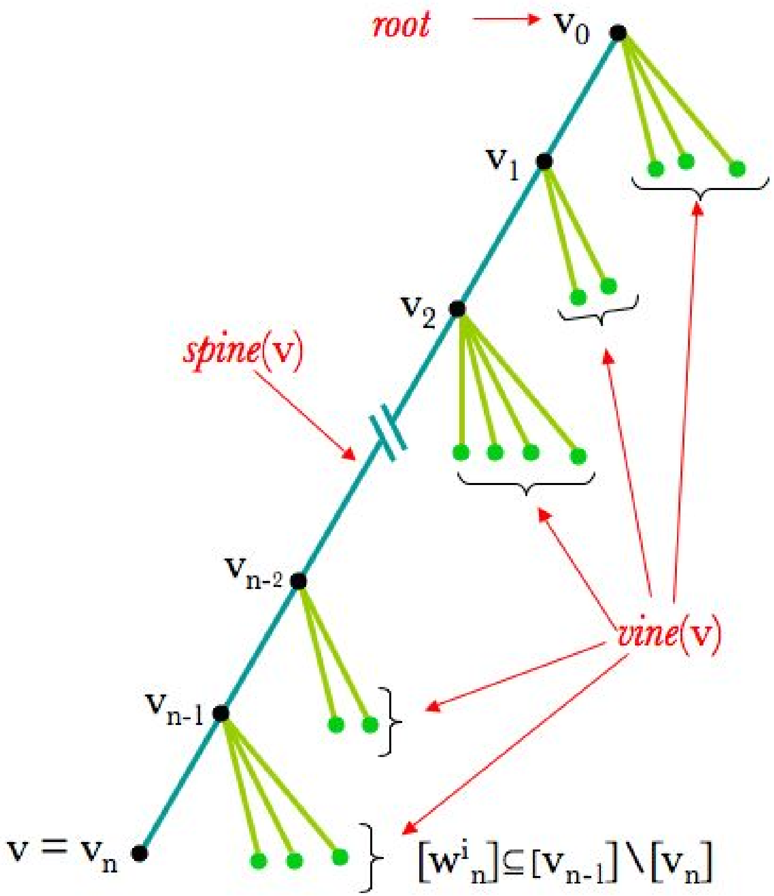
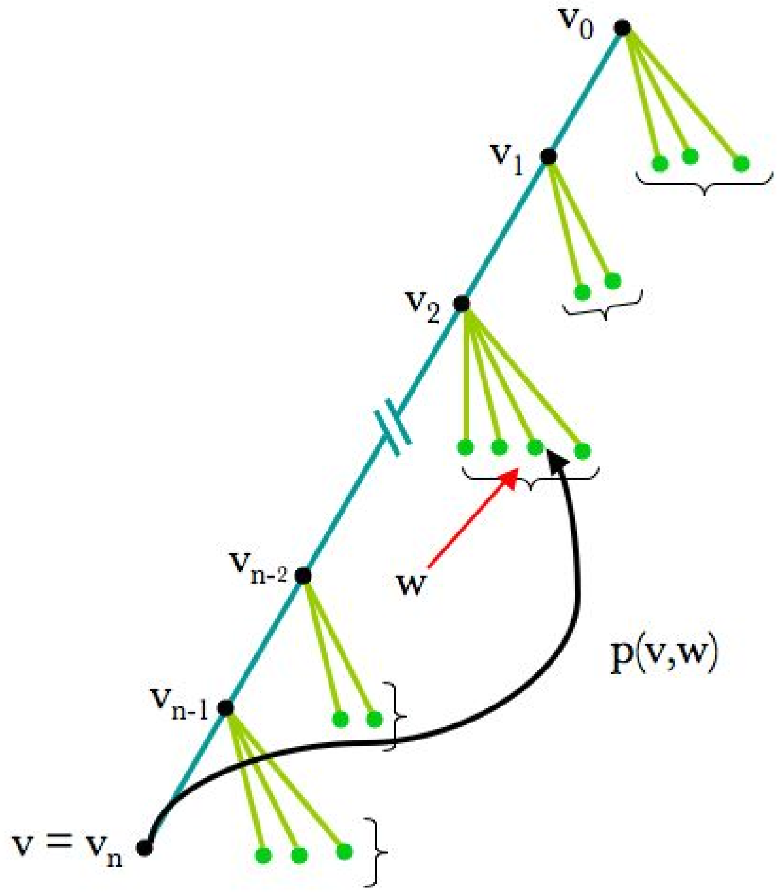
$$\Delta_s \chi_v = \sum_{w \in \mathcal{V}(v)} p(v,w)(\chi_w - \chi_v)$$

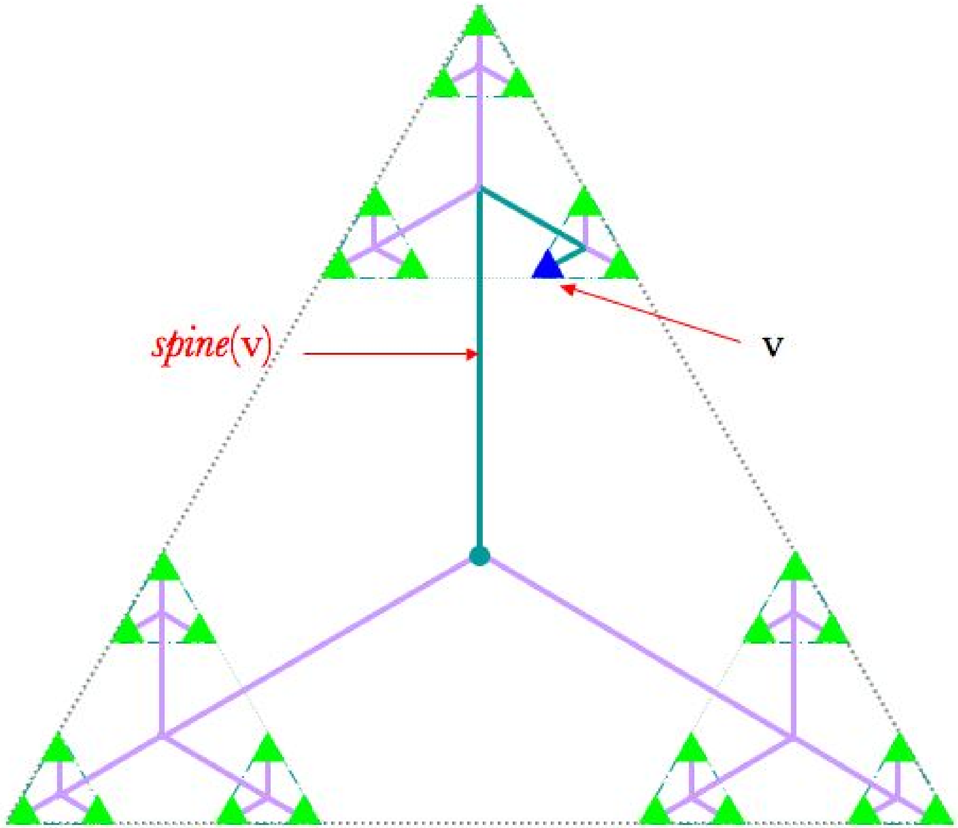
where $p(v,w) > 0$ represents the *probability for X_t to jump from v to w per unit time*.



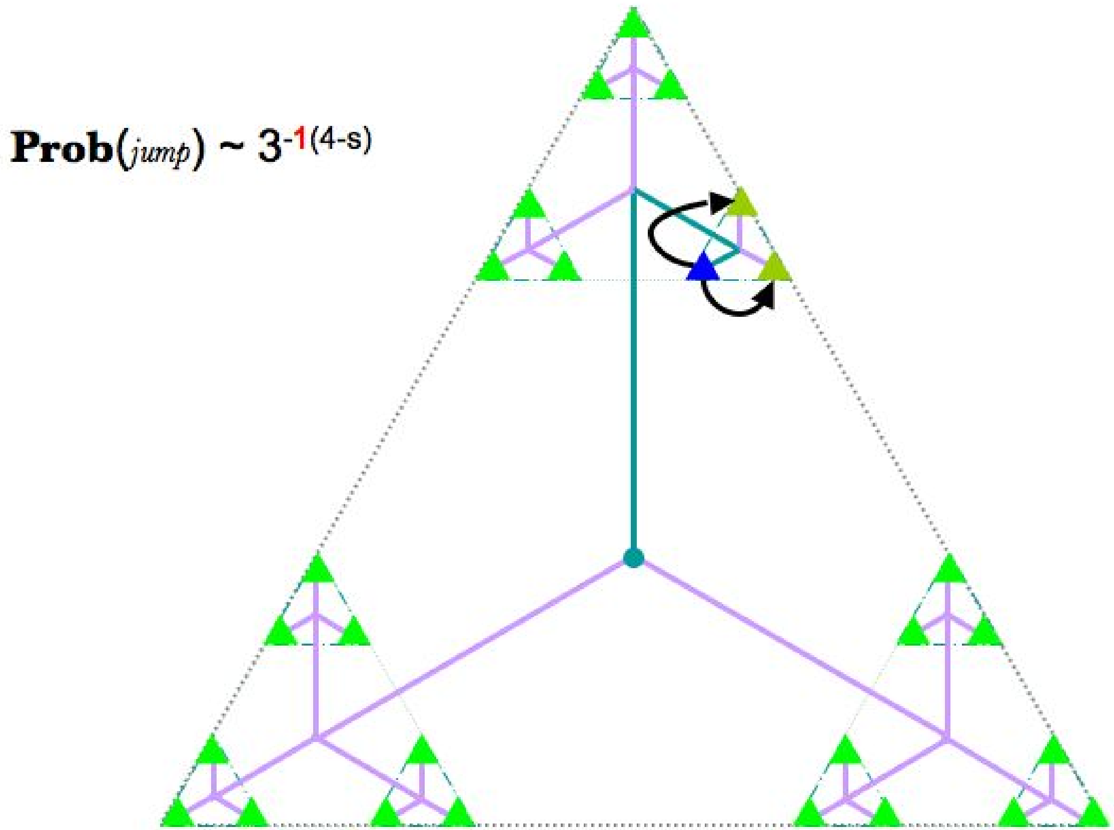
The vine of a vertex v



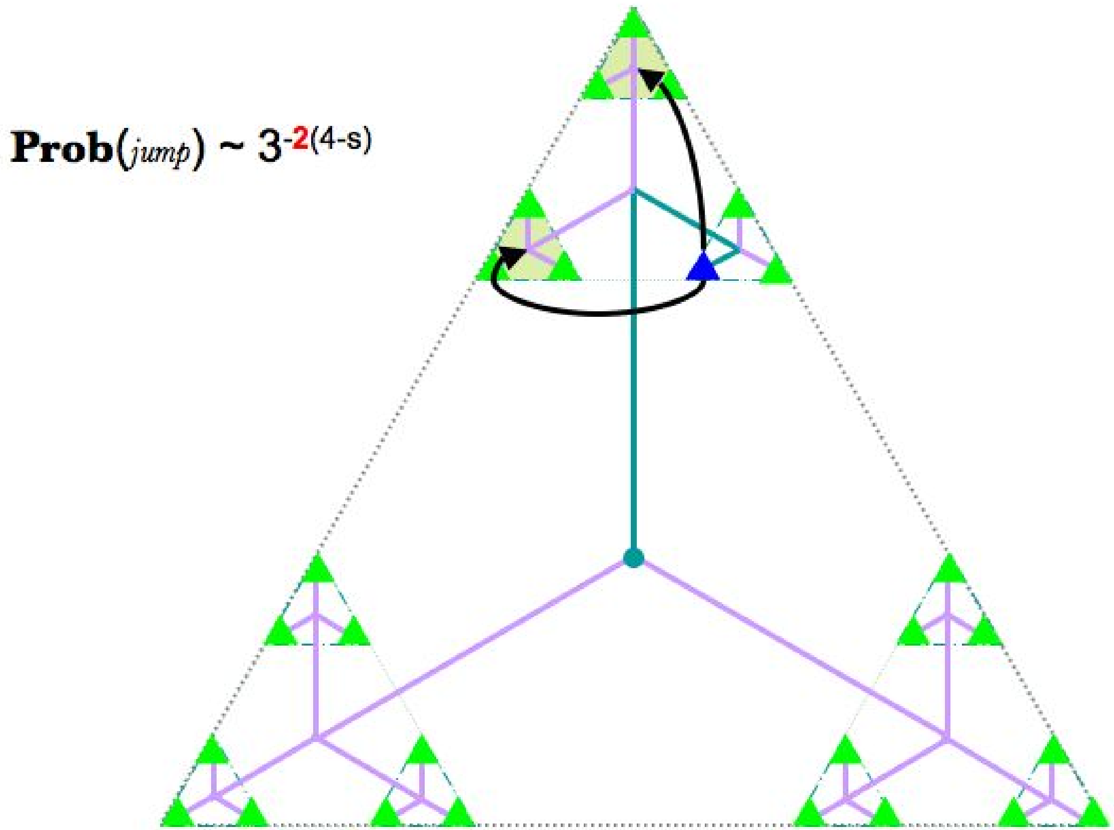
Jump process from v to w



The tree for the triadic ring $\mathbb{Z}(3)$

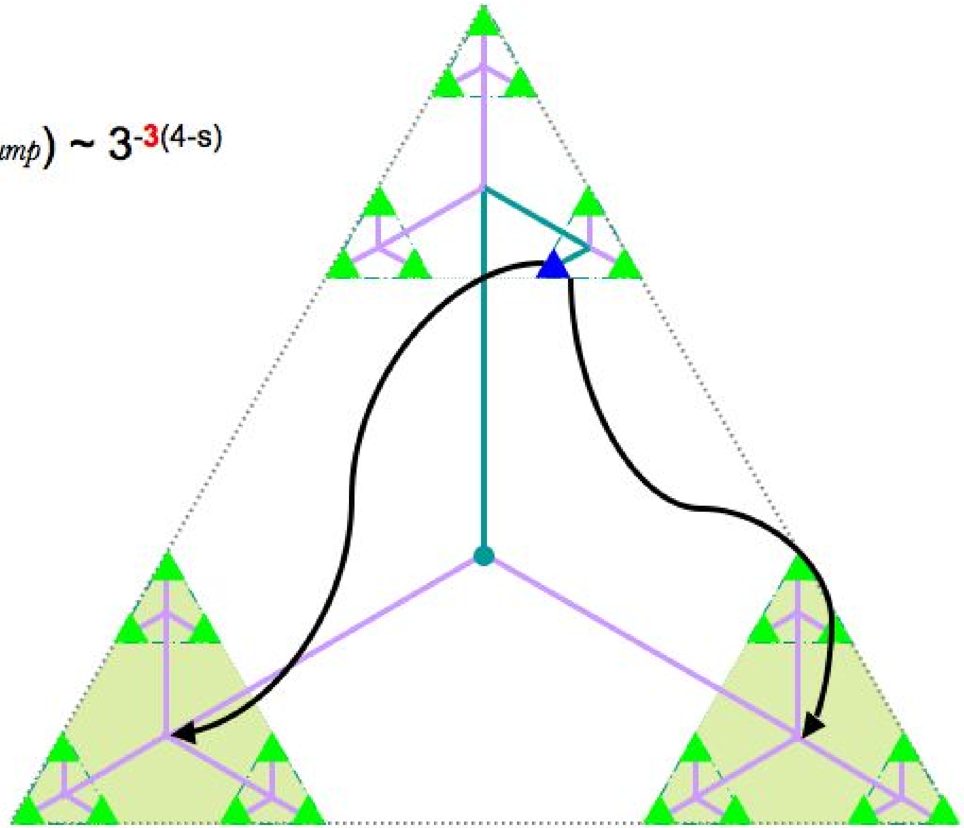


Jump process in $\mathbb{Z}(3)$



Jump process in $\mathbb{Z}(3)$

$\mathbf{Prob}(\mathit{jump}) \sim 3^{-3(4-s)}$



Jump process in $\mathbb{Z}(3)$

Concretely, if $\hat{w}$ denotes the *father* of w (which belongs to the spine)

$$p(v, w) \;=\; 2\delta(\hat{w})^{s-2} \;\frac{\mu([v])}{Z_{\hat{w}}}$$

where $Z_{\hat{w}}$ is the normalization constant for the measure $\nu_{\hat{w}}$ on the set of choices at $\hat{w}$, namely

$$Z_{\hat{w}} = \sum_{u \neq u' \in \mathrm{Ch}(\hat{w})} \mu([u])\mu([u'])$$

IV.4)- Eigenspaces

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$$\chi_v = \sum_{w \in \mathrm{Ch}(v)} \chi_w \quad \in \mathcal{E}_v \,.$$

IV.4)- Eigenspaces

Let v be a vertex of the Michon graph with $\mathrm{Ch}(v)$ as its set of children. Let $\mathcal{E}_v$ be the linear space generated by the *characteristic function* χ_w of the $[w]$'s with $w \in \mathrm{Ch}(v)$. In particular

$$\chi_v = \sum_{w \in \mathrm{Ch}(v)} \chi_w \quad \in \mathcal{E}_v \, .$$

Theorem *For any $s \in \mathbb{R}$, the eigenspaces of $-\Delta_s$ are the spaces of the form $\{\chi_v\}^\perp \subset \mathcal{E}_v$, namely, the orthogonal complement of χ_v is $\mathcal{E}_v$.*

IV.5)- The Triadic Cantor Set

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- The eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ of Δ_s can be computed explicitly

More precisely, the eigenvalues are

$$\lambda_n = -2\left(1 + 3^{s_0+2-s} + \cdots + 3^{(s_0+2-s)(n-2)} + 2 \cdot 3^{(s_0+2-s)(n-1)}\right)$$

with $n \geq 1$ *and with multiplicity*

$$g_n = 2^{n-1}$$

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- The eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ of Δ_s can be computed explicitly
- The eigenfunctions can also be computed explicitly

In the triadic Cantor set a vertex v at level n of the hierarchy, can be labeled by a finite string 0110001 of 0's and 1's of length n.

The eigenfunctions are given by the Haar functions *defined by*

$$\varphi_\omega = \sum_{v \in \{0,1\}^n} (-1)^{\omega \cdot v} \chi_v$$

where $\omega \in \{0, 1\}^{\mathbb{N}}$ *and* $|\omega| \leq n$ *if* $|\omega|$ *denotes the maximum index* k *such that* $\omega_k = 1$.

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- The *density of state* $\mathcal{N}(\lambda) = \#\{n \in \mathbb{N}\,;\, \lambda_n \leq \lambda\}$ satisfies the Weyl asymptotics (where $k > 0$ is explicit)

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- If $s = s_0$ then $\mathcal{N}(\lambda) \sim \lambda^{s_0/2}$ suggesting that s_0 is the right dimension for the *noncommutative Riemannian manifold* (C, d).

In addition, the stochastic process has an *anomalous diffusion*

$$\mathbb{E}\{d(X_{t_0}, X_{t_0+t})^2\} \stackrel{t\downarrow 0}{=} D\,t\ln(1/t)\,(1+o(1))$$

for some explicit positive D.

V - To conclude

- Ultrametric Cantor sets can be described as *Riemannian manifolds*, through Noncommutative Geometry.
- An analog of the *tangent unit sphere* is given by *choices*
- The *upper box dimension* plays the role of the dimension
- A *volume measure* is defined through the Dixmier trace
- A *Laplace-Beltrami operator* is defined with compact resolvent and Weyl asymptotics
- It generates a *jump process* playing the role of the *Brownian motion*.
- This process exhibits *anomalous diffusion*.